Dual Divisors

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We show that $1, 3, 9, 11, 33, 99$ are the only positive integers with the property that divisibilty by them is *preserved* under reversal of digits in the dividend.

Non-palindromic positive integers in decimal from $ab\dots cd\,(a \neq 0)$ that divide their reverses $dc\dots ba$ are called *reverse divisors*. There are infinitely many of these, the first being 1089, and they are all identified in [1]. In Section 3 of that article, Roger Webster notes that 1, 3, 9, 11, 33, 99 have the property that whenever they divide a positive integer, they *also* divide its reverse. Here we show that *these* are the only such positive integers.

1. Basics

Throughout the discussion, a *number* n will mean a positive integer $ab\dots cd$ $(a \neq 0)$ in *decimal* form, with *reverse* $dc\dots ba$ denoted by n^R. Such a number is called a *dual divisor* if it has the property that whenever it divides a number, it *also* divides the reverse of the number. Thus our objective is to prove that 1, 3, 9, 11, 33, 99 are the *only* dual divisors.

Theorem 1.1 The numbers 1, 3, 9, 11, 33, 99 are dual divisors.

Proof Clearly, 1 is dual divisor. That 3, 9, 11 *are*, follows from the facts that a number is divisible by 3 (or 9) if and only if the sum of its digits is divisible by 3 (or 9), and is divisible by 11 if and only if the *alternating* sum of its digits is divisible by 11. Since 3 (or 9) is *coprime* to 11, a number is divisible by 33 (or 99) if and only if it is both divisible by 3 (or 9) and 11. It follows that a number is divisible by 33 (or 99) if and only if its reverse is, showing that 33 and 99 are dual divisors. See [2]. □

Before embarking on the intricate proof that 1, 3, 9, 11, 33, 99 are the only dual divisors, we pause for breath. Many of our arguments involve showing that certain numbers are *not* dual divisors. To prove that a number n is *not* a dual divisor, it suffices to find a number k such that $(k \times n)^R$ is *not* divisible by n. By way of an interesting example, to show that the reverse divisor 109989 is *not* a dual divisor, we note that $(911 \times 109989)^R$ is *not* divisible by 109989, whereas for *all* $k < 911$, $(k \times 109989)^R$ *is*.

A word of caution. Our results are stated concisely, and our arguments, whilst elementary, are both formal and detailed. To help readers unaccustomed to such situations, we have included more discussion in some proofs than is necessary for the more seasoned reader.

Lemma 1.2 Every number has a multiple whose first digit is 1.

Proof Let n be a d-digit number, so $n < 10^d$. Let r be the *smallest* number such that $r \times n > 10^d$, so $(r-1) \times n \leq 10^d$. Then $10^d < r \times n < 2 \times 10^d$, whence the first digit of $r \times n$ is 1. □

Corollary 1.3 No dual divisor is even or ends in 5.

Proof Let n be a number that is either even or ends in 5. Then by Lemma 1.2, there is a number k such that $(k \times n)^R = \ \dots 1$, which is *not* divisible by n. Thus n divides $k \times n$, but not $(k \times n)^R$, and n is *not* a dual divisor. □

Lemma 1.4 No number whose first and final digits are 7 is a dual divisor.

Proof Let $n = 7\dots 7$ be a d-digit $(d \geq 1)$ number whose first and final digits are 7. Then the $(d+1)$-digit number $m = (2 \times n)^R = 4\dots 1$ is *not* divisible by n, for if it were, the quotient would end in 3, but $3 \times n < m$ and $13 \times n > m$. Thus n divides $2 \times n$ but not $(2 \times n)^R$, and so n is *not* a dual divisor. □

Theorem 1.5 A dual divisor is *either* a reverse divisor *or* a palindrome beginning with a 1, 3 or 9.

Proof Suppose that the number n is a dual divisor. Since n divides itself, it must divide n^R. If $n^R \neq n$, then n is a reverse divisor. If $n^R = n$, then n is a palindrome, which begins with a 1, 3 or 9 by Corollary 1.3 and Lemma 1.4. □

In the remainder of the article, we progressively eliminate numbers of the forms specified in Theorem 1.5 from being dual divisors, until only the six dual divisors given in Theorem 1.1 remain.

