Friends in High Places

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A history of amicable pairs from Pythagorean times until the present day. Number mysticism, the work of Thabit ibn Qurra and Euler, the effect of electronic computation on the search for pairs, and some open conjectures.

Photograph by David Brown

Sgurr nan Ceannaichean (see Authors' note)

Pythagoras was born c. 580 BC on the Aegean island of Samos. After years spent traveling, he eventually settled in Crotona, a Greek seaport in south west Italy. Here he founded the celebrated Pythagorean School, a secret brotherhood with its own strict observances, which continued to flourish long after its leader's death c. 500 BC, spreading his teachings throughout the Greek world.

The Pythagoreans' belief that *natural numbers* held the key to the universe led them to a *number mysticism*, in which numbers were ascribed various qualities. For example, odd numbers were thought of as *male* and even ones *female*. One represented *reason*, two *opinion*, three *harmony*, four *justice*, five *marriage*, and six *creation* etc. Today raffles, the Lottery, and TV shows like *Deal or No Deal* engender a culture of *lucky numbers*.

This fanciful number mysticism led the Pythgaoreans to take the first steps in developing *number theory*, the abstract study of integers, with their early recognition of *even*, *odd*, *prime*, and *composite* numbers. They labeled numbers *abundant*, *deficient*, or *perfect* according to whether the sum of their *proper* divisors was *more than*, *less than*, or, respectively, *equal to* the number itself.

An undoubted highlight of their mathematics was the discovery of the pair of distinct numbers 220 and 284, each of which is the sum of the *proper* divisors of the other, i.e.

$$220 = 1 + 2 + 4 + 71 + 142,$$
$$284 = 1 + 2 + 4 + 5 + 10 + 11 + 20 + 22 + 44 + 55 + 110.$$

Such a pair is said to be *amicable* or *friendly*. The smaller number 220 is abundant and the larger 284 deficient, a result true for any amicable pair. To the number mystics, 220 and 284, each being composed of the parts of the other, symbolized perfect friendship, superstition holding that two talismans bearing them would seal friendship between the wearers. Indeed, Pythagoras described a friend as ‘another I such as are 220 and 284’.

The numbers came to take on a mystical aura, playing a role in magic, astrology, sorcery, and horoscopes. Even so, commentators failed to find a single high profile appearance of this *friendliest* number pair. All the literature offers is two low profile appearances: Jacob’s gift of 220 goats and 220 sheep to Esau (Genesis 32: 14), seen as a mystical way of securing friendship; and the testing of their erotic effects by eleventh-century Arab El Madschriti, who baked confections in the shapes of the two numbers, with a friend eating the smaller number, while he ate the larger, the outcome going unrecorded!

The highest profile peak in the British Isles is Ben Nevis, one of the Munros, Scottish mountains over 3000 ft, named after Sir Hugh Munro. (Munro made his ascents in the hours of darknesss, so as not to disturb the local lairds, and perhaps this is why he failed to scale all the mountains named after him!) Scottish hills between 2500 ft and 3000 ft are known as Corbetts, after J. Rooke Corbett, the first Sassenach to conquer all the Munros. (Whilst attending St John’s College, Cambridge University, from 1895 to 1898, Corbett walked from Manchester to Cambridge at the beginning of each term, and back again at the end!) Although unacquainted personally, these two climbers were kindred spirits in a common quest, who may be poetically described as *friends in high places*, a friendship sealed in Pythagorean lore, for there are 220 Corbetts and 284 Munros!

Although the amicable pair (220 : 284) was known in Pythagorean times, no new pair appeared until the ninth century. Legend has it that:

> Once upon a time there lived a sultan, who prided himself on his great intellect. When one of his prisoners begged to be freed, the sultan replied: ‘Set me a problem and you may remain free until I have solved it.’ The prisoner challenged the sultan to find an amicable pair of numbers beyond (220 : 284). He then went his way and lived happily ever after, for the sultan never did solve the problem.

Progress in finding new amicable pairs only occurred when mathematicians joined the search.

The first serious contribution to the study of amicable numbers is the following remarkable result of Arab polymath Thabit ibn Qurra (826–901).

