Poincaré and his infamous Conjecture

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Disclaimer. *Topology is a deep and beautiful subject. To provide precise statements and proofs assumes a certain degree of sophistication in the discipline. The current article makes no such assumption. Consequently, technicalities are omitted. Experts may shudder, but it is hoped that an audience with little background will be grateful. For a more comprehensive account, the reader may consult [2] or [4].*

1 Topology

Elementary geometry was well established in ancient Greece by 300 BC, but there's a newer idea called *topology*. Imagine doing your homework not on paper, but on a sheet of stretchy rubber. Once you've finished, what would happen if you stretched the rubber? Concepts like *straight* and *parallel* become meaningless – even *size* is challenged. Yet many interesting properties *do* make sense after stretching.

Let's illustrate with an example. Picture a map of the London Underground. Does it tell you how far it is from King's Cross to Euston, or the direction from one to the other? The answers are no, but millions of travellers navigate everyday with this *topological* map, which conveys how the stations are linked together. If you draw the Underground map on a rubber sheet, and then stretch the rubber, it will still tell you how the stations are configured.

In topology, we imagine everything is made from rubber, and **we allow ourselves to stretch and squash, but not to cut or glue**. In terms of the rubber Underground map, this means that you can distort the picture smoothly. Cutting the map might break links between stations, whilst gluing two Underground lines adds extra links. The confusion this would cause London's population is why topologists forbid cutting and gluing!

If one rubber object can be manipulated to look just like another, we say that the two are *topologically the same*. For example, a rubber football could be squashed to look like a rugby ball, so they are topologically the same. Similarly, a handkerchief and a spinnaker sail are topologically the same. Can you suggest further pairs of objects which are topologically the same?

Note that when we talk about a football, we mean the skin or the *surface* of the ball, and not the air inside. Similarly, we distinguish between a *circle* and a *solid circle* or *disc*, shown in Figure 1.

Figure 1: A circle and a solid circle, (disc).

You may like to ask if the following phrase, common with French school children, is accurate from a topologist's point of view. How about a geometer's?

> *Qu'est-ce qu'un cercle? Ce n'est point carré.*
> (What is a circle? It's not a square.)

1.1 Elastic loops

Think again of a football. Tie a loop around it with a thread. Topology allows us to imagine an *elastic loop.* We can stretch and squash until it forms a smaller and smaller loop on the football. Eventually it becomes so small that it's a mere speck! We then say that we have *topologically removed* the elastic loop from the football, as in Figure 2. Notice as we manipulate the elastic, we never pull it away from the football. All points of the loop remain in contact with the football throughout.

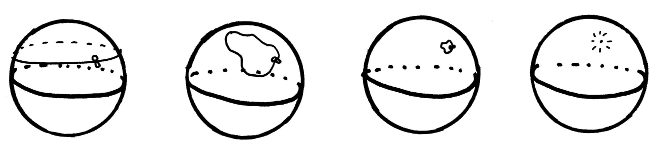

Figure 2: Topologically removing an elastic loop from a football.

Consider now the surface of the earth – ignoring the rock and magma inside (as we did with the air inside the football). The surface of the earth is topologically the same as the football, so should we expect to be able to repeat our experiment with an elastic loop tied around the equator? The answer is provided by the following result.

Theorem. *If two objects X and Y are topologically the same, then we have precisely one of the following situations.*

(i) You can topologically remove any elastic loop from X and any elastic loop from Y.

(ii) You can find an elastic loop λ on X and an elastic loop μ on Y so that neither λ nor μ can be topologically removed.

By contrast, consider a ring donut – its mathematical name is a *torus*. (Make certain to distinguish between a torus and a solid torus.) You can find an elastic loop which *cannot* be topologically removed from your torus – no matter how much you stretch or squash it. See Figure 3.

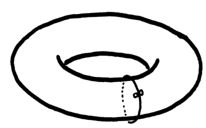

Figure 3: An elastic loop that can never be topologically removed from the torus.

One goal of modern topology is to determine if two given objects are topologically the same. The Theorem is a powerful tool in this pursuit. Try to use it to decide if a football is topologically the same as a torus.

2 Poincaré and Manifolds

2.1 The "Father of Topology"

Henri Poincaré (1854 – 1912) came from a distinguished French family: his cousin Raymond was President of the French Republic during World War I. But Henri Poincaré preferred mathematics and physics over politics – highlights of his career include studying the relative motion of a system of three planets, *special relativity* (a brilliant subject attributed to Einstein) and the *Poincaré-Hopf* theorem. The latter is commonly known as the *hairy ball* theorem, saying that you can't continuously comb a dog's hair without either a parting or a bald spot! His 1895 masterpiece *Analysis Situs* earned Poincaré the title "father of topology".

