THE EQUIVARIANT K-THEORY AND COBORDISM RINGS OF DIVISIVE WEIGHTED PROJECTIVE SPACES

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Abstract. We apply results of Harada, Holm and Henriques to prove that the Atiyah-Segal equivariant complex K-theory ring of a divisive weighted projective space (which is singular for non-trivial weights) is isomorphic to the ring of integral piecewise Laurent polynomials on the associated fan. Analogues of this description hold for other complex-oriented equivariant cohomology theories, as we confirm in the case of homotopical complex cobordism, which is the universal example. We also prove that the Borel versions of the equivariant K-theory and complex cobordism rings of more general singular toric varieties, namely those whose integral cohomology is concentrated in even dimensions, are isomorphic to rings of appropriate piecewise formal power series. Finally, we confirm the corresponding descriptions for any *smooth*, compact, projective toric variety, and rewrite them in a face ring context. In many cases our results agree with those of Vezzosi and Vistoli for algebraic K-theory, Anderson and Payne for operational K-theory, Krishna and Uma for algebraic cobordism, and Gonzalez and Karu for operational cobordism; as we proceed, we summarize the details of these coincidences.

1. Introduction

Throughout this work G is a compact Lie group, and $G\circlearrowright Y$ a G-space; when G is understood, we rewrite the latter as Y. Our aim is to investigate the G-equivariant complex K-theory and complex cobordism rings of certain special families of Y, for which G is a torus T^n. Given the recent proliferation of K-theory and cobordism functors [AnPa], [C], [GoKa], [KU], it is important to specify precisely which we use, and to comment on their relationship with other versions as we proceed. Our underlying philosophy is closest to algebraic topology and homotopy theory.

So far as K-theory is concerned, we focus mainly on the unreduced Atiyah-Segal G-equivariant ring $K_G^*(Y)$ [Se], graded over the integers for later convenience. If Y is compact, then $K_G^0(Y)$ is constructed from equivalence classes of G-equivariant complex vector bundles; otherwise, it is given by equivariant homotopy classes $[Y, \mathrm{Fred}(\mathcal{H}_G)]_G$, where $\mathcal{H}_G$ is a Hilbert space containing infinitely many copies of each irreducible representation of G [AS]. For the 1-point space $*$ with trivial G-action, we write the *coefficient ring* $K_G^*(*)$ as K_G^*. It is isomorphic to $R(G)[z, z^{-1}]$, where $R(G)$ denotes the complex representation ring of G, and realises K_G^0; the *Bott periodicity element* z has cohomological dimension -2. The equivariant projection $Y \to *$ induces the structure of graded K_G^*-algebra on $K_G^*(Y)$, for any $G\circlearrowright Y$.

For complex cobordism, our primary interest is tom Dieck's G-equivariant ring $MU_G^*(Y)$ [tD], defined by equivariant stable homotopy classes $[Y, MU_G]_G$ of maps into the Thom spectrum MU_G. Although the coefficient ring MU_G^* remains undetermined, considerable information is available when $G = T^n$; Sinha [Si], for example, has made extensive calculations, and solved the case $n = 1$. The equivariant projection $Y \to *$ induces the structure of graded MU_G^*-algebra on $MU_G^*(Y)$, for any $G\circlearrowright Y$. In fact $MU_G^*(-)$ is universal amongst unreduced complex-oriented G-equivariant cohomology theories, at least for abelian G [CGK02]. The most natural link between cobordism and K-theory arises within this framework, and is provided by the equivariant Todd genus $td\colon MU_G^*(Y) \to K_G^*(Y)$.

Given the universality of $MU_G^*(-)$, we may follow the lead of [HaHeHo] and consider arbitrary complex-oriented cohomology theories $E_G^*(-)$ for abelian G. In these situations, readers will

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lose little by interpreting E as whichever of complex K-theory, complex cobordism, or integral cohomology takes their fancy. When more detail is required, we shall treat K-theory first and cobordism second; both depend on the more familiar case of integral cohomology, which we recall as necessary.

From this point onwards, we insist that G be a torus.

If Y is an n–dimensional toric variety [Fu], it is automatically endowed with an action of T^n. The problem of describing the ring $K^*_{T^n}(Y)$ has already been addressed, albeit indirectly, in the symplectic context [HaL]. From the algebraic viewpoint, however, it is more natural to study algebraic vector bundles over Y, and to compute the corresponding algebraic K-theory. Influential contributions along these lines include Brion [B], Dupont [D], Kaneyama [Kan], Klyachko [Kl], Liu and Yao [LY], Morelli [Mor], and Payne [P], although the work of Vezzosi and Vistoli [VV] is closest to ours in spirit, and leads to answers that are isomorphic to $K^*_{T^n}(Y)$ for all smooth Y. The results of [VV] are also cited in the appendix to [RKR], with appeal to arguments of Franz. The same arguments are likely to provide an alternative approach to our own results, but have yet to be fully documented. We emphasise our insistence on integer coefficients, bearing in mind that several of the above authors tensor their K-groups with $\mathbb{Q}$ or $\mathbb{C}$ throughout.

The techniques used by symplectic geometers apply to a wider class of T^n-spaces, and are based on symplectic reduction, Morse theory [HoM], and GKM graphs [GKM]. Algebraic geometers, on the other hand, tend to restrict attention (at least over $\mathbb{Z}$) to smooth toric varieties Y, possibly non-compact, and express their invariants in terms of the underlying fan Σ_Y.

Our aim is to combine features of each viewpoint, and describe $K^*_{T^n}(Y)$ in terms of Σ_Y for a certain family of *singular* toric varieties. Following [BFR2], we refer to these as *divisive weighted projective spaces*, and denote them by $\mathbb{P}(\chi)$. Recent work [BFNR, Theorem 1.2] shows that *any* weighted projective space is homotopy equivalent to one which is divisive, but such equivalences need not be equivariant, by the concluding remarks of [BFR1, Section 5].

We are motivated by related calculations of the Borel equivariant cohomology of more general singular examples, in which $H^*_{T^n}(Y;\mathbb{Z})$ is identified with the graded ring of piecewise polynomials on Σ_Y [BFR1]. In its simplest form, our main result states the following.

Theorem 1.1. *For any divisive weighted projective space, $K^0_{T^n}(\mathbb{P}(\chi))$ is isomorphic as $K^0_{T^n}$-algebra to the ring of piecewise Laurent polynomials on Σ_Y; furthermore, $K^1_{T^n}(\mathbb{P}(\chi))$ is zero.*

A precise statement is proven as Theorem 5.5.1 in Section 5.

More recently, Anderson and Payne [AnPa] have introduced equivariant *operational* algebraic K-theory, and identified the rings $\mathrm{op}K^\circ_{T^n}(Y)$ with piecewise Laurent polynomials on Σ_Y. Their calculations are valid for *all* toric varieties, and therefore agree with ours on divisive weighted projective spaces.

Turning to $MU^*_{T^n}(\mathbb{P}(\chi))$, we note first that algebraic geometers have developed a successful theory of *algebraic cobordism* during the last 15 years, by working over the Lazard ring L^*. They have also introduced equivariant versions that are related to $MU^*_{T^n}(-)$. As described by Krishna and Uma [KU], for example, these theories are complete; so their coefficient rings cannot be isomorphic to $MU^*_{T^n}$. Nevertheless, the equivariant algebraic cobordism ring of many toric varieties Y may be expressed in terms of piecewise formal power series on Σ_Y [KU]. As we explain below, this is isomorphic to the Borel equivariant cobordism ring $MU^*(ET^n \times_{T^n} Y)$ in cases such as smooth Y, or products of weighted projective space.

Our conclusions for complex cobordism are based on the fact that $MU^*_{T^n}$ is an algebra over the Lazard ring L^*, graded cohomologically. So we refer to $MU^*_{T^n}$ as the ring of *T^n-cobordism forms*, and express our second result accordingly.

Theorem 1.2. *For any divisive weighted projective space, $MU^*_{T^n}(\mathbb{P}(\chi))$ is isomorphic as $MU^*_{T^n}$-algebra to the ring of piecewise cobordism forms on Σ_Y; in particular, $MU^*_{T^n}(\mathbb{P}(\chi))$ is zero in odd dimensions.*

A precise statement is proven as Theorem 5.5.2 in Section 5.

Most recently, inspired by [AnPa], Gonzalez and Karu [GoKa] have defined equivariant *operational* algebraic cobordism. For any quasiprojective toric variety Y, they identify their operational ring with the ring of piecewise formal power series on Σ_Y, and therefore with $MU^*(ET^n \times_{T^n} Y)$ for smooth Y, or products of weighted projective spaces.

We introduce weighted projective spaces as singular toric varieties in Section 2, focusing on divisive examples $\mathbb{P}(\chi)$ and their invariant CW-structures. In Section 3 we recall the generalised GKM-theory that allows us to compute $K^*_{T^n}(Y)$ and $MU^*_{T^n}(Y)$ for certain stratified T^n-spaces Y, and confirm that the theory applies to $\mathbb{P}(\chi)$. In order to rewrite the outcome in the context of Theorems 1.1 and 1.2, we devote Section 4 to describing diagrams of algebras, and piecewise structures on arbitrary fans. We combine the two viewpoints in Section 5, and deduce a version of Theorems 1.1 and 1.2 that holds for a wider class of equivariant cohomology theories. In Section 6 we relate $K^*_{T^n}(\mathbb{P}(\chi))$ and $MU^*_{T^n}(\mathbb{P}(\chi))$ to the Borel equivariant K-theory and cobordism of $\mathbb{P}(\chi)$, in terms of piecewise formal power series, the Chern character, and the Boardman homomorphism. Finally, in Section 7, we extend our conclusions to *smooth* toric varieties, and rewrite the resulting piecewise algebras in the context of face rings.

Before we begin, it is convenient to introduce notation and conventions that we shall use without further comment.

We write S^1 for the circle as a topological space, and $T < \mathbb{C}^\times$ for its realisation as the group of unimodular complex numbers under multiplication. The $(n+1)$–dimensional compact torus T^{n+1} is a subgroup of the algebraic torus $(\mathbb{C}^\times)^{n+1}$, and acts on $\mathbb{C}^{n+1}$ by coordinate-wise multiplication. This is the *standard action*; it preserves the unit sphere $S^{2n+1} \subset \mathbb{C}^{n+1}$, and the corresponding orbit space may be identified with the standard n-simplex $\Delta^n \subset \mathbb{R}^{n+1}_{\geqslant}$ in the positive orthant. For any integer $k > 0$ we write $\mathbb{Z}/k$ for the integers modulo k, and $C_k < T$ for its realisation as the subgroup generated by a primitive kth root of unity.

Readers who require background information and further references on equivariant topology may consult [AlPu], and the survey articles in [Ma]. For fans, toric varieties, and their topological aspects, we suggest [Fr], [Fu] and [Od].

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2. Weighted projective space

Weighted projective spaces are amongst the simplest and most elegant examples of *toric orbifolds* [Fu], and we devote this section to summarising their definition and basic properties.

A *weight vector* χ is a sequence $(\chi_0, \dots, \chi_n)$ of $n+1$ positive integers; χ determines a subcircle $T(\chi) < T^{n+1}$ by

$$T(\chi) \;=\; \{(t^{\chi_0}, \dots, t^{\chi_n}) : |t| = 1\},$$

which acts on S^{2n+1} with finite stabilizers. Then the weighted projective space $\mathbb{P}(\chi)$ is defined to be the orbit space $S^{2n+1}/T(\chi)$. Each point of $\mathbb{P}(\chi)$ may be written as an equivalence class $[z] = [z_0, \dots, z_n]$, where the z_j are known as *homogenous coordinates*. Permutations of the z_j induce self-homeomorphisms of $\mathbb{P}(\chi)$, so we may reorder the weights as required; it is often convenient to assume that they are non-decreasing. Of course $\mathbb{P}(\chi)$ may equally well be exhibited as the orbit space $(\mathbb{C}^{n+1} \setminus \{0\})/\mathbb{C}^\times(\chi)$, and therefore as a complex algebraic variety.