Thabit’s rule. *If* n *is a natural number such that all the three numbers*

$$p = 3(2)^n - 1, \qquad q = 3(2)^{n+1} - 1, \qquad r = 9(2)^{2n+1} - 1$$

are prime, then the numbers $2^{n+1}pq$ *and* $2^{n+1}r$ *form an amicable pair.*

For $n = 1, 3, 6$, Thabit's Rule produces the first, second, and third *known* amicable pairs, namely

$$2^2 \cdot 5 \cdot 11 = 220 \quad \text{and} \quad 284 = 2^2 \cdot 71,$$
$$2^4 \cdot 23 \cdot 47 = 17\,296 \quad \text{and} \quad 18\,416 = 2^4 \cdot 1151,$$
$$2^7 \cdot 191 \cdot 383 = 9\,363\,584 \quad \text{and} \quad 9\,437\,056 = 2^7 \cdot 73\,727,$$

but yields no other amicable pairs for $n \leq 191\,600$. The first is the classic one, the only one known at the time. Thabit deserves the credit for his breakthrough in discovering the second, for although not writing it down, the geometric example he used in proving his rule, when interpreted numerically, is *precisely* it. Al-Banna (1256–1321) and al-Farisi (1260–1320) also have legitimate claims on the latter. The honour of unearthing the third falls to Yazdi (c. 1600).

Knowledge of the amicable pair $(220 : 284)$ and its role in number mysticism reached Europe via the Arabs, and by 1550 the pair had already appeared in the works of Chuquet, Stifel, Cardan, and Tartaglia, although Thabit's rule and its consequences were unknown in the West. So it was when Fermat and Descartes began their own search for amicable pairs. Both rediscovered Thabit's rule, and in letters to Mersenne, each claimed a new pair: Fermat $(17\,296 : 18\,416)$ in 1636 and Descartes $(9\,363\,584 : 9\,437\,056)$ in 1638.

Euler's transient appearance on the scene revolutionized the search. On his entrance in 1737, just three pairs had been found in over two thousand years. On his exit in 1740 this had risen, solely by his hands, to 62, even discounting the few that later turned out to be *unfriendly*. He investigated when number pairs of a particular structure, say (apq, ars), where p, q, r, and s are distinct primes and *not* divisors of a, are amicable. This particular structure led to his smallest pair

$$(2620 : 2924) = (2^2 \cdot 5 \cdot 131 : 2^2 \cdot 17 \cdot 43),$$

laying to rest expectation that any new pairs would be inordinately large. His discovery of the pair

$$(12\,285 : 14\,595) = (3^3 \cdot 5 \cdot 7 \cdot 13 : 3 \cdot 5 \cdot 7 \cdot 139)$$

laid to rest expectation that both members of an amicable pair are even.

One general result to emerge from Euler's work is the following rule.

Euler's rule. *If natural numbers k and n with $k \leq n$ are such that all the three numbers*

$$p = (2^k + 1)2^{n+1-k} - 1, \qquad q = (2^k + 1)2^{n+1} - 1, \qquad r = (2^k + 1)^2 2^{2n+2-k} - 1$$

are prime, then the numbers $2^{n+1}pq$ and $2^{n+1}r$ form an amicable pair.

Case $k = 1$ is Thabit's rule. Only two pairs (k, n) with $k > 1$ are known to satisfy the above conditions, namely $(7, 7)$ and $(11, 39)$, neither of which Euler seems to have known. The first yields an amicable pair, each of whose members has 10 digits, the second yields an amicable pair, each of whose members has 40 digits.

Euler's success in developing systematic methods for finding amicable pairs encouraged a brave few to take up the search for themselves, but the thoroughness of his investigations left few easy avenues to explore. So successful had he been, that only four new pairs were discovered in the next century and a half, contributing to a grand total of 66 by the end of the nineteenth century. In 1972 Lee and Madachy (see references 1–4) published a historical survey of amicable pairs, listing all 1108 known at the time.