2. Reverse divisors and dual divisors

We now prove that *no* reverse divisor is a dual divisor. Our analysis relies upon properties of reverse divisors established in [1]. Reverse divisors *either* have quotient 4 *or* quotient 9, which means that, in the former case, the reverse is *four* times the original number, and in the latter, *nine* times. We also use the facts that reverse divisors with quotient 4 are *even*, and that reverse divisors with quotient 9 have the form $10\ldots 89$. For each non-negative integer r, we denote by 0_r the string of r consecutive 0's.

Theorem 2.1 No reverse divisor is a dual divisor.

Proof Reverse divisors with quotient 4 are even, and so by Corollary 1.3 *cannot* be dual divisors. Let, then, $n = 10A89$ be a d-digit $(d \geq 4)$ reverse divisor with quotient 9, where A is a string of $(d-4)$ digits. We establish the theorem by proving $(k \times n)^R$ is not divisible by n for $k = 10^{d-1}+1$. Consider the number

$$m = ((10^{d-1}+1) \times n)^R = (10A890_{d-1} + 10A89)^R = 98A^R009A^R01.$$

Next, remembering that n is a reverse divisor with quotient 9, we observe that

$$9 \times 10^{d-1} \times n = 10^{d-1} \times n^R = 98A^R010_{d-1}.$$

Since the number $(9 \times 10^{d-1} \times n) - m = 10^{d-1} - 9A^R01$ has $(d-2)$ digits, it cannot be divisible by n, whence neither can m be. Thus n is *not* a dual divisor. □

3. Palindromes and dual divisors

Here we show that there are *no* palindromic dual divisors with *three* or more digits which begin with a 1, 3 or 9.

Palindromes of the form 1...1 and dual divisors

Lemma 3.1 No palindrome of the form $1a\ldots 1$ $(a \neq 1)$ is a dual divisor.

Our proof considers each of the *nine* possible cases for a. The case $a = 0$ (a good warm up for the proof of Lemma 3.2) is the hardest, and could be left until *after* the reader has mastered the other cases.

Proof Let n be a d-digit $(d \geq 3)$ palindrome of the form $1a\ldots 1$ $(a \neq 1)$.

Case $a = 0$: If $n = 101$, then $m = (109 \times n)^R = 90,011$ is not divisible by n. If $n = 10A01$, where A is a palindromic string of $(d-4)$ digits $(d \geq 4)$, then $9 \times n = 9B09$, for some string B of $(d-3)$ digits. We now establish this case by proving $(k \times n)^R$ is not divisible by n for $k = 9 \times 10^{d-1} + 1$. Consider the number

$$m = ((9 \times 10^{d-1} + 1) \times n)^R = (9B090_{d-1} + 10A01)^R = 10A^R001B^R9 = 10A001B^R9.$$

Then the number $(10^{d-1} \times n) - m = 10^{d-1} - 1B^R9$ has $(d-1)$ digits, and so is not divisible by n. Thus neither is m, and n is *not* a dual divisor.

Case $a = 2$: Then $m = (5 \times n)^R = 5\ldots 6$ is not divisible by n, for if it were, it would be $6n$, but $6n > m$. Thus n is *not* a dual divisor.

Case $a = 3$: Then $m = (4 \times n)^R = 4\ldots 5$ is not divisible by n, for if it were, it would be $5n$, but $5n > m$. Thus n is *not* a dual divisor.

Case $a = 4$: Then the inequalities $2n < (3 \times n)^R = 3\ldots 4 < 3n$ show n is *not* a dual divisor.

Cases $a \geq 5$: Then the inequalities $n < (2 \times n)^R = 2\ldots 3 < 2n$ show n is *not* a dual divisor. □

Lemma 3.2 Let n be a d-digit $(d \geq 5)$ palindrome of the form $n = 11A11$, where A is a $(d-4)$-digit palindromic string containing a digit other than 1. Then n is not a dual divisor.