The finite stabilisers ensure that $\mathbb{P}(\chi)$ is an *orbifold*, which is singular unless $\chi = (d, \dots, d)$ for some positive integer d. The residual action of the torus $T^n \cong T^{n+1}/T(\chi)$ turns $\mathbb{P}(\chi)$ into a *toric* orbifold, with quotient polytope an n-simplex. If $\chi = (1, \dots, 1)$, then $T(\chi)$ is the diagonal circle $T_\delta < T^{n+1}$, and $\mathbb{P}(\chi)$ reduces to the standard projective space $\mathbb{C}P^n$. In this case, $K^*_{T^n}(\mathbb{C}P^n)$ is computed in [GW, Section 4.1].

Orbifolds may be studied from a number of different perspectives, and recent articles have focused on their interpretation as *groupoids* [Moe] and as *stacks* [L]. Several invariants of of these richer structures have been defined, such as the orbifold fundamental group [ALR]. Nevertheless, in this work we remain firmly in the topological world, and study the underlying topological space $\mathbb{P}(\chi)$; in other contexts, it is known as the *coarse moduli space* of the stack. There has been a recent surge of interest [BFR1], [BFNR] in its T^n-equivariant topological invariants.

By construction, $\mathbb{P}(d\chi)$ is equivariantly homeomorphic to $\mathbb{P}(\chi)$ for any positive integer d, although they differ as orbifolds. For our purposes, it therefore suffices to assume that the greatest common divisor $\gcd(\chi)$ is 1; in orbifold terminology, this is tantamount to restricting attention to *effective* cases. If $\gcd(\chi) = 1$, then [Do] provides an equivariant homeomorphism

$$\mathbb{P}(d\chi_0, \dots, d\chi_{j-1}, \chi_j, d\chi_{j+1}, \dots, d\chi_n) \;\cong\; \mathbb{P}(\chi)$$

for any $0 \le j \le n$, and any positive integer d such that $\gcd(d, \chi_j) = 1$. Further simplification is therefore possible by insisting that χ be *normalised*, in the sense that

$$\gcd(\chi_0, \dots, \widehat{\chi_j}, \dots, \chi_n) \;=\; 1 \tag{2.1}$$

for every $0 \le j \le n$.

We may impose additional restrictions on the weights, with important topological consequences.

Definitions 2.2. The weight vector χ and the weighted projective space $\mathbb{P}(\chi)$ are

1. *weakly divisive* if χ_j divides χ_n for every $0 \le j < n$
2. *divisive* if χ_{j-1} divides χ_j for every $1 \le j \le n$.

A divisive χ is automatically weakly divisive, and is necessarily non-decreasing. Moreover, χ is divisive precisely when the reverse sequence $\chi_n, \ldots, \chi_0$ is *well ordered*, in the sense of Nishimura and Yosimura [NY].

Remark 2.3 ([BFR2, Theorem 3.7, Corollary 3.8])**.** If $\mathbb{P}(\chi)$ is weakly divisive, then it is homeomorphic to the Thom space of a complex line bundle over $\mathbb{P}(\chi')$, where $\chi' = (\chi_0, \dots, \chi_{n-1})$; if it is divisive, then it is homeomorphic to an n-fold iterated Thom space of complex line bundles over the one-point space $*$.

In case $\chi_0 = 1$, there exists a canonical isomorphism $j\colon T^n \to T^{n+1}/T(\chi)$, defined by setting $j(u_1, \dots, u_n) = [1, u_1, \dots, u_n]$; the resulting action of T^n on $\mathbb{P}(\chi)$ satisfies

$$(u_1, \dots, u_n) \cdot [z_0, z_1, \dots, z_n] \;=\; [z_0, u_1 z_1, \dots, u_n z_n]. \tag{2.4}$$

From this point onwards, we therefore make the following assumptions.

Assumptions 2.5.

1. *The weight vector χ is both normalised and divisive, so $\chi_0 = \chi_1 = 1$.*
2. *The residual action $T^n \circlearrowright \mathbb{P}(\chi)$ is given by the isomorphism j and* (2.4)*.*

For *any* weighted projective space, Kawasaki's calculations [Kaw] imply the existence of a homotopy equivalent CW-complex with a single cell in every even dimension. This has been realized in [BFNR], but current evidence suggests that an explicit cellular decomposition for the general case is unpleasantly complicated [ON]. Nevertheless, Remark 2.3 provides an easy solution for divisive χ. Given any $1 \le i \le n$, it is convenient to write D^{2i} for the closed unit disc

$$\{w : |w_{n-i+1}|^2 + \cdots + |w_n|^2 \le 1\} \;\subset\; \mathbb{C}^i,$$

and $g_i\colon D^{2i} \to \mathbb{P}(\chi)$ for the map given by

$$g_i(w) \;=\; \big[0, \dots, 0, (1 - |w_{n-i+1}|^2 - \cdots - |w_n|^2)^{1/2}, w_{n-i+1}, \dots, w_n\big]. \tag{2.6}$$

For $i = 0$, let $D^0 = \{0\}$ and $g_0(0) = [0, \dots, 0, 1]$.

Proposition 2.7. *For every divisive $\mathbb{P}(\chi)$, the g_i are characteristic maps for a CW-structure that contains exactly $n + 1$ cells.*

Proof. For $1 \le i \le n$, the restriction of g_i to the interior of D^{2i} is a homeomorphism onto

$$C_i \;=\; \{[z] : z_0 = \cdots = z_{n-i-1} = 0,\ z_{n-i} \ne 0\} \;\subset\; \mathbb{P}(\chi), \tag{2.8}$$

which is therefore an open $2i$-cell. Furthermore, g_i maps the boundary of D^{2i} onto the subspace $\{[z] : z_0 = \cdots = z_{n-i} = 0\}$, which is the union of all lower dimensional cells. The zero cell is $C_0 = \{[0, \dots, 0, 1]\}$. $\square$

Corollary 2.9. *The CW-structure is invariant under the residual action of T^n.*

Proof. The action (2.4) automatically preserves the conditions of (2.8). □

Combining (2.4) and (2.6) shows that the characteristic map g_i induces the action

$$(2.10)\qquad (u_1,\dots,u_n)\cdot(w_{n-i+1},\dots,w_n) \;=\; (u_{n-i+1}u_{n-i}^{-\chi_{n-i+1}/\chi_{n-i}}w_{n-i+1},\dots,u_n u_{n-i}^{-\chi_n/\chi_{n-i}}w_n)$$

of T^n on D^{2i}, for each $1\leq i\leq n$ (taking $u_0=1$ in case $i=n$). This is the unit disc $D(\rho_i)$ of an i-dimensional unitary representation ρ_i of T^n.

We denote the CW-structure of Proposition 2.7 by $\mathbb{P}(\chi)=e^0\cup e^2\cup\cdots\cup e^{2n}$, where e^{2i} is the closure of C_i in $\mathbb{P}(\chi)$; the centers $[0,\dots,0,1,0,\dots,0]$ of the cells are precisely the fixed points of the residual action.

3. Generalized GKM-theory

In this section we recall the generalized GKM-theory of [HaHeHo], and explain its application to Corollary 2.9. This leads to a description of $E^*_{T^n}(\mathbb{P}(\chi))$ for divisive χ and several examples of E_{T^n}, including equivariant complex K-theory and cobordism.

Following [HaHeHo, §3], we require the space $G\circlearrowright Y$ to be equipped with a G-invariant stratification $Y=\bigcup_{i\in I}Y_i$, and write $Y_{<i}$ for the subspace $\bigcup_{j<i}Y_j\subset Y_i$ for every $i\in I$. We insist that each Y_i contains a subspace F_i, whose neighbourhood is homeomorphic to the total space V_i of a G-equivariant E_G-oriented vector bundle $\rho_i:=(V_i,\pi_i,F_i)$. As usual, the *equivariant Euler class* $e_G(\rho_i)$ is defined in $E_G^{\dim V_i}(F_i)$ by restricting the equivariant Thom class of ρ_i to the zero section, for every $i\in I$.

We recall the following four assumptions of [HaHeHo], which insure that $E^*_G(Y)$ may be computed by their methods. As we shall see, they are satisfied by every divisive $\mathbb{P}(\chi)$.

Assumptions 3.1.

1. *Each subquotient $Y_i/Y_{<i}$ is homeomorphic to the Thom space $Th(\rho_i)$, with attaching maps $\varphi_i: S(\rho_i)\to Y_{<i}$.*
2. *Every ρ_i admits a G-equivariant direct sum decomposition $\bigoplus_{j<i}\rho_{i,j}$ into E_G-oriented subbundles $\rho_{i,j}=(V_{i,j},\pi_{i,j},F_i)$.*
3. *There exist G-equivariant maps $f_{i,j}\colon F_i\to F_j$ such that the restrictions $f_{i,j}\circ\pi_{i,j}|_{S(\rho_{i,j})}$ and $\varphi_i|_{S(\rho_{i,j})}$ agree for every $j<i$.*
4. *The Euler classes $e_G(\rho_{i,j})$ are not divisors of zero in $E^*_G(F_i)$ for any $j<i$, and are pairwise relatively prime.*

Note that the $\rho_{i,j}$ may have dimension 0. Assumption 3.1.4 means that $e_G(\rho_{i,j})$ divides y for each j if and only if $e_G(\rho_i)$ divides y, for any $y\in E^*_G(F_i)$.

Now let $\iota^*\colon E^*_G(Y)\to\prod_i E^*_G(F_i)$ be the homomorphism induced by the inclusion $\coprod_i F_i\subset Y$.

Theorem 3.2 ([HaHeHo, Theorem 3.1])**.** *Let Y be a G-space satisfying the four Assumptions 3.1; then ι^* is monic, and has image*

$$\mathit{\Gamma}_Y:=\left\{y=(y_i): e_G(\rho_{i,j})\text{ divides } y_i-f^*_{i,j}(y_j)\text{ for all } j<i\right\}\;\leq\;\prod_i E^*_G(F_i).$$ □

As in several of the examples in [HaHeHo], our application to Corollary 2.9 involves a T^n-invariant skeletal filtration. Specifically, $Y_i=\cup_{j\leq i}e^{2j}$ is the $2i$–skeleton of $\mathbb{P}(\chi)$ for $0\leq i\leq n$, and the $F_i\subset Y_i$ contain only the centers of the cells e^{2i}. These are the fixed points of the T^n-action, and the T^n-equivariant bundles ρ_i reduce to i–dimensional complex representations, which are canonically E_{T^n}-oriented. Assumption 3.1.1 is then satisfied, where the rôles of the ϕ_i are played by the restrictions of the g_i of Proposition 2.7 to S^{2i-1}. The equivariant Euler classes $e_{T^n}(\rho_i)$ lie in the coefficient ring $E^*_{T^n}$.

In order to check Assumption 3.1.2, we refer back to (2.10). Each ρ_i decomposes as a sum $\bigoplus_{j<i}\rho_{i,j}$ of 1–dimensionals, where $\rho_{i,j}$ is defined by

$$(3.3)\qquad (u_1,\dots,u_n)\cdot w_{n-j} \;=\; u_{n-j}u_{n-i}^{-\chi_{n-j}/\chi_{n-i}}w_{n-j}$$

for $0 \leq j < i \leq n$. These decompositions respect the canonical E_{T^n}-orientations, by definition.

For Assumption 3.1.3, the maps $f_{i,j}$ are necessarily constant and equivariant, so the restrictions to $S(\rho_{i,j})$ of $f_{i,j} \circ \pi_{i,j}$ and g_i agree, for every $j < i$.