Mathematicians like to characterise things. For example, the integers which are divisible by 5 can be characterised as those with final digit either 0 or 5. Poincaré wished to characterise "high-dimensional" footballs, in terms of topologically removing elastic loops. He never found this characterisation, but he made a guess as to what it might be. This guess became known as the *Poincaré Conjecture*, and engaged mathematicians until very recently. Before we state it, we need to discuss what we mean by a "high-dimensional" football!

2.2 Manifolds

Topologists work with many different objects. Among the nicest types are *manifolds*. You may have an intuitive idea of what a *surface* is, perhaps the surface of a football or the surface of the earth. Manifolds generalise this notion to other dimensions. We describe a manifold as an n-manifold if it is a generalisation to dimension n (a positive integer).

Most people have an intuitive idea about the following statements.

- A straight line is one-dimensional.
- A plane is two-dimensional.
- Space is three-dimensional.

Once you feel happy with these statements, manifolds are not much harder:

- A 1-manifold looks one-dimensional (like part of a straight line) if you zoom in at any point.
- A 2-manifold looks two-dimensional (like part of a plane) if you zoom in at any point.
- A 3-manifold looks three-dimensional (like a piece of space) if you zoom in at any point.

For example, pick a point on a circle (remember the difference between a circle and a solid circle). Zoom in on that point, at great magnification. As in Figure 4, the result will look like a straight line segment – one dimensional. So a circle is a 1-manifold.

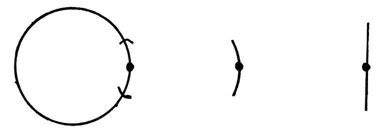

Figure 4: A small portion of a circle looks like a straight line segment when we zoom in closely enough.

Consider what happens when you zoom in closely to the earth's surface, or a football, or a torus. You should convince yourself that each of these objects is a 2-manifold. Try also to find some objects which are *not* manifolds. For example, what happens when you zoom in at the centre of a letter X?

Let's now list a few manifolds. The circle and football are examples of what topologists call *spheres*. We use the notation S^1 for a circle – S for sphere and 1 because a circle is a 1-manifold. The football is known as S^2. In Euclidean space, they can be described symbolically:

$$S^1 = \{(x, y) \mid x^2 + y^2 = 1\}; \qquad S^2 = \{(x, y, z) \mid x^2 + y^2 + z^2 = 1\}.$$

Can you spot a pattern and guess how one might describe something called S^3? The answer is

$$S^3 = \{(w, x, y, z) \mid w^2 + x^2 + y^2 + z^2 = 1\},$$

and this object is a 3-manifold, called the 3-sphere. (Naturally enough, S^2 is called the 2-sphere and S^1 the 1-sphere. What's less obvious is that there is also a 0-sphere and an n-sphere for each positive integer n.)

Picturing S^3 is tricky, and takes practice, but 3-manifolds are worthwhile objects. Pieces of the universe appear three dimensional when we look closely at them, so the universe can be modelled as a 3-manifold. A natural question to ask is *which* 3-manifold. The answer is not known, but see [5] for a more comprehensive discussion.

3 The Poincaré Conjecture

We now rephrase Poincaré's goal as *a characterisation of S^3 in terms of topologically removing elastic loops.* In 1900, Poincaré made a first guess, which he proved wrong[1] in 1904. Inspired, this motivated Poincaré to probe further, leading to:

[1] The breakthrough arose when he considered something now called *Poincaré dodecahedral space.*

Conjecture (The Poincaré Conjecture, 1904)**.** *If a* 3*-manifold has the property that any elastic loop can be topologically removed, then it is topologically the same as* S^3*.*

Remark. *Recall our initial disclaimer. The Poincaré Conjecture is a precise statement, whereas the statement above is more suited to this article.*

In other words, Poincaré asked if S^3 is characterised by being the only 3-manifold with the property that all elastic loops can be topologically removed from it.[2]

3.1 Early attempts at proof

It became apparent that the Poincaré conjecture would be difficult to settle, and mathematicians began to despair. Characterisations of S^4, S^5, ... were considered, and by 1961, the analogous result was known for S^n with $n \geq 5$. In 1982, S^4 was dealt with, but S^3 remained an unsolved mystery.

One might expect that as n increases, questions about S^n become harder to answer, but beyond $n = 2$ the opposite is true! In higher dimensions, the topologist has more space, more freedom, in which to manoeuvre. Consider the following illustration.

Suppose someone asks you to vacate a telephone box, *without* opening the door. Naturally, you can't do it. Now, suppose on the beach that someone draws a circle in the sand around you. You can easily escape without breaking the circle – see Figure 5.