Proof **Case** $9 \times A$ **has** $(d-4)$ **digits**: Since A contains a digit other than 1, the number $9 \times A$ contains a digit other than 9. Thus $B = (9 \times A) + 1$ contains $(d-4)$ digits. Consider the number

$$m = ((9 \times 10^{d-1} + 1) \times n)^R = (99(9 \times A)990_{d-1} + 11A11)^R = 11A^R100B^R99 = 11A100B^R99.$$

Then the number $(10^{d-1} \times n) - m = 10^{d-1} - B^R99$ has $(d-1)$ digits, and so is not divisible by n. Thus neither is m, and n is *not* a dual divisor.

Case $9 \times A$ **has** $(d-3)$ **digits**: Then $9 \times n = 10C99$, where both C and $D = C+1$ have $(d-3)$ digits. Consider the number

$$m = ((9 \times 10^{d-1} + 1) \times n)^R = (10C990_{d-1} + 11A11)^R = 11A^R100D^R01 = 11A100D^R01.$$

Then the number $(10^d \times n) - m = 10^d - D^R01$ is a d-digit number ending in 9. But no d-digit number ending in 9 can be divisible by n, for $9 \times n$ has $(d+1)$ digits, and $l \times n$ $(l < 9)$ does not end in 9. Thus $(10^d \times n) - m$, and hence m, is not divisible by n, and n is *not* a dual divisor. □

Lemma 3.3 Let n be a d-digit $(d \geq 3)$ number containing only 1's. Then n is not a dual divisor.

Proof The $(d+1)$-digit number $m = (19 \times n)^R = 90\ldots12$ is not divisible by n, since the smallest l for which $l \times n$ ends in 12 is 92, but $92 \times n > m$. Thus n is *not* a dual divisor. □

Summary 3.4 No palindrome beginning with a 1, having three or more digits, is a dual divisor.

Palindromes of the form 3...3 and dual divisors

Lemma 3.5 No palindrome of the form $3a\ldots3$ $(a \neq 3)$ is a dual divisor.

Proof Let n be a d-digit $(d \geq 3)$ palindrome of the form $3a\ldots3$ $(a \neq 3)$.

Case $a = 0$: If $n = 303$, then $m = (43 \times n)^R = 92{,}031$ is not divisible by n. If $n = 30A03$, where A is a palindromic string of $(d-4)$ digits, then $3 \times n = 9B09$, for some string B of $(d-3)$ digits. Now

$$m = ((3 \times 10^{d-1} + 1) \times n)^R = 30A^R021B^R9 = 30A021B^R9,$$

and $(10^{d-1} \times n) - m$ has $(d-1)$ digits, and so is not divisible by n. Thus neither is m, and n is *not* a dual divisor.

Cases $a = 1, 2$: Then $m = (4 \times n)^R$, a $(d+1)$-digit number of one of the forms $2\ldots21$ or $2\ldots31$, is not a multiple of n. For if $k \times n = m$, then k would end in 7. But the $(d+1)$-digit number $17 \times n = 5\ldots1 > m$, whilst $7 \times n$ ends in either 91 or 61. Thus n is *not* a dual divisor.

Case $a = 4$: Then the $(d+1)$-digit number $m = (3 \times n)^R = 92\ldots01$ is not a multiple of n, because $26 \times n < m < 28 \times n$, and $27 \times n$ ends 61 *not* 01. Thus n is *not* a dual divisor.

Cases $a \geq 5$: Then the d-digit number $m = (2 \times n)^R = 6\ldots7$ is not a multiple of n, since $n < m < 2 \times n$. Thus n is *not* a dual divisor. □

Lemma 3.6 Let n be a d-digit $(d \geq 5)$ palindrome of the form $n = 33A33$, where A is a $(d-4)$-digit palindromic string containing a digit other than 3. Then n is not a dual divisor.

Proof **Case** $3 \times A$ **has** $(d-4)$ **digits**: Since A contains a digit other than 3, the number $3 \times A$ contains a digit other than 9. Thus $(3 \times A) + 1$ contains $(d-4)$ digits. Consider the number

$$m = ((3 \times 10^{d-1} + 1) \times n)^R = 33A^R320B^R99 = 33A320B^R99, \text{ where } B = (3 \times A) + 1.$$

Then the number $(10^{d-1} \times n) - m$ has $(d-1)$-digits, and so is not divisible by n. Thus neither is m, and n is *not* a dual divisor.