Before confirming Assumption 3.1.4, recall [Hu] that the complex representation ring of T^n is isomorphic to the Laurent polynomial algebra

$$R(T^n) \;\cong\; S^{\pm}_{\mathbb{Z}}(\alpha) \;:=\; \mathbb{Z}[\alpha_1, \ldots, \alpha_n]_{(\alpha_1 \cdots \alpha_n)} \tag{3.4}$$

on generators α_j, which represent the 1–dimensional irreducibles defined by projection onto the jth coordinate circle. In particular, (3.3) states that

$$\rho_{i,j} \;\cong\; \alpha_{n-j}\alpha_{n-i}^{-\chi_{n-j}/\chi_{n-i}} \tag{3.5}$$

(taking $\alpha_0 = 1$ in case $i = n$). Since equivariant Euler classes behave exponentially, e_{T^n} is determined on any representation by its value on the monomials $\alpha^J := \alpha_1^{j_1} \ldots \alpha_n^{j_n}$, which form an additive basis for $R(T^n)$ as J ranges over $\mathbb{Z}^n$.

In the case of K-theory, the coefficient ring $K^*_{T^n}$ is isomorphic to $R(T^n)$ in even dimensions, and is zero in odd [Se]. Periodicity may be made explicit by incorporating the Bott element z into (3.4) and writing

$$K^*_{T^n} \;\cong\; S^{\pm}_{K^*}(\alpha) \;\cong\; R(T^n)[z, z^{-1}]\,, \tag{3.6}$$

where α^J (for any J) and z have cohomological dimensions 0 and -2 respectively. The notation reflects the fact that the coefficient ring K^* is isomorphic to $\mathbb{Z}[z, z^{-1}]$. Then $e_{T^n}(\alpha^J) = 1 - \alpha^{\pm J}$, where both choices of sign occur in the literature. Some authors even prefer $z^{-1}(1 - \alpha^{\pm J})$, to achieve greater consistency with cobordism and cohomology by realizing the Euler class in cohomological dimension 2; here, for notational convenience, we employ $1 - \alpha^J$. The kernel of the augmentation $K^*_{T^n} \to K^*$ is the ideal $(1 - \alpha_1, \ldots, 1 - \alpha_n)$.

In the case of complex cobordism, the coefficient ring $MU^*_{T^n}$ is an algebra over L^*, and is freely generated as L^*-module by even-dimensional elements [C]. The Euler classes $e_{T^n}(\alpha^J)$ are non-zero elements $e(\alpha^J)$ of $MU^2_{T^n}$, and feature prominently in the calculations in [Si] and elsewhere; they generate the positive–dimensional subring $MU^{>0}_{T^n}$. The kernel of the augmentation $MU^*_{T^n} \to L^*$ is the ideal $(e(\alpha_1), \ldots, e(\alpha_n))$ [CM].

In the case of Borel equivariant integral cohomology, the coefficient ring $H\mathbb{Z}^*_{T^n}$ is isomorphic to the polynomial algebra

$$H^*(BT^n; \mathbb{Z}) \;\cong\; S_{\mathbb{Z}}(x) \;:=\; \mathbb{Z}[x_1, \ldots, x_n] \tag{3.7}$$

on 2–dimensional generators x_j. Then $e_{T^n}(\alpha^J) = \sum_J j_k x_k$ for any J; in particular, the equation $e_{T^n}(\alpha_i) = x_i$ may be taken to define x_i for every $1 \leq i \leq n$. The kernel of of the augmentation $H^*_{T^n} \to H^*$ is the ideal $(x_1, \ldots, x_n)$.

So from (3.5), we deduce that

$$e_{T^n}(\rho_{i,j}) \;=\; \begin{cases} 1 - \alpha_{n-j}\alpha_{n-i}^{-\chi_{n-j}/\chi_{n-i}} & \text{in } K^0_{T^n} \\ e(\alpha_{n-j}\alpha_{n-i}^{-\chi_{n-j}/\chi_{n-i}}) & \text{in } MU^2_{T^n} \\ x_{n-j} - (\chi_{n-j}/\chi_{n-i})x_{n-i} & \text{in } H\mathbb{Z}^2_{T^n} \end{cases} \tag{3.8}$$

for $0 \leq j < i \leq n$.

In each of these three cases, the ambient ring is an integral domain; for $MU^*_{T^n}$, this is proven in [Si, Corollary 5.3]. So none of the Euler classes of (3.8) are divisors of zero. The following criteria address the remaining parts of Assumption 3.1.4.

Criteria 3.9 ([HaHeHo, Lemma 5.2]). *Given any finite set of non-zero α^J, their equivariant Euler classes are pairwise relatively prime in $K^*_{T^n}$ or $MU^*_{T^n}$ whenever no two J are linearly dependent in $\mathbb{Z}^n$; the additional condition that no prime p divides any two J is required in $H\mathbb{Z}^*_{T^n}$.* □

For $\rho_{i,j}$ with $i < n$, (3.5) shows that J has only two non-zero entries, namely 1 in position $n - j$ and $-\chi_{n-j}/\chi_{n-i}$ in position $n - i$; for $\rho_{n,j}$, there is a single 1 in position $n - j$. So Criteria

3.9 confirm that the Euler classes $e_{T^n}(\rho_{i,j})$ are pairwise relatively prime in all three cases, and therefore that Assumption 3.1.4 also holds.

We may now conclude our first description of $E^*_{T^n}(\mathbb{P}(\chi))$.

Proposition 3.10. *For any divisive weighted projective space, $E^*_{T^n}(\mathbb{P}(\chi))$ is isomorphic as $E^*_{T^n}$-algebra to the subring*

$$\Gamma_{\mathbb{P}(\chi)} \;=\; \{y : e_{T^n}(\rho_{i,j}) \text{ divides } y_i - y_j \text{ for all } j < i\} \;\leq\; \prod_i E^*_{T^n},$$

in each of the cases $K^*_{T^n}(\mathbb{P}(\chi))$, $MU^*_{T^n}(\mathbb{P}(\chi))$ and $H\mathbb{Z}^*_{T^n}(\mathbb{P}(\chi))$.

Proof. Our preceding analysis shows that Theorem 3.2 applies directly to the skeletal filtration. Compatibility with the $E^*_{T^n}$-algebra structure follows immediately. □

Proposition 3.10 shows that $E^*_{T^n}(\mathbb{P}(\chi))$ is zero in odd dimensions.

The idea behind Theorem 5.5 is to convert Proposition 3.10 into a form more directly related to the properties of the fan $\Sigma_\chi := \Sigma_{\mathbb{P}(\chi)}$.

4. Piecewise algebra

Before stating Theorem 5.5, we introduce certain algebraic and geometric constructions associated to arbitrary fans by the theory of diagrams. They are motivated by modern approaches to homotopy theory, and provide a common language in which to address the cases under discussion.

A rational fan Σ in $\mathbb{R}^n$ determines a small category $\text{CAT}(\Sigma)$, whose objects are the cones σ and morphisms their inclusions $i_{\sigma,\tau}\colon \sigma \subseteq \tau$. The zero cone $\{0\}$ is initial, and the maximal cones admit only identity morphisms. The opposite category $\text{CAT}^{op}(\Sigma)$ has morphisms $p_{\tau,\sigma}\colon \tau \supseteq \sigma$, and $\{0\}$ is final.

For $0 \leq d \leq n$, the set of d–dimensional cones is denoted by $\Sigma(d) \subseteq \Sigma$. The elements of $\Sigma(1)$ are known as *rays*, and are represented by primitive vectors $v_1, \ldots, v_m$, where m denotes the cardinality of $\Sigma(1)$ henceforth. Each cone may be identified by its generating rays $v_{j_1}, \ldots, v_{j_k}$, and interpreted as a subset $\sigma \subseteq \Sigma(1)$. The cardinality $k = |\sigma|$ coincides with the dimension $d = \dim(\sigma)$ if and only if the cone σ is simplicial.

Every d-dimensional σ gives rise to an $(n-d)$–dimensional subspace $\mathbb{R}_{\sigma^\perp} \subseteq \mathbb{R}^n$, by forming the orthogonal complement of its linear hull $\mathbb{R}_\sigma$. The rationality of σ implies that $\mathbb{R}_{\sigma^\perp} \cap \mathbb{Z}^n$ has rank $(n-d)$, and admits a basis $w_1, \ldots, w_{n-d}$ of integral vectors; it is unique up to the action of $GL(n-d,\mathbb{Z})$, and determines the linear forms

$$w_c^{tr} x \;=\; w_{c,1}x_1 + \cdots + w_{c,n}x_n \quad \text{for} \quad 1 \leq c \leq n-d. \tag{4.1}$$

The intersection of their kernels is $\mathbb{R}_\sigma$, and there exists a splitting $\mathbb{R}^n \cong \mathbb{R}_{\sigma^\perp} \times \mathbb{R}_\sigma$. It is convenient to interpret $\mathbb{R}^n$ as the Lie algebra of T^n and write the associated splitting as

$$T^n \;\cong\; T_{\sigma^\perp} \times T_\sigma, \tag{4.2}$$

where the Lie algebra of $T_{\sigma^\perp}$ is spanned by the w_c, for any cone σ. Thus $T_{\sigma^\perp} = \{1\}$ for top-dimensional cones, and $T_{\{0\}^\perp} = T^n$.

Definitions 4.3. A *Σ-diagram* in a category C is a covariant functor $F\colon \text{CAT}(\Sigma) \to \text{C}$; similarly, a *$\Sigma^{op}$-diagram* (or *contravariant Σ-diagram*) is a covariant functor $F\colon \text{CAT}^{op}(\Sigma) \to \text{C}$.

We are interested in diagrams for which one or both of $\lim F$ and $\operatorname{colim} F$ exist in C.

Definitions 4.3 are motivated by a familiar diagram in TOP, which underlies the construction of the toric variety X_Σ as a topological T^n-space. It is denoted by $U\colon \text{CAT}(\Sigma) \to T^n\text{-TOP}$, and uses the dual cones $\sigma^\vee$ in the lattice $M := (\mathbb{Z}^n)^\vee$; it is given by

$$U(\sigma) \;=\; U_\sigma \;:=\; \operatorname{Hom}(\sigma^\vee \cap M, \mathbb{C}^\times \cup \{0\}) \quad \text{and} \quad U(i_{\sigma,\tau}) \;=\; j_{\sigma,\tau}\,, \tag{4.4}$$

where $\operatorname{Hom}(-)$ denotes the affine variety of semigroup homomorphisms, and $j_{\sigma,\tau}\colon U_\sigma \to U_\tau$ is induced by $i^\vee_{\sigma,\tau}\colon \tau^\vee \to \sigma^\vee$. The standard description of X_Σ, as given by [Fu, §1.4], for example, may then be expressed as the colimit $\operatorname{colim} U$ in $T^n\text{-TOP}$.

In fact $T^n \circlearrowright U_\sigma$ is T^n-equivariantly homotopy equivalent to $T^n \circlearrowright T^n/T_\sigma$ for every cone σ [CLS, Proposition 12.1.9, Lemma 3.2.5]. So U is objectwise equivariantly equivalent to the diagram $V\colon \mathrm{CAT}(\Sigma) \to T^n$-TOP, given by

$$V(\sigma) \;=\; T^n/T_\sigma \quad \text{and} \quad V(i_{\sigma,\tau}) \;=\; r_{\sigma,\tau}\,, \tag{4.5}$$

where $r_{\sigma,\tau}$ is the projection induced by the inclusion $T_\sigma \leq T_\tau$. Since U is cofibrant, it follows that $\operatorname{hocolim} V$ is equivariantly homotopy equivalent to $\operatorname{colim} U = X_\Sigma$. Diagram (4.5) first appeared in [WZZ], and more recently in [Fr].

We may now describe our basic examples of Σ^{op}-diagrams in the category GCALG_E of graded commutative $E^*_{T^n}$-algebras.