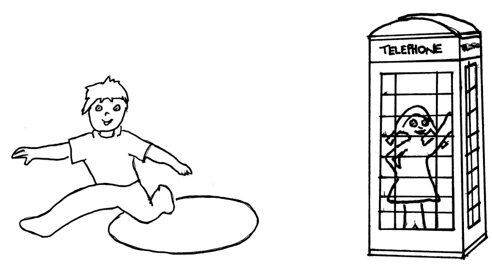

Figure 5: Escaping from S^1 (a circle) is easy; escaping from S^2 (topologically the same as a telephone box with the door glued shut) is not!

[2]You may have thought of Euclidean 3-space as a 3-manifold in which any elastic loop can be topologically removed. This is *not* a counter example to the Poincaré Conjecture, since a technicality, omitted in the spirit of the remark, precludes it.

The two situations are analogous – escape from within a sphere – but the spheres (S^1 and S^2) are of different dimensions. It's the third dimension that we inhabit that allows you to jump over the circle in the sand. If we had an extra, fourth, dimension available, we could use it to escape from the telephone box. Likewise, in a 3-manifold there isn't enough "room" to do all that one can in higher dimensions. For a delightful account of dimension related phenomena, [1] cannot be recommended too highly.

So the Poincaré Conjecture remained a conjecture, but it grew in stature as a holy grail for mathematicians. In 2000, The Clay Mathematics Institute listed seven problems, judged to be the most difficult and important mathematical problems of the time – and offered a prize of $1,000,000 for the first solution to each.[3] The inclusion of the Poincaré Conjecture thrust the quest for a proof, or counter-example, into the limelight. The race was on, and the conclusion does not disappoint!

3.2 Grigory Perelman and the first proof

In 2002 and 2003, almost one hundred years after the Poincaré Conjecture was formulated, the Russian mathematician Grigory Perelman uploaded three papers onto the internet. This was peculiar, because papers are usually submitted to learned journals, but Perelman himself is somewhat mysterious too.

Perelman lives in St. Petersburg, where he has resigned from his job, and has not made attempts to publish his three papers formally. As a child, he won the International Mathematical Olympiad with a perfect score, and went on to earn a reputation as a mathematician who thought deeply and seldom made mistakes.

Perelman's three papers did not mention the Poincaré Conjecture. They talked about the *Geometrization Conjecture*. In 1983, the American mathematician William Thurston conjectured that any 3-manifold can be decomposed into pieces, each with one of eight possible structures. Crucially, the Poincaré Conjecture is a consequence of the Geometrization Conjecture, and mathematicians started to believe that Perelman's papers provided the final steps of a proof of the Geometrization Conjecture.[4]

Despite no formal publication, most of the mathematical community have now accepted that Perelman's work is correct, if lacking details. In

[3] Prospective prize-winners can find out about the *Millennium Problems* in [2].

[4] The method of proof was devised by Richard Hamilton, another American mathematician. The key idea is called *Ricci flow,* which "smoothes out" structures on manifolds until they resemble one of the eight specified in the Geometrization Conjecture.

2006, Perelman was awarded a Fields Medal (the most prestigious prize in mathematics), but the delightful Russian responded with the words "I refuse", making him the first person ever to turn down such an honour.

Many other mathematicians were involved in the work, notably the Chinese mathematician Shing-Tung Yau. Drama followed when *The New Yorker* magazine alleged in [3] that Yau denounced Perelman's contribution in an attempt to divert credit to himself. Yau responded with formal steps for defamation.

Now that the dust is settling, the consensus is that Perelman's work proves the Geometrization Conjecture, and hence the Poincaré Conjecture. Others have expanded the details, resulting in a 327 page paper which gives a complete account. But the problem which tortured mathematicians for almost a century was ultimately laid to rest by a stroke of genius from a Russian recluse.

And as for the $1,000,000 prize? Officially, a solution should be published in a "refereed mathematics publication of worldwide repute", something which Grigory Perelman seems to have no interest in doing. Controversy exists as to whether the rules should be reinterpreted in this electronic age, and speculation continues over what Perelman might do if he is ever offered the million dollar prize.

References

[1] Edwin A. Abbott. *Flatland: A Romance of Many Dimensions.* Dover Publications Inc., 1992.

[2] Keith Devlin. *The Millennium Problems: The Seven Greatest Unsolved Mathematical Puzzles of Our Time.* Granta Books, 2005.

[3] Sylvia Nasar and David Gruber. Manifold Destiny: A legendary problem and the battle over who solved it. *The New Yorker* magazine, 28 August 2006. Available at `http://www.newyorker.com/fact/content/articles/060828fa_fact2`.

[4] Donal O'Shea. *The Poincaré Conjecture: In Search of the Shape of the Universe.* Walker & Company, New York, 2007.

[5] Jeffrey R. Weeks. *The Shape of Space.* Second edition, Marcel Dekker Inc., New York, 2002.