Electronic computation has transformed the search for amicable pairs, from a trickle into an avalanche. In their early history, only a single pair was found in a thousand years, in their recent history even one three-page article could boast *A Million New Amicable Pairs* (see reference 5). Since 1995 Jan Munch Pedersen has serviced the Internet site *Known Amicable Pairs*, http://amicable.homepage.dk/knwnc2.htm, which lists, in increasing order of lowest members, all 11 994 387 (as of September 2007) known amicable pairs, together with their discoverers, years of discovery, and prime factorizations. Its opening entries are

1. Pythagoras (500 BC): $220 = 2^2 \cdot 5 \cdot 11$ and $284 = 2^2 \cdot 71$,
2. Paganini (1866): $1184 = 2^5 \cdot 37$ and $1210 = 2 \cdot 5 \cdot 11^2$,
3. Euler (1747): $2620 = 2^2 \cdot 5 \cdot 131$ and $2924 = 2^2 \cdot 17 \cdot 43$.

The first and third occasion no surprise, for we have met with them before, but what of the second? Thereby hangs a tale. For two thousand years, mathematicians of the calibres of Fermat, Descartes, and Euler had scoured the heavens with their sophisticated instruments looking for amicable pairs, but failed to see what lay at their very feet, the second smallest pair (1184 : 1210). Who, then, made this startling discovery so late in the history of amicable pairs, thus ensuring a place in number theory's *hall of fame*? Yet another fairytale ending: a sixteen-year-old Italian schoolboy, Nicolo Paganini.

We have seen that the first three entries in Pedersen's 2007 list of amicable pairs have a tale to tell, but what of the last? It is the 11 994 387th and as of 2009 the largest one known. It begins

11 994 387. Jobling & Walker (2005): 1760... and 1826...,

where in *both* cases the '...' represent 24 069 further digits! It is certainly no exaggeration to say that these two members of the largest known amicable pair are indeed *friends in high places*, very high places!

Many intriguing conjectures about the structure of amicable pairs still await resolution. Amongst them include the following.

(i) The members of an amicable pair are either both even or both odd.

(ii) The members of an odd amicable pair are divisible by three.

(iii) The members of an amicable pair have a nontrivial common divisor.

Alongside attempts to solve such conjectures, the ongoing search for the largest amicable pair forges quietly ahead, unlike that for the largest known prime, which proceeds in the full blaze of media publicity. There is, however, a subtle difference in the nature of the two searches: Euclid proved that there are infinitely many primes, but no such result has been established for amicable pairs. Researchers all agree that there are infinitely many amicable pairs, but this is only a conjecture, albeit the most famous in the saga of friendly numbers.

A tongue-in-cheek comparison may be drawn between this conjecture and Fermat's Last Theorem, itself only a conjecture for three centuries. Investigators trying to prove the latter were no doubt encouraged by Fermat's own claim, that he had a truly wonderful proof, but it would not fit in the margin. Likewise, the few stalwarts attempting to prove that there are infinitely many amicable pairs may take succour from the following casual remark tucked away in Euler's 1747 paper, 'There is no doubt that infinitely many [amicable pairs] may be given'. Perhaps the conjecture that there are infinitely many amicable pairs should be dubbed Euler's Last Theorem!

Authors' note

This article was prompted by a letter of 31 October 2008 from David Gibson, Senior Officer of The Mountaineering Council of Scotland, confirming that there were 220 Corbetts and 284 Munros. Unfortunately, on the day this article was completed, *The Independent* of 11 September 2009 reported that, after re-measurement in July 2009, the height of the lowest Munro, *Sgurr nan Ceannaichean*, had been found to be 2996.82 ft. Consequently, it has been demoted to a mere Corbett, so now there are 221 Corbetts and 283 Munros. Maybe a topic for another article!

References

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5 M. Garcia, A million new amicable pairs, *J. Integer Sequences* **4** (2001), # 01.2.6.

Roger Webster *is a lecturer at Sheffield University specializing in the history of mathematics. He is a G & S devotee, revelling in this Gilbertian situation, in which a Munro lost its standing in Scottish mountaineering aristocracy, because a certain Lord High Surveyor decreed it to have dropped five feet, even though it had not moved at all!*

Gareth Williams *is a staff tutor in mathematics at the Open University specializing in topology, and is in great demand as a popularizer of mathematics. A keen mountaineer, having scaled many Munros and Corbetts, he was delighted to receive a holiday postcard, depicting a mountain scene, from his co-author beginning 'Hi 220, . . . ' and ending 'Cheers 284'.*