Definition 4.6. For any complex-oriented equivariant cohomology theory $E^*_{T^n}(-)$, the diagram $EV\colon \mathrm{CAT}^{op}(\Sigma) \to \mathrm{GCALG}_E$ has

$$EV(\sigma) \;=\; E^*_{T^n}(T^n/T_\sigma) \quad \text{and} \quad EV(p_{\tau,\sigma}) \;=\; r^*_{\sigma,\tau}\,. \tag{4.7}$$

The limit $P_E(\Sigma)$ of EV is the $E^*_{T^n}$-algebra of *piecewise coefficients* on Σ.

Remarks 4.8. By definition, $P_E(\Sigma)$ is an $E^*_{T^n}$-subalgebra of $\prod_\sigma EV(\sigma)$, so every piecewise coefficient f has one component f^σ for each cone σ of Σ. If σ is top dimensional, then $T_\sigma = T^n$ and f^σ is a genuine element of $E^*_{T^n}$; on the other hand, $T_{\{0\}} = \{1\}$ and $f^{\{0\}}$ lies in E^*. The components of f are compatible under the homomorphisms $j^*_{\tau,\sigma}$, and are congruent modulo the augmentation ideal. Sums and products of piecewise coefficients are taken conewise, and $E^*_{T^n} \leq P_E(\Sigma)$ occurs as the subalgebra of *global coefficients*, whose components agree on every cone. In particular, it contains the global constants 0 and 1, which act as zero and unit respectively.

In many cases, EV and P_E may be described more explicitly, as follows.

Suppose that ρ has codimension 1, and that w_1 is a primitive vector generating $\mathbb{R}_{\rho^\perp}$. The splitting (4.2) ensures that the natural action of T^n on T^n/T_ρ may then be identified with the unit circle $S(\eta)$ of the irreducible representation $\eta := \alpha^{w_1}$, on which the circle $T_{\rho^\perp}$ acts freely and the $(n-1)$–torus T_ρ acts trivially. The inclusion of $S(\eta)$ into the unit disc $D(\eta)$ determines the equivariant cofiber sequence

$$S(\eta) \;\longrightarrow\; D(\eta) \;\longrightarrow\; S^\eta,$$

where S^η denotes the one-point compactification $T^n \circlearrowright D(\eta)/S(\eta)$. Applying $E^*_{T^n}(-)$ yields the long exact sequence

$$\ldots \;\longrightarrow\; E^*_{T^n}(S^\eta) \;\xrightarrow{\cdot e}\; E^*_{T^n}(D(\eta)) \;\longrightarrow\; E^*_{T^n}(S(\eta)) \;\longrightarrow\; \ldots\,. \tag{4.9}$$

Since $D(\eta)$ is equivariantly contractible and the Thom isomorphism applies to the Thom space S^η, the homomorphism $\cdot e$ may be interpreted as multiplication by the Euler class $e_{T^n}(\eta)$. So $\cdot e$ is monic in each of our three cases, and (4.9) becomes short exact, yielding isomorphisms

$$E^*_{T^n}/(e_{T^n}(\eta)) \;\cong\; E^*_{T^n}(S(\eta)) \;\cong\; E^*_{T^n}(T^n/T_\rho) \;=\; EV(\rho)$$

of $E^*_{T^n}$-algebras.

This calculation extends to lower dimensional cones τ by iteration. If τ has dimension k, then the natural action of T^n on T^n/T_τ may be identified with the product $S(\eta_1) \times \cdots \times S(\eta_{n-k})$, where η_c denotes the irreducible α^{w_c} for $1 \leq c \leq n-k$. The $(n-k)$–torus $T_{\tau^\perp}$ acts freely, and the k–torus T_τ acts trivially, yielding isomorphisms

$$E^*_{T^n}/(e_{T^n}(\eta_1), \ldots, e_{T^n}(\eta_{n-k})) \;\cong\; E^*_{T^n}(S(\eta_1) \times \cdots \times S(\eta_{n-k})) \;\cong\; E^*_{T^n}(T^n/T_\tau) \;=\; EV(\tau). \tag{4.10}$$

If $\sigma \subset \tau$ has dimension $d < k$, then $\mathbb{R}_{\sigma^\perp}$ arises from $\mathbb{R}_{\tau^\perp}$ by adjoining additional basis vectors $w_{n-k+1}, \ldots, w_{n-d}$, and the projection

$$q_{\tau,\sigma}\colon E^*_{T^n}/(e_{T^n}(\eta_1), \ldots, e_{T^n}(\eta_{n-k})) \;\longrightarrow\; E^*_{T^n}/(e_{T^n}(\eta_1), \ldots, e_{T^n}(\eta_{n-d}))$$

corresponds to $r^*_{\tau,\sigma}\colon E^*_{T^n}(T^n/T_\tau) \to E^*_{T^n}(T^n/T_\sigma)$ under (4.10).

We conclude that (4.7) may be rewritten as

$$EV(\sigma) \;=\; E^*_{T^n}/(e_{T^n}(\eta_1),\dots,e_{T^n}(\eta_{n-d})) \quad\text{and}\quad EV(p_{\tau,\sigma}) \;=\; q_{\tau,\sigma}, \tag{4.11}$$

and proceed to describing the examples $E = K$, MU, and H in these terms.

For $E = K$, we work with graded commutative algebras over the Laurent polynomial ring $S^{\pm}_{K^*}(\alpha)$ of (3.6).

Example 4.12. The *Laurent polynomial diagram* $KV\colon \mathrm{CAT}^{op}(\Sigma) \to \mathrm{GCALG}_K$ has

$$KV(\sigma) \;=\; S^{\pm}_{K^*}(\alpha)/J_\sigma \quad\text{and}\quad KV(p_{\tau,\sigma}) \;=\; q_{\tau,\sigma}\,, \tag{4.13}$$

where J_σ denotes the ideal generated by the Euler classes $1-\alpha^{w_c}$ arising from the w_c of (4.1) for $1 \le c \le n-d$. In this case, $P_K(\alpha;\Sigma)$ is the $S_{K^*}(\alpha)$-algebra of *piecewise Laurent polynomials* on Σ.

For $E = MU$, we work with graded commutative algebras over $MU^*_{T^n}$, whose structure is unknown. We therefore rely on the fact that every element of $MU^*_{T^n}$ is an even-dimensional linear combination of generators over L^*, and refer to $MU^*_{T^n}$ as the ring of *T^n-cobordism forms.* Such forms may not be representable by genuine T^n-manifolds, as exemplified by the Euler class $e(\alpha^J)$, whose homological dimension is -2. This phenomenon arises from the lack of equivariant transversality, and the consequent failure of the Pontryagin-Thom construction to be epimorphic.

Example 4.14. The *cobordism form diagram* $MUV\colon \mathrm{CAT}^{op}(\Sigma) \to \mathrm{GCALG}_{MU}$ has

$$MUV(\sigma) \;=\; MU^*_{T^n}/J_\sigma \quad\text{and}\quad MUV(p_{\tau,\sigma}) \;=\; q_{\tau,\sigma}\,, \tag{4.15}$$

where J_σ denotes the ideal generated by the Euler classes $e(\alpha^{w_c})$ for $1 \le c \le n-d$. In this case, $P_{MU}(\Sigma)$ is the $MU^*_{T^n}$-algebra of *piecewise cobordism forms* on Σ.

For $E = H$, we work with graded commutative algebras over the polynomial algebra $S_{\mathbb{Z}}(x)$ of (3.7).

Example 4.16. The *polynomial diagram* $HV\colon \mathrm{CAT}^{op}(\Sigma) \to \mathrm{GCALG}_H$ has

$$HV(\sigma) \;=\; S_{\mathbb{Z}}(x)/J_\sigma \quad\text{and}\quad HV(p_{\tau,\sigma}) \;=\; q_{\tau,\sigma}\,,$$

where J_σ denotes the ideal generated by the Euler classes $w_c^{tr}x$ of (4.1) for $1 \le c \le n-d$. In this case, $P_H(x;\Sigma)$ is the $S_{\mathbb{Z}}(x)$-algebra of *piecewise polynomials* on Σ.

In [BFR1], $P_H(x;\Sigma)$ is referred to as $PP_{\mathbb{Z}}(x;\Sigma)$.

In Section 6 we invest $H^*_{T^n}(-)$ with various commutative graded rings of coefficients R, which are zero in odd dimensions. The standard example $\mathbb{Z}$ is concentrated in dimension 0, but we also consider $K\mathbb{Q}^* := \mathbb{Q}[z,z^{-1}]$, where z has cohomological dimension -2, and

$$H\wedge MU_* \;=\; H\wedge MU^{-*} \;:=\; H_*(MU) \;\cong\; S_{\mathbb{Z}}(b_j : j \ge 1),$$

where b_j has cohomological dimension $-2j$ for every j. The corresponding spectrum is denoted by $E = HR$, and the analogue of diagram (4.13) by HRV.

In these circumstances, the equivariant coefficient ring $HR^*_{T^n} = H^*(BT^n;R)$ must be identified with the completed tensor product $H^*(BT^n)\,\widehat{\otimes}\,R$. When $R = \mathbb{Z}$, the outcome is $H^*(BT^n)$; but for $K\mathbb{Q}^*$ or $H\wedge MU^*$, the ring

$$H^*(BT^n)\,\widehat{\otimes}\,R \;\cong\; R[[x]] \tag{4.17}$$

is an algebra of formal power series. It follows that $HRV(\sigma) \cong R[[x]]/J_\sigma$, and that $P_{HR}(x;\Sigma)$ is the $R[[x]]$-algebra of *piecewise formal power series* on Σ.

We require two further Σ^{op}-diagrams, obtained by applying Definition (4.6) to the Borel equivariant cohomology theories $E^*(ET^n \times_{T^n} -)$. In these cases the coefficients $E^*(BT^n)$ are also rings of formal power series.

The first such example identifies $K^*(BT^n)$ with $K^*[[\gamma]]$, on 0–dimensional indeterminates γ_j for $1 \le j \le n$.

Example 4.18. The *Borel K-theory diagram* $K_BV\colon \text{CAT}^{op}(\Sigma) \to \text{GCALG}_{K_B}$ has

$$K_BV(\sigma) \;=\; K^*[[\gamma]]/J_\sigma, \tag{4.19}$$

where J_σ denotes the ideal generated by the Euler classes $(1+\gamma)^{w_c} - 1$ for $1 \le c \le n-d$ and $K_BV(p_{\tau,\sigma})$ is the natural projection. The limit $P_{K_B}(\gamma;\Sigma)$ is the $K^*[[\gamma]]$-algebra of piecewise formal power series on Σ.

The second example identifies $MU^*(BT^n)$ with $L^*[[u]]$, on indeterminates u_j of cohomological dimension 2 for $1 \le j \le n$.

Example 4.20. The *Borel cobordism diagram* $MU_BV\colon \text{CAT}^{op}(\Sigma) \to \text{GCALG}_{MU_B}$ has

$$MU_BV(\sigma) \;=\; L^*[[u]]/J_\sigma, \tag{4.21}$$

where J_σ denotes the ideal generated by the Euler classes $[w_{c,1}](u_1) +_U \cdots +_U [w_{c,n}](u_n)$ (expressed in terms of the universal formal group law U [Haz] over L^*), and $MU_BV(p_{\tau,\sigma})$ is the natural projection. The limit $P_{MU_B}(u;\Sigma)$ is the $L^*[[u]]$-algebra of piecewise formal power series on Σ.

In fact $P_{MU}(\Sigma)$ and $P_{MU_B}(u;\Sigma)$ are the universal piecewise coefficient and piecewise formal power series algebras on Σ respectively, for complex-oriented $E^*_{T^n}(-)$ and $E^*(ET^n \times_{T^n} -)$. The cases $P_K(\alpha;\Sigma)$ and $P_{K_B}(\gamma;\Sigma)$ correspond to the *multiplicative* formal group law, classified by the equivariant Todd genus. Similarly, $P_{HR}(x;\Sigma)$ corresponds to the *additive* formal group law, classified by the Thom genus.