Case $3 \times A$ **has** $(d-3)$ **digits**: Then $3 \times n = C99$, where C and $C+1$ have $(d-1)$ digits. Consider

$$m = ((3 \times 10^{d-1} + 1) \times n)^R = 33A^R320D^R = 33A320D^R, \text{ where } D = C+1.$$

Then $(10^d \times n) - m$ is a d-digit number beginning with a 9. But no d-digit number beginning with a 9 can be divisible by n, for n itself begins with a 3, $2 \times n$ begins with a 6, and $3 \times n$ has $(d+1)$ digits. Thus $(10^d \times n) - m$, and hence m, is not divisible by n, and n is *not* a dual divisor. □

Lemma 3.7 Let n be a d-digit $(d \geq 3)$ number containing only 3's. Then n is not a dual divisor.

Proof The $(d+1)$-digit number $m = (13 \times n)^R = 92\ldots34$ is not divisible by n. For if $m = l \times n$, then l must end in 8. But $18 \times n < m < 28 \times n$. Thus n is *not* a dual divisor. □

Summary 3.8 No palindrome beginning with a 3, having three or more digits, is a dual divisor.

Palindromes of the form 9...9 and dual divisors

Lemma 3.9 No palindrome of the form $9a\ldots9$ $(a \neq 9)$ is a dual divisor.

Proof Let n be a d-digit ($d \geq 3$) palindrome of the form $9a \ldots 9$ ($a \neq 9$).

Case $a = 0$: If $n = 909$, then $m = (21 \times n)^R = 98,091$ is not divisible by n. If $n = 90A09$, where A is a palindromic string of $(d-4)$ digits, consider the number

$$m = ((10^{d-1} + 1) \times n)^R \ = \ 90A^R081A^R09 \ = \ 90A081A09.$$

Then $(10^{d-1} \times n) \ - \ m$ has $(d-1)$ digits, and so is not divisible by n. Thus neither is m, and n is *not* a dual divisor.

Cases $1 \leq a \leq 8$: Then the $(d+1)$-digit number $m \ = \ (2 \times n)^R$, which has one of the forms $8 \ldots 81$ or $8 \ldots 91$, is not divisible by n. For if $m = l \times n$, then l ends in 9. But *none* of the 8 possible endings for $9 \times n$ is 81 or 91, and $19 \times n > m$. Thus n is *not* a dual divisor. $\square$.

Lemma 3.10 Let n be a d-digit ($d \geq 5$) palindrome of the form $n = 99A99$, where A is a $(d-4)$-digit palindromic string containing a digit other than 9. Then n is not a dual divisor.

Proof Since A contains a digit other than 9, $A + 1$ also has $(d-4)$ digits. Consider the number

$$m = ((10^{d-1} + 1) \times n)^R = 99A^R980B^R99 = 99A980B^R99, \ \text{ where } \ B = A + 1.$$

Then $(10^{d-1} \times n) \ - \ m$ has $(d-1)$ digits, and so is not divisible by n. Thus neither is m, and n is *not* a dual divisor. $\square$

For each non-negative integer r, we denote by I_r the string of r consecutive 9's.

Lemma 3.11 The number I_d ($d \geq 3$) is not a dual divisor.

Proof We note that the number $m = \ (I_d \times 10_{d-2}11)^R \ = \ 98I_{d-1}0_{d-1}1$ is not divisible by I_d, for

$$(990_{d-2}8) \times I_d \ = \ 98I_{d-2}80I_{d-2}2 \ < \ m \ < \ 98I_{d-1}0I_{d-2}1 \ = \ (990_{d-2}9) \times I_d. \ \square$$

Summary 3.12 No palindrome beginning with a 9, having three or more digits, is a dual divisor.

Theorems 1.1, 1.5, 2.1 combine with Summaries 3.4, 3.8, 3.12 to bring us to our goal.

Conclusion There are precisely six dual divisors : **1, 3, 9, 11, 33, 99**.

References

1. Roger Webster and Gareth Williams, On the trail of Reverse Divisors: 1089 and all that follow. *Math. Spectrum* **45** (2012/2013) Number 3, pp. 96 - 102.

2. David Sharpe and Roger Webster, Reversing Digits: Divisibility by 27, 81 and 121. *Math. Spectrum* **45** (2012/2013) Number 2, pp. 69 - 71.

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