Remark 4.22. A map of fans $\xi\colon \Sigma' \to \Sigma$ may be interpreted as an $n \times n'$ integer matrix ξ, for which the image $\xi(\sigma')$ of any cone σ' is contained in some cone σ. Let $\xi^\dagger(\sigma')$ be the minimal such σ. In each of the above cases, ξ induces a natural transformation $(\xi^\dagger, \xi^*)$ of diagrams, and therefore a homomorphism ξ^* of limits.

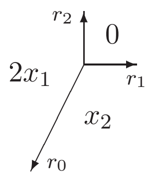
For example, in the case of HV, the homomorphism $\xi^*\colon HV(\sigma) \to HV(\sigma')$ is given in terms of the coordinate functions x and x' by the matrix ξ^{tr}; it is well-defined because $w \in \xi^\dagger(\sigma')^\perp$ implies that $\xi^{tr}w \in (\sigma')^\perp$. In the case of KV, the homomorphism $\xi^*\colon KV(\sigma) \to KV(\sigma')$ is induced by $\xi^*(\alpha^J) = (\alpha')^{\xi^{tr}J}$, and is well-defined for similar reasons.

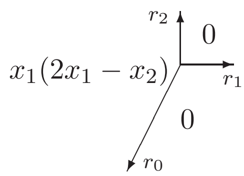
The construction of each piecewise algebra is therefore functorial (although care is required to check that $\xi \mapsto \xi^\dagger$ preserves composition). In particular, isomorphic fans yield isomorphic algebras.

Piecewise algebraic structures are natural generalizations of their global counterparts, and provide simple qualitative descriptions of algebras that may well be difficult to express in quantitative terms. For example see [BFR1, Section 4], where $P_H(x;\Sigma_{(1,2,3,4)})$ is computed in terms of generators and relations. The simplest non-trivial divisive example is the following.

Example 4.23. The fan $\Sigma_{(1,1,2)}$ in $\mathbb{R}^2$ has seven cones: $\{0\}$; the three rays through $r_0 = (-1,-2)$, $r_1 = (1,0)$, and $r_2 = (0,1)$; and three 2-dimensional cones generated by all pairs of rays.

So $P_H(x)$ is an $S_{\mathbb{Z}}(x)$-algebra, where $x = (x_1, x_2)$. Calculations confirm that $P_H(x)$ is generated as a ring by four piecewise polynomials, namely the global linear functions x_1 and x_2, together with the elements

$p \;=$ 

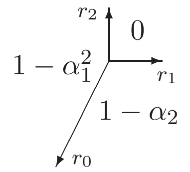
and $q \;=$ 

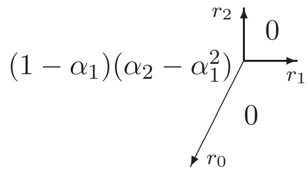
of degree 2 and 4 respectively. In fact $P_H(x)$ is isomorphic to $\mathbb{Z}[x_1, x_2, p, q]/I_1$, where I_1 is the ideal

$$\big(\, p(p - x_2) - 2q,\;\; q(p - 2x_1),\;\; q(q - x_1(2x_1 - x_2))\,\big)\,.$$

As $S_{\mathbb{Z}}(x)$–module, $P_H(x)$ has basis $\{1, p, q\}$.

Similarly, $P_K(\alpha)$ is an $S^{\pm}_{\mathbb{Z}}(\alpha)$-algebra, where $\alpha = (\alpha_1, \alpha_2)$. Calculations confirm that $P_K(\alpha)$ is generated as a ring by the global functions $\alpha_1^{\pm}$ and $\alpha_2^{\pm}$, together with the elements

$\epsilon =$ 

and $\zeta =$ 

of grading (and virtual dimension) 0. In fact $P_K(\alpha)$ is isomorphic to $S^{\pm}_{\mathbb{Z}}(\alpha_1, \alpha_2)[\epsilon, \zeta]/I_2$, where I_2 is the ideal

$$\bigl(\epsilon(\epsilon + \alpha_2 - 1) - (1 + \alpha_1)\zeta,\ \ \zeta(\epsilon + \alpha_1^2 - 1),\ \ \zeta(\zeta - (1 - \alpha_1)(\alpha_2 - \alpha_1^2))\bigr)\,.$$

As $S^{\pm}_{\mathbb{Z}}(\alpha)$-module, $P_K(\alpha)$ has basis $\{1, \epsilon, \zeta\}$. An equivalent calculation of Anderson and Payne [AnPa, Example 1.6] interprets the latter in terms of an $R(T^2)$-module basis for the algebra of *piecewise exponential functions* on $\Sigma_{(1,1,2)}$.

5. Cohomological applications

In this section we prove Theorem 5.5 by translating the GKM-theoretic content of Proposition 3.10 into the piecewise algebraic language of Section 4.

Our motivation lies in the results of [BFR1], which state that the Borel equivariant cohomology ring $H^*_{T^n}(X_\Sigma; R)$ is isomorphic to $P_{HR}(x; \Sigma)$ for any projective toric variety (smooth or singular) whose integral cohomology is concentrated in even dimensions. This may be thought of as a statement of compatibility with limits, in the sense that equivariant cohomology maps the homotopy colimit $\operatorname{hocolim} V \simeq X_\Sigma$ to the algebraic limit $\lim HRV = P_{HR}(x; \Sigma)$.

It also follows from [BFR1] that the sequence

$$0 \longrightarrow (S_R(x)) \longrightarrow P_{HR}(x; \Sigma) \longrightarrow H^*(X_\Sigma; R) \longrightarrow 0 \tag{5.1}$$

is short exact. So Kawasaki's calculations [Kaw] confirm that these properties hold for every weighted projective space. Furthermore, $P_{HR}(x; \Sigma)$ is isomorphic to the *face ring* (or *Stanley-Reisner algebra*) $R[x; \Sigma]$ for any smooth fan. In other words, $P_{HR}(x; \Sigma_\chi)$ reduces to the face ring whenever $\chi_j = 1$ for every $0 \le j \le n$.

Working over a field immediately simplifies the situation; for example,

$$H^*_{T^n}(X_\Sigma; \mathbb{Q}) \;\cong\; P_{H\mathbb{Q}}(x; \Sigma) \;\cong\; \mathbb{Q}[x; \Sigma] \tag{5.2}$$

holds for *any* fan Σ.

We now explain how to interpret the GKM description of Proposition 3.10 as the limit of an appropriate contravariant Σ_χ diagram. To proceed, we must therefore identify Σ_χ more explicitly. For general weights χ, this may be difficult; in the divisive case, however, it is easy to specify the rays $r_0, \ldots, r_n$ precisely. Bearing in mind that χ is normalised (2.1), we set

$$\begin{pmatrix} r_0 & \ldots & r_n \end{pmatrix} \;=\; \begin{pmatrix} -1 & 1 & 0 & \ldots & 0 \\ -\chi_2 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -\chi_n & 0 & 0 & \ldots & 1 \end{pmatrix} \tag{5.3}$$

as an $n \times (n+1)$ matrix. The cones σ_A of Σ_χ are generated by rays $\{r_i : i \notin A\}$, as A ranges over all non-empty strictly increasing subsequences $a_1, \ldots, a_d$ of $0, \ldots, n$. In particular, the n–dimensional cones are $\sigma_0, \ldots, \sigma_n$, and $\sigma_A \cap \sigma_{A'} = \sigma_{A \cdot A'}$ holds for any A and A', where $A \cdot A'$ is given by juxtaposition and reordering.

In order to study the diagrams KV, MUV and HV of Section 4, we must first identify the linear forms of (4.1) for Σ_χ.

For every $0 \le k < l \le n$, the $(n-1)$–dimensional cone $\sigma_{k,l}$ is generated by the columns of the $n \times (n-1)$-matrix obtained from (5.3) by deleting columns k and l. So a basis for $\sigma_{k,l}^{\perp}$ consists of a single primitive integral vector w, orthogonal to all remaining columns. If $1 \leq k$, then

$$w = (0, \dots, 0, -\chi_l/\chi_k, 0, \dots, 0, 1, 0, \dots, 0)$$

(non-zero in positions k and l) satisfies the conditions; if $k = 0$, then $w = (0, \dots, 0, 1, 0 \dots, 0)$ suffices.

With reference to Examples 4.12, 4.14, and 4.16, we may now deduce the following.

Lemma 5.4. *For any cone $\sigma_{k,l}$ and any $0 \leq k < l \leq n$ in Σ_χ, the principal ideals $J_{k,l} := J_{\sigma_{k,l}}$ in $S^{\pm}_{\mathbb{Z}}(\alpha)$, $MU^*_{T^n}$, and $S_{\mathbb{Z}}(x)$ are generated by $1 - \alpha_l \alpha_k^{-\chi_l/\chi_k}$, $e\big(\alpha_l \alpha_k^{-\chi_l/\chi_k}\big)$, and $x_l - (\chi_l/\chi_k)x_k$ respectively, where $x_0 = 0$ and $\alpha_0 = 1$ if $k = 0$.* □

Lemma 5.4 summarizes the input required to prove Theorem 5.5.

Theorem 5.5. *For any divisive weighted projective space $\mathbb{P}(\chi)$:*

1. *$K^*_{T^n}(\mathbb{P}(\chi))$ is isomorphic as $S^{\pm}_{K^*}(\alpha)$-algebra to $P_K(\alpha; \Sigma_\chi)$;*
2. *$MU^*_{T^n}(\mathbb{P}(\chi))$ is isomorphic as $MU^*_{T^n}$-algebra to $P_{MU}(\Sigma_\chi)$;*
3. *$H^*_{T^n}(\mathbb{P}(\chi); R)$ is isomorphic as $S_R(x)$-algebra to $P_{HR}(x; \Sigma_\chi)$.*

Proof. We give the details for **1**.

Invoking (3.8) and Proposition 3.10, we must identify the algebra $\Gamma_{\mathbb{P}(\chi)}$ with $\lim_{\text{GCALG}} KV$. The former is given by

$$\Big\{ g = g(\alpha) : 1 - \alpha_{n-j} \alpha_{n-i}^{-\chi_{n-j}/\chi_{n-i}} \text{ divides } g_i - g_j \text{ for all } 0 \leq j < i \leq n \Big\} \;\leq\; \prod_i S^{\pm}_{K^*}(\alpha),$$

and the universal properties of the latter suggest that we proceed by finding compatible homomorphisms $h_{a_0,\dots,a_d} \colon \Gamma_{\mathbb{P}(\chi)} \to KV(\sigma_{a_0,\dots,a_d})$ for every cone $\sigma_{a_0,\dots,a_d}$ in Σ_χ. It follows from Example 4.12 and Lemma 5.4 that

$$KV(\sigma_{a_0,\dots,a_d}) \;=\; S^{\pm}_{K^*}(\alpha)/J_{a_0,\dots,a_d},$$

where $J_{a_0,\dots,a_d}$ denotes the ideal generated by the polynomials $1 - \alpha_l \alpha_k^{-\chi_l/\chi_k}$ as k, l ranges over the length 2 subsequences of $0 \leq a_0, \dots, a_n \leq n$.

Given any g in $\Gamma_{\mathbb{P}(\chi)}$, we first consider cones of dimension n, and define $h_k(g) := g_{n-k}$ in $S^{\pm}_{K^*}(\alpha)$ for every $0 \leq k \leq n$. On cones of dimension $n-1$, we let

$$h_{k,l}(g) \;:=\; g_{n-k} \;\equiv\; g_{n-l} \quad \text{mod } \; 1 - \alpha_l \alpha_k^{-\chi_l/\chi_k}$$

in $KV(\sigma_{k,l}) = S^{\pm}_{K^*}(\alpha)/(1 - \alpha_l \alpha_k^{-\chi_l/\chi_k})$, for every $0 \leq k < l \leq n$. This is well-defined, because $1 - \alpha_l \alpha_k^{-\chi_l/\chi_k}$ divides $g_{n-k} - g_{n-l}$ in $S^{\pm}_R$. The definition extends to

$$h_{a_0,\dots,a_d}(g_i) \;:=\; g_{n-a_0} \;\equiv\; \dots \;\equiv\; g_{n-a_d} \quad \text{mod } \; J_{a_0,\dots,a_d}$$

for any $2 \leq d \leq n$, because the $g_{n-a_0}, \dots, g_{n-a_d}$ satisfy precisely the required pairwise divisibility conditions in $S^{\pm}_{K^*}(\alpha)$. Moreover, $h_{a_0,\dots,a_d}$ is a homomorphism of $S^{\pm}_{K^*}(\alpha)$-algebras, by definition.

In order to confirm the compatibility of the h_A over $\text{CAT}^{op}(\Sigma_\chi)$, we note that every morphism takes the form $\sigma_A \supseteq \sigma_{A \cdot A'}$. The corresponding projection $q_{\sigma_A, \sigma_{A \cdot A'}} \colon LP(\sigma_A) \to LP(\sigma_{A \cdot A'})$ is induced by the inclusion $J_A \leq J_{A \cdot A'}$, which adjoins the expressions $1 - \alpha_l \alpha_k^{-\chi_l/\chi_k}$ as k, l ranges over the length 2 subsequences of A': so compatibility is assured. We have therefore constructed a homomorphism $h \colon \Gamma_{\mathbb{P}(\chi)} \to \lim_{\text{GCALG}} KV$ of $S^{\pm}_{K^*}(\alpha)$-algebras.

We conclude by showing that h is automatically an isomorphism. Given distinct elements g and g' of $\Gamma_{\mathbb{P}(\chi)}$, there must exist at least one k such that $g_k \neq g'_k$ as elements of $S^{\pm}_{K^*}(\alpha)$; hence $h_k(g) \neq h_k(g')$ in $KV(\sigma_k)$, and h is monic. Similarly, any element (g_A) of $\lim_{\text{GCALG}} KV$ determines (g_a) in $\Gamma_{\mathbb{P}(\chi)}$, by restricting to n-dimensional cones; thus $h(g_a) = (g_A)$, and h is epic.

The entire argument applies to **2** and **3**, with minor modifications. For $H^*_{T^n}(\mathbb{P}(\chi); R)$, the statement is also a special case of [BFR1, Proposition 2.2]. □

Informally, the connection between T^n-equivariant bundles over X_Σ and piecewise Laurent polynomials on Σ is easy to make. Every such bundle determines a representation of T^n on the fibre at each fixed point, and therefore on each maximal cone. These representations must be compatible over any T^{n-1}-invariant S^2 containing two fixed points, and therefore on cones of codimension 1.

The close relationship between GKM theory and piecewise algebra has long been known over fields of characteristic zero, and Theorem 5.5 is an instance of its extension to integral situations.

6. Completion and Borel cohomology

In this section we introduce the completions of $K^*_{T^n}(\mathbb{P}(\chi))$ and $MU^*_{T^n}(\mathbb{P}(\chi))$ at their augmentation ideals, and discuss their relationships with the Borel equivariant cohomology ring of $\mathbb{P}(\chi)$ under the Chern character and the Boardman homomorphism respectively. We express our results in terms of certain natural transformations between diagrams of Section 4.

Definitions 6.1. The *K-theory completion transformation* $\wedge := \wedge^K \colon KV \to K_BV$ is defined on objects by the ring homomorphisms

$$S^{\pm}_{K^*}(\alpha)/J_\sigma \longrightarrow K^*[[\gamma]]/J_\sigma,$$

where $\wedge(\alpha_j) = 1-\gamma_j$ and $\wedge(\alpha_j^{-1}) = 1+\sum_{i\geq 1}\gamma_j^i$ for $1 \leq j \leq n$; on morphisms, $\wedge$ maps the first natural projection to the second.

The *cobordism completion transformation* $\wedge := \wedge^{MU} \colon MUV \to MU_BV$ is defined on objects by the ring homomorphisms

$$MU^*_{T^n}/J_\sigma \longrightarrow L^*[[u]]/J_\sigma,$$

where $\wedge(e_{T^n}(\alpha_j)) = u_j$ and $\wedge(e_{T^n}(\alpha_j^{-1})) = [-1](u_j)$ for $1 \leq j \leq n$; on morphisms, $\wedge$ maps the first natural projection to the second.

So $\wedge^K$ is induced by the homomorphism

$$S^{\pm}_{K^*}(\alpha) \longrightarrow K^*[[\gamma]] \tag{6.2}$$

representing completion at the augmentation ideal I [AM, Chapter 10]. It is well-defined because $\wedge(\alpha_j^{-1}) = (\wedge(\alpha_j))^{-1}$ for $1 \leq j \leq n$ and $\wedge(\alpha^{w_c}) = (1-\gamma)^{w_c}$ for $1 \leq c \leq n-d$, so that $\wedge$ maps J_σ to J_σ. The augmentation $S^{\pm}_{K^*}(\alpha) \to K^*$ assigns to each virtual representation its dimension.

Similarly, $\wedge^{MU}$ is induced by the homomorphism

$$MU^*_{T^n} \longrightarrow L^*[[u]] \tag{6.3}$$

representing completion at the augmentation ideal I. It is well-defined because $\wedge(\alpha^{w_c}) = [w_{c,1}](u_1) +_U \cdots +_U [w_{c,n}](u_n)$ for $1 \leq c \leq n-d$, so that $\wedge$ maps J_σ to J_σ. The augmentation $MU^*_{T^n} \to L^*$ forgets the T^n-action on each equivariant cobordism class.

Definitions 6.4. The *Chern transformation* $ct \colon K_BV \to H(K\mathbb{Q}^*)V$ is given on objects by the homomorphisms

$$K^*[[\gamma]]/J_\sigma \longrightarrow S_{K\mathbb{Q}^*}(x)/J_\sigma,$$

where $ct(\gamma_j) = 1-e^{zx_j}$ for $1 \leq j \leq n$, and ct embeds the scalars K^* as $K^*\otimes 1$ in $K\mathbb{Q}^* := K^*\otimes\mathbb{Q}$; on morphisms, ct maps the first natural projection to the second.

The *Boardman transformation* $bt \colon MU_BV \to H(H\wedge MU^*)V$ is given on objects by the homomorphisms

$$L^*[[u]]/J_\sigma \longrightarrow S_{H\wedge MU^*}(x)/J_\sigma,$$

where $bt(u_j) = \sum_{i\geq 0} b_i x_j^{i+1}$ for $1 \leq j \leq n$, and bt embeds the scalars L^* in $H\wedge MU^*$ via the Hurewicz homomorphism; on morphisms, bt maps the first natural projection to the second.

So ct and bt are induced by the respective homomorphisms

$$K^*[[\gamma]] \longrightarrow S_{K\mathbb{Q}^*}(x) \quad \text{and} \quad L^*[[u]] \longrightarrow S_{H\wedge MU^*}(x), \tag{6.5}$$

and are well-defined because J_σ maps to J_σ in each instance.

The commutativity of the diagrams required for the naturality of $\wedge$, *ct*, and *bt* follow directly from the definitions. They therefore induce morphisms of limits, and so define homomorphisms

$$\wedge^K\colon P_K(\alpha;\Sigma) \longrightarrow P_{K_B}(\gamma;\Sigma) \quad \text{and} \quad ct\colon P_{K_B}(\gamma;\Sigma) \longrightarrow P_{H(K\mathbb{Q}^*)}(x;\Sigma) \tag{6.6}$$

and

$$\wedge^{MU}\colon P_{MU}(\Sigma) \longrightarrow P_{MU_B}(u;\Sigma) \quad \text{and} \quad bt\colon P_{MU_B}(u;\Sigma) \longrightarrow P_{H(H\wedge MU^*)}(x;\Sigma) \tag{6.7}$$

of piecewise structures. In particular, $\wedge^K$ and $\wedge^{MU}$ may be viewed as *conewise completions*; but completion commutes with limits, so they coincide with the respective completions of $P_K(\alpha;\Sigma)$ and $P_{MU}(\Sigma)$ at their augmentation ideals I. Similarly, *ct* and *bt* are the *conewise Chern* and *conewise Boardman homomorphism* respectively.

Note that $\wedge^K$ and $\wedge^{MU}$ are morphisms of algebras over the respective completion homomorphisms (6.2) and (6.3) of scalars. Furthermore, *ct* and *bt* arise from conewise rational isomorphisms, and are therefore rational isomorphisms themselves; they are also morphisms of algebras over the homomorphisms (6.5). In other words, (6.6) and (6.7) describe the extensions of (6.2), (6.3), and (6.5) to the piecewise setting.

Remarks 6.8. The composition $ct \circ \wedge^K$ is a natural transformation $cc\colon KV \to H(K\mathbb{Q}^*)V$. It is induced by the homomorphism

$$cc\colon S^{\pm}_{K^*}(\alpha) \longrightarrow S_{K\mathbb{Q}^*}(x), \tag{6.9}$$

which satisfies $cc(\alpha_j) = e^{zx_j}$ for all $1 \le j \le n$. On limits,

$$cc\colon P_K(\alpha;\Sigma) \longrightarrow P_{H(K\mathbb{Q}^*)}(x;\Sigma)$$

identifies $P_K(\alpha;\Sigma)$ with a subring of piecewise formal exponential functions (the viewpoint adopted by [AnPa], and anticipated in Example 4.23). It is a morphism of algebras over the homomorphism (6.9) of scalars.

Similarly, $bt \circ \wedge^{MU}$ is a natural transformation $bc\colon MUV \to H(H\wedge MU^*)V$. It is induced by the homomorphism

$$bc\colon MU^*_{T^n} \longrightarrow S_{H\wedge MU^*}(x), \tag{6.10}$$

which satisfies $bc(e(\alpha_j)) = \sum_{i\ge 0} b_i x_j^{i+1}$ for all $1 \le j \le n$. On limits,

$$bc\colon P_{MU}(\Sigma) \longrightarrow P_{H(H\wedge MU^*)}(x;\Sigma)$$

identifies $P_{MU}(\Sigma)$ with a subring of piecewise formal power series; it is a morphism of algebras over the homomorphism (6.10) of scalars.

Using the isomorphism of Theorem 5.5.2, we may now interpret the homomorphisms (6.6) and (6.7) topologically. We deal first with K-theory.

For any $T^n \circlearrowright Y$, the Borel equivariant K-theory $K^*(ET^n \times_{T^n} Y)$ is an algebra over the coefficient ring $K^*(BT^n)$, which acts via the projection map π of the T^n-bundle

$$Y \longrightarrow ET^n \times_{T^n} Y \xrightarrow{\ \pi\ } BT^n. \tag{6.11}$$

Atiyah and Segal define a homomorphism $\lambda\colon K^*_{T^n}(Y) \to K^*(ET^n \times_{T^n} Y)$ by assigning the vector bundle $(ET^n \times \theta)/T^n$ to each T^n-equivariant vector bundle θ over Y. For compact spaces such as X_Σ, they prove [AS, Theorem 2.1] that λ is completion at the augmentation ideal I. If, for example, Y is the 1-point T^n-space $*$, then $\lambda\colon K^*_{T^n} \to K^*(BT^n)$ corresponds to the completion map (6.2), and identifies the coefficients $K^*(BT^n)$ with $K^*[[\gamma]]$, as in Example 4.18. Furthermore, γ_j is the K-theoretic Euler class of the jth canonical line bundle over BT^n, for $1 \le j \le n$. In general, λ may be interpreted as converting the $S^{\pm}_{K^*}(\alpha)$-algebra structure of $K^*_{T^n}(Y)$ to the $K^*[[\gamma]]$-algebra structure of $K^*(ET^n \times_{T^n} Y)$.

The *Chern character* $ch\colon K^*(ET^n \times_{T^n} Y) \to H^*(ET^n \times_{T^n} Y; K\mathbb{Q}^*) =: H^*_{T^n}(Y; K\mathbb{Q}^*)$ is the natural transformation of cohomology theories induced by the Hurewicz morphism

$$K \simeq S^0 \wedge K \xrightarrow{\ i\wedge 1\ } H\wedge K \tag{6.12}$$

of complex-oriented ring spectra, where i denotes the unit of the integral Eilenberg-Mac Lane spectrum H. On coefficient rings, it embeds $K^* \cong \mathbb{Z}[z, z^{-1}]$ in $H \wedge K^* \cong \mathbb{Q}[z, z^{-1}] = K\mathbb{Q}^*$ by $\mathbb{Z} < \mathbb{Q}$, and on $\mathbb{C}P^\infty$ it embeds $K^*(\mathbb{C}P^\infty) \cong K^*[[\gamma_1]]$ in $H^*(\mathbb{C}P^\infty; K\mathbb{Q}^*) \cong S_{K\mathbb{Q}^*}(x_1)$ by $ch(\gamma_1) = 1 - e^{zx_1}$. Further properties of ch may be found in [Hi, Chapter 5], for example; in particular, it is always a rational isomorphism.

Theorem 6.13. *For any divisive weighted projective space, $K^*(ET^n \times_{T^n} \mathbb{P}(\chi))$ is isomorphic as $K^*[[\gamma]]$-algebra to $P_{K_B}(\gamma; \Sigma_\chi)$; with respect to this identification, the Atiyah-Segal completion map $\lambda\colon K^*_{T^n}(\mathbb{P}(\chi)) \to K^*(ET^n \times_{T^n} \mathbb{P}(\chi))$ corresponds to the conewise completion homomorphism $\wedge^K$, and the Chern character $ch\colon K^*(ET^n \times_{T^n} \mathbb{P}(\chi)) \to H^*_{T^n}(\mathbb{P}(\chi); K\mathbb{Q}^*)$ corresponds to the conewise Chern transformation ct.*

Proof. Theorem 5.5.2 shows that λ corresponds to $\wedge$, and has target $P_{K_B}(\gamma; \Sigma_\chi)$; so the latter is necessarily isomorphic to $K^*(ET^n \times_{T^n} \mathbb{P}(\chi))$ as $K^*[[\gamma]]$-algebra.

Moreover, $ch\colon K^*(BT^n) \to H^*(BT^n; K\mathbb{Q}^*)$ maps γ_j to $1 - e^{zx_j}$ for every $1 \le j \le n$, which agrees precisely with ct as specified in (6.5). Since ch is natural with respect to the inclusion $\coprod BT^n \subset ET^n \times_{T^n} \mathbb{P}(\chi)$ induced by the fixed point set, the result follows. □

We may extend the isomorphism of Theorem 6.13 to any projective toric variety X_Σ for which $H^*(X_\Sigma; \mathbb{Z})$ is torsion free and concentrated in even dimensions. We interpret $P_{K_B}(\gamma; \varsigma)$ as a $K^*[[\gamma]]$-subalgebra of $\prod K^*[[\gamma]]$, where the latter contains one factor for each maximal cone of Σ, and hence for each T^n-fixed point.

Theorem 6.14. *The Borel equivariant K-theory of any such X_Σ is isomorphic as $K^*[[\gamma]]$-algebra to $P_{K_B}(\gamma; \Sigma)$.*

Proof. By [BFR1, Proposition 2.2], $H^*_{T^n}(X_\Sigma; \mathbb{Z})$ is also torsion free and even dimensional, and isomorphic to $P_H(x; \Sigma)$. So the Chern character $ch\colon K^*(ET^n \times_{T^n} X_\Sigma) \to H^*_{T^n}(X_\Sigma; K\mathbb{Q}^*)$ is monic, and identifies $K^*(ET^n \times_{T^n} X_\Sigma)$ with a subring of $P_{H(K\mathbb{Q}^*)}(x; \Sigma)$. By the naturality of ch, the inclusion of the fixed point set and the projection of (6.11) induce a commutative diagram

$$\begin{array}{ccccc} \prod K^*[[\gamma]] & \xleftarrow{K^*(\iota)} & K^*(ET^n \times_{T^n} X_\Sigma) & \xleftarrow{K^*(\pi)} & K^*[[\gamma]] \\ {\scriptstyle ch}\big\downarrow & & {\scriptstyle ch}\big\downarrow & & \big\downarrow{\scriptstyle ch} \\ \prod S_{K\mathbb{Q}^*}(x) & \xleftarrow[H^*(\iota)]{} & P_{H(K\mathbb{Q}^*)}(x; \Sigma) & \xleftarrow[H^*(\pi)]{} & S_{K\mathbb{Q}^*}(x) \end{array} \tag{6.15}$$,

in which $H^*(\iota)$, $H^*(\pi)$, $K^*(\pi)$, and all maps ch, are monic. It follows that $K^*(\iota)$ is also monic, and that we may identify the elements γ_j in $K^*(ET^n \times_{T^n} X_\Sigma)$, for $1 \le j \le n$. The image of $K^*(\iota)$ automatically lies in the subalgebra $P_{K_B}(\gamma; \Sigma) < \prod K^*[[\gamma]]$, because ι factors through the equivariant 1-skeleton of X_Σ; so it remains to show that the image is the entire subalgebra.

Note that the augmentation ideal I of $K^*[[\gamma]]$ is generated by the γ_j, and its image $ch(I) = (x)$ is generated by the x_j. Furthermore, the filtration by powers of I coincides with the skeletal filtration for $K^*(BT^n)$.

Choose $f := f(\gamma)$ in $P_{K_B}(\gamma; \Sigma)$, and assume that its augmentation is zero without loss of generality. So $ch(f) = f(1 - e^{zx})$ is a piecewise formal power series in $\prod S_{K\mathbb{Q}^*}(x)$, and may be rewritten as a piecewise polynomial in the variables x over $K\mathbb{Q}^*$. As such, it takes the form $H^*(\iota)(f')$ for some element $f' = f'(x)$ in $P_{H(K\mathbb{Q}^*)}(x)$. Since $f'(0) = 0$, it has filtration $q_1 \ge 1$ with respect to (x); so the integrality properties of ch ensure the existence of an element $f_1 = f_1(x)$ in $K^*(ET^n \times_{T^n} X_\Sigma)$, such that $ch(f_1) \equiv f'$ modulo I^{q_1+1}. Hence $K^*(\iota)(f_1) \equiv f$ modulo I^{q_1+1} in $PFS_{K^*}(\gamma; \Sigma)$.

Now iterate this procedure on $f - K^*(\iota)(f_1)$, to obtain a sequence of elements $(f_n = f_n(x))$ in $K^*(ET^n \times_{T^n} X_\Sigma)$ for which

$$K^*(\iota)(f_n) \;\equiv\; f - K^*(\iota)(f_1 + \cdots + f_{n-1}) \;\text{ modulo }\; I^{q_n+1},$$

with $q_1 < \cdots < q_n$. Since $K^*(ET^n \times_{T^n} X_\Sigma)$ and $P_{K_B}(\gamma;\Sigma)$ are I-adically complete, it follows that $f_1 + \cdots + f_n + \cdots$ converges to an element f° in the former, and that $K^*(\iota)(f^\circ) = f$ in the latter. So $K^*(\iota)$ is epic, as required. □

We now turn to the cobordism versions of our previous two results.

In [tD], tom Dieck introduces the *bundling transformation* $\alpha\colon MU^*_{T^n}(Y) \to MU^*(ET^n \times_{T^n} Y)$, which is proven in [CM] to be completion at the augmentation ideal. If $Y = *$, then α reduces to the homomorphism (6.3), and identifies $MU^*(BT^n)$ with $MU^*[[u]]$, as in Example 4.20. Each u_j is the cobordism Euler class of the jth canonical line bundle over BT^n, for $1 \le j \le n$.

The Boardman homomorphism $bh\colon MU^*(ET^n \times_{T^n} Y) \to H^*_{T^n}(Y; H \wedge MU^*)$ is induced by the Hurewicz morphism

$$MU \;\simeq\; S^0 \wedge MU \;\xrightarrow{i\wedge 1}\; H \wedge MU,$$

by analogy with (6.12). On coefficient rings it embeds L^* in $H \wedge MU^*$ by the Hurewicz homomorphism, and on $\mathbb{C}P^\infty$ it embeds $MU^*(\mathbb{C}P^\infty) \cong L^*[[u_1]]$ in $H \wedge MU^*(\mathbb{C}P^\infty) \cong H_*(MU)[[x_1]]$ by $bh(u_1) = \sum_{i\ge 0} b_i x_1^{i+1}$. Further properties of bh may be found in [A], for example.

We are now in a position to state the cobordism versions of Theorems 6.13 and 6.14; they are verified by substituting bh for ch in each of the proofs.

Theorem 6.16. *For any divisive weighted projective space, $MU^*(ET^n \times_{T^n} \mathbb{P}(\chi))$ is isomorphic as $L^*[[u]]$-algebra to $P_{MU_B}(u;\Sigma)$; with respect to this identification, the bundling transformation $\alpha\colon MU^*_{T^n}(\mathbb{P}(\chi)) \to MU^*(ET^n \times_{T^n} \mathbb{P}(\chi))$ corresponds to the conewise completion homomorphism $\wedge^{MU}$, and the Boardman homomorphism $bh\colon MU^*(ET^n \times_{T^n} \mathbb{P}(\chi)) \to H^*_{T^n}(\mathbb{P}(\chi); H \wedge MU^*)$ corresponds to the conewise Boardman transformation bt.* □

Theorem 6.17 applies to projective toric varieties X_Σ whose integral cohomology is free and even.

Theorem 6.17. *The Borel equivariant MU-theory of any such X_Σ is isomorphic as $L^*[[u]]$-algebra to $P_{MU_B}(u;\Sigma)$.* □

For the proof, we interpret $P_{MU_B}(u;\Sigma)$ as an $L^*[[u]]$-subalgebra of $\prod L^*[[u]]$.

Note that Theorems 6.14 and 6.17 apply to many more toric varieties than Theorem 5.5; in particular, they hold for all smooth examples, and for iterated products of arbitrary weighted projective spaces. They provide evidence for the conjecture that $K^*_{T^n}(X_\Sigma)$ and $MU^*_{T^n}(X_\Sigma)$ are isomorphic to $P_K(\alpha;\Sigma)$ and $P_{MU}(\Sigma)$ respectively, for *any* projective toric variety whose integral cohomology is free and even. Without further proof, however, the most we can claim is that each pair of algebras share the same completion.

Combining Theorem 6.17 with [KU, Theorem 6.4] confirms that the equivariant algebraic cobordism ring of a smooth projective toric variety Y is isomorphic to $MU^*(ET^n \times_{T^n} Y)$. This fact also follows from [KK, Theorem 3.7]. Most recently, Gonzalez and Karu have defined *operational* equivariant algebraic cobordism, and [GoKa, Theorem 7.2] proves that their ring is isomorphic to $MU^*(ET^n \times_{T^n} Y)$ for any quasiprojective toric variety Y to which Theorem 6.17 applies, singular or otherwise. No analogous coincidences arise in Sections 5 or 7, because the coefficient rings $L^*[[u]]$ of the algebraic theories are complete.

7. The smooth case

A version of Theorem 5.5 for *smooth* fans may well be known to experts, but statements are difficult to find in the literature. There are, however, explicit references such as [VV] and [AnPa] to analogous results in equivariant *algebraic* and *operational* K-theory respectively. The goal of this section is to outline a proof (and certain consequences) of Theorem 5.5 for smooth polytopal fans Σ, thereby confirming that all three forms of K-theory agree on the corresponding X_Σ. In this context, *polytopal* indicates that Σ is the normal fan of a compact simple polytope P_Σ, and therefore *complete*. We expect that our results may be extended to more general fans by applying the methods of [Fr].

Our proof relies on the work of Harada and Landweber [HaL], which deals with symplectic manifolds (M,ω) equipped with a Hamiltonian action of T^n. In [HaL, Definition 4.1], such an action is defined to be *GKM* whenever the fixed point set is finite and the isotropy weights at each fixed point p are pairwise linearly independent. The latter requirement may be restated in the notation of Section 3 by decomposing the n-dimensional representation ρ_p of T^n on $T_p(M)$ as a sum $\bigoplus_{j=1}^n \alpha^{J_{p,j}}$ of 1-dimensionals, and insisting that the $J_{p,j}$ are pairwise linearly independent in $\mathbb{Z}^n$ for each fixed point p.

Let Σ be smooth and polytopal, so that X_Σ is smooth, compact and projective. In fact, X_Σ is also a symplectic toric manifold in the sense of [CdS, Part XI], with respect to the induced symplectic structure arising from its projective embedding. The associated moment map Φ can be identified with the orbit map $X_\Sigma \to X_\Sigma/T^n \cong P_\Sigma$; moreover, P_Σ is a *Delzant polytope* [D, CdS]. The fixed points of the action are precisely the inverse images of the vertices of P_Σ; in particular, the set of fixed points is finite. Furthermore, since P_Σ is Delzant, the edges incident on any vertex t of P_Σ are specified by n primitive integral vectors $J_{t,s_1},\dots,J_{t,s_n}$, which form a basis for the standard lattice $\mathbb{Z}^n$ in the ambient $\mathbb{R}^n$. Then $T^n \circlearrowright X_\Sigma$ is given in a neighborhood of t by the representation $\alpha_t := \bigoplus_{j=1}^n \alpha^{J_{t,s_j}}$, and the action is GKM.

We may now apply [HaL, Theorem 4.4] to $T^n \circlearrowright X_\Sigma$, by noting that Φ has compact domain, and hence has a component which is proper and bounded below.

Theorem 7.1. *For any smooth polytopal fan Σ, the inclusion of the fixed point set induces an isomorphism of $K^*_{T^n}(X_\Sigma)$ with*

$$\Gamma_\Sigma \;=\; \left\{y : 1-\alpha^{J_{t,s}} \text{ divides } y_s - y_t \text{ for all } s \prec t\right\} \;\leq\; \prod_t K^*_{T^n},$$

where t ranges over the vertices of P_Σ. □

Following our observations on MU_{T^n}-Euler classes in Section 3, Theorem 7.1 may equally well be applied to $MU^*_{T^n}(X_\Sigma)$.

Corollary 7.2. *For any smooth polytopal fan Σ:*

1. *$K^*_{T^n}(X_\Sigma)$ is isomorphic as $S^{\pm}_{K^*}(\alpha)$-algebra to $P_K(\alpha;\Sigma)$;*
2. *$MU^*_{T^n}(X_\Sigma)$ is isomorphic as $MU^*_{T^n}$-algebra to $P_{MU}(\Sigma)$;*
3. *$H^*_{T^n}(X_\Sigma;R)$ is isomorphic as $S_R(x)$-algebra to $P_{HR}(x;\Sigma)$.* □

Corollary 7.2 follows from Theorem 7.1 by adapting the proof of Theorem 5.5.

Since Σ is smooth, each algebra of Corollary 7.2 admits an alternative description in terms of the face ring $R[\Sigma]$, inspired by the isomorphism $H^*_{T^n}(X_\Sigma;R) \cong R[\Sigma]$ mentioned in Section 5. The face ring associates 2–dimensional indeterminates y_j to the the rays v_j of Σ for $1 \leq j \leq m$, and may first have been described as a limit in [PRV].

Given any set ω of rays, it is convenient to denote the set of variables $\{y_j : v_j \in \omega\}$ by $y(\omega)$, and to abbreviate the monomial $\prod_{y(\omega)} y_j$ to y_ω.

Definition 7.3. The *face diagram* $S_R\colon \text{CAT}^{op}(\Sigma) \to \text{GCALG}$ has

$$S_R(\sigma) \;=\; S_R(y(\sigma)) \quad\text{and}\quad S_R(p_{\tau,\sigma}) \;=\; f_{\tau,\sigma}\,, \tag{7.4}$$

where $f_{\tau,\sigma}\colon S_R(\tau) \to S_R(\sigma)$ is induced by annihilating those y_i for which $v_i \in \tau \setminus \sigma$.

The face ring $R[\Sigma] = R[y;\Sigma]$ is the limit of S_R, and is additively generated by those monomials $y^J := y_1^{j_1}\cdots y_m^{j_m}$ whose support $\prod_{j_k\neq 0} y_k$ is y_σ for some cone σ, where $J=(j_1,\dots,j_m)$ lies in $\mathbb{Z}^m_{\geq}$. There is therefore a canonical isomorphism

$$S_R(y)/(y_\omega : \omega \notin \Sigma) \;\xrightarrow{\cong}\; R[y;\Sigma]$$

of graded $S_R(y)$ algebras.

The version required for $K^*_{T^n}(X_\Sigma)$ involves 0–dimensional indeterminates β_j, for $1 \leq j \leq m$.

Definition 7.5. The *Laurent face diagram* $S^{\pm}_{K^*}\colon \text{CAT}^{op}(\Sigma) \to \text{GCALG}$ has

$$S^{\pm}_{K^*}(\sigma) \;=\; S^{\pm}_{K^*}(\beta(\sigma)) \quad\text{and}\quad S^{\pm}_{K^*}(p_{\tau,\sigma}) \;=\; f_{\tau,\sigma}\,, \tag{7.6}$$

where $f_{\tau,\sigma}\colon S^{\pm}_{K^*}(\tau) \to S^{\pm}_{K^*}(\sigma)$ is induced by mapping those β_j to 1 for which $v_j \in \tau \setminus \sigma$.

In this case the *Laurent face algebra* $F_K[\Sigma] = F_K[\beta; \Sigma]$ is the limit of $S^{\pm}_{K^*}$, and is additively generated by those monomials $(1-\beta)^J := (1-\beta_1)^{j_1} \cdots (1-\beta_m)^{j_m}$ whose support $\prod_{j_k \neq 0}(1-\beta_k)$ is $(1-\beta)_\sigma$ for some cone σ, where $J = (j_1, \dots, j_m)$ lies in $\mathbb{Z}^m$. There is therefore a canonical isomorphism

$$S^{\pm}_{K^*}(\beta)/((1-\beta)_\omega : \omega \notin \Sigma) \xrightarrow{\cong} F_K[\beta; \Sigma]$$

of graded $S^{\pm}_{K^*}(\beta)$-algebras.

A similar construction is possible for the MU_{T^n}-analogue $F_{MU}[\Sigma]$, and for the versions involving formal power series used below. In the case of cohomology, $F_{HR}[\Sigma]$ coincides with the standard face ring $R[\Sigma]$, so we retain the latter notation.

We show that $P_K(\alpha; \Sigma)$ is isomorphic to $F_K[\beta; \Sigma]$ by appealing to the defining diagrams; in the context of algebraic K-theory, a proof has long been available [VV].

By analogy with (5.3), we consider the $n \times m$ matrix

$$\xi \;=\; \xi_\Sigma \;:=\; \begin{pmatrix} v_1 & \dots & v_m \end{pmatrix}. \tag{7.7}$$

This notation is consistent with Remark 4.22, for we may view ξ as a map $\Sigma' \to \Sigma$ of fans; the rays of Σ' are the standard basis vectors in $\mathbb{R}^m$, and its cones $\sigma' := \{e_{i_1}, \dots, e_{i_k}\}$ correspond bijectively to the cones $\sigma := \{v_{i_1}, \dots, v_{i_k}\}$ of Σ. For any n–dimensional σ we write ξ_σ for the $n \times n$ submatrix of ξ whose columns generate σ. The smoothness of Σ guarantees that every ξ_σ is invertible over $\mathbb{Z}$. So ξ defines an epimorphism $\mathbb{Z}^m \to \mathbb{Z}^n$, and ξ^{tr} induces monomorphisms $S_R(x) \to S_R(y)$ and $S^{\pm}_{K^*}(\alpha) \to S^{\pm}_{K^*}(\beta)$ of graded rings; the latter maps α^J to $\beta^{\xi^{tr}J}$ for any $J \in \mathbb{Z}^n$.

Proposition 7.8. *For any smooth polytopal fan Σ, the matrix ξ induces a natural isomorphism*

$$\xi^*\colon P_K(\alpha; \Sigma) \longrightarrow F_K[\beta; \Sigma]$$

of algebras over $\xi^{tr}\colon S^{\pm}_{K^*}(\alpha) \to S^{\pm}_{K^*}(\beta)$.

Proof. The epimorphism $\xi\colon \mathbb{Z}^m \to \mathbb{Z}^n$ maps the generator e_j to the ray v_j, for all $1 \le j \le m$. It therefore induces an isomorphism $\xi^*(\sigma)\colon KV(\sigma) \to S^{\pm}_{K^*}(\sigma)$ of algebras over ξ^{tr}, defined by $\xi^*(\sigma)(\alpha^J) = \beta^{\xi(\sigma)^{tr}J}$ for any $J \in \mathbb{Z}^n$. Moreover, $f_{\tau,\sigma} \cdot \xi^*(\tau) = \xi^*(\sigma) \cdot q_{\tau,\sigma}$ for every morphism $\tau \supseteq \sigma$ in $\mathrm{CAT}(\Sigma)$, because $\xi_\tau \supseteq \xi_\sigma$ as submatrices of ξ.

So ξ^* is a natural isomorphism of diagrams, and induces the required isomorphism of limits. □

Corollary 7.9. *For any smooth polytopal fan Σ, there is an isomorphism $K^*_{T^n}(X_\Sigma) \to F_K[\beta; \Sigma]$ of algebras over ξ^{tr}.*

Proof. Combine Corollary 7.2 with Proposition 7.8. □

Example 7.10. If ξ is the $n \times (n+1)$ matrix $(-1 \;\; I_n)$, then X_Σ is $\mathbb{C}P^n$ and there is an isomorphism

$$K^*_{T^n}(\mathbb{C}P^n) \;\longrightarrow\; S^{\pm}_{K^*}(\beta_0, \dots, \beta_n) \Big/ \Big(\prod_{j=0}^{n} (1-\beta_j) \Big)$$

of algebras over ξ^{tr}. An equivalent formula appears, for example, in [GW].

Finally, for any smooth polytopal fan Σ, we describe the Borel equivariant K-theory of X_Σ in terms of the face ring $F_K[[\delta; \Sigma]]$, whose indeterminates are 0–dimensional. This is an algebra over $K^*[[\delta]]$, and is the limit of the diagram that assigns $K^*[[\delta_1, \dots, \delta_m]]/(\delta_\omega : \omega \notin \sigma)$ to each cone $\sigma \in \Sigma$. It is also the completion of $F_K[\beta; \Sigma]$ at the augmentation ideal $(1-\beta_1, \dots, 1-\beta_m)$, where $\delta_j = 1 - \beta_j$ for all $1 \le j \le m$.

We retain the notation of (7.7), observing that ξ^{tr} of Proposition 7.8 extends to a homomorphism $\xi^{tr}\colon K^*[[\gamma]] \to K^*[[\delta]]$.

Proposition 7.11. *For any smooth polytopal fan Σ, there are isomorphisms*

$$K^*(ET^n \times_{T^n} X_\Sigma) \longrightarrow P_{K_B}(\gamma;\Sigma) \xrightarrow{\xi^*} F_K[[\delta;\Sigma]]$$

of algebras over $K^*[[\gamma]]$ and ξ^{tr} respectively.

Proof. The first isomorphism is the completion of Corollary 7.2.2, and the second is the completion of Proposition 7.8. □

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