Reverse divisors with quotient 4

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The results and proofs of [1, Section 2] for reverse divisors with quotient 4

Theorem 1 Let $ab\dots cd$ $(a \neq 0)$ be a reverse divisor with quotient 4. Then it has the form $21\dots78$.

Proof Since $4 \times (ab\dots cd) = dc\dots ba$, a is even, whence $a = 2$. Thus d is 8 or 9. But $d = 9$ implies $a = 6$, so $d = 8$. Thus b is 0, 1 or 2, of which only $b = 1$ is consistent with ba being divisible by 4. Thus $b = 1$, implying c is 4, 5, 6 or 7, of which only $c = 7$ is consistent with $4 \times (\dots c8) = \dots 12$. Thus $c = 7$. □

Corollary 2 The only four digit reverse divisor with quotient 4 is 2178. □

Theorem 3 Let $21ab\dots cd78$ be an n-digit $(n \geq 5)$ reverse divisor with quotient 4. Then

$$4 \times (99\dots99 - ab\dots cd) = 99\dots99 - dc\dots ba,$$

where $99\dots99$ is the $(n-4)$-digit number comprising solely of 9's.

Proof By [1, Theorem 1.1], the n-digit $(n \geq 5)$ number $2199\dots9978$, whose central $(n-4)$ digits are 9's, is a reverse divisor with quotient 4, whence $4 \times (2199\dots9978) = 8799\dots9912$. By hypothesis, $4 \times (21ab\dots cd78) = 87dc\dots ba12$. Subtracting these equalities, then dividing by 100, gives the result. □

Theorem 3 can be used to show, even without a calculator, that the only 5-, 6-, 7-digit reverse divisors with quotient 4 are the basic ones: 21978, 219978, 2199978. A routine computer search reveals that the only *non-basic* 8- and 9-digit reverse divisors with quotient 4 are 21782178 and 217802178. Thus the number of reverse divisors with quotient 4 under a billion is eight.

Our study of the beginnings and endings of reverse divisors below culminates in the key Theorem 8.

Theorem 4 A reverse divisor with quotient 4 ends in either 178 or 978.

Proof A reverse divisor with quotient 4 that ends $b78$ begins $(87b\dots) \div 4 = 21a\dots$, whence

$$a = 7 \text{ if } b = 0, 1; \qquad a = 8 \text{ if } b = 2, 3, 4, 5; \qquad a = 9 \text{ if } b = 6, 7, 8, 9. \tag{†}$$

Since $4 \times (\dots b78) = \dots a12$, the units digit of $3 + 4b$ is a. Thus either $b = 1$ $(a = 7)$ or $b = 9$ $(a = 9)$. □

Theorem 5 A reverse divisor with quotient 4 ending 178 both begins and ends with 2178.

Proof A reverse divisor with quotient 4 ending $b178$ begins $(871b\dots) \div 4 = 217a\dots$, whence (†) holds. Since $4 \times (\dots b178) = \dots a712$, the units digit of $4b$ is a. Thus $b = 2$ and $a = 8$. □

For each non-negative integer r, denote by I_r resp. 0_r the string of r consecutive 9's resp. 0's, and by T_r the basic reverse divisor $T_r = 21I_r78$ with quotient 4. Thus T_0 is 2178 itself.

Lemma 6 A reverse divisor with quotient 4 ending 978 begins $21I_r7$ and ends $1I_r78$ for some $r \geq 1$.

Proof A reverse divisor with quotient 4 ending 978 must end bI_r78 for some $b \neq 9$ and $r \geq 1$. It begins $(87I_rb\dots) \div 4 = 21I_ra\dots$, whence

$$a = 7 \text{ if } b = 0, 1; \qquad a = 8 \text{ if } b = 2, 3, 4, 5; \qquad a = 9 \text{ if } b = 6, 7, 8.$$

Since $4 \times (\dots bI_r78) = \dots aI_r12$, the units digit of $3 + 4b$ is a. Thus $b = 1$ and $a = 7$. □

Theorem 7 A reverse divisor with quotient 4 ending 978, both begins and ends $21I_r78$ for some $r \geq 1$.

Proof By Lemma 6, a reverse divisor with quotient 4 ending 978, ends $b1I_r78$ for some b and $r \geq 1$. It begins $(87I_r1b\dots) \div 4 = 21I_r7a\dots$, whence (†) holds. Since $4 \times (\dots b1I_r78) = \dots a7I_r12$, the units digit of $4b$ is a. Thus $b = 2$ and $a = 8$. □

Theorems 4, 5, 7 together yield Theorem 8 below, the pivotal result in decomposing a general reverse divisor with quotient 4 into basic ones.

Theorem 8 Each reverse divisor with quotient 4 both begins and ends with the same basic reverse divisor with quotient 4. □

Below we write Q^R to denote the reverse of a string of digits Q. Thus the condition for such a Q, with leading digit non-zero, to be a reverse divisor with quotient 4, becomes $4 \times Q = Q^R$.

Lemma 9 Let M be a non-basic reverse divisor with quotient 4. Then $M = T_r 0_s V 0_s T_r$ for some $r, s \geq 0$, where V is *either* a reverse divisor with quotient 4, *or* 0_t for some $t \geq 0$.

Proof Since M is non-basic, Theorem 8 shows that $M = T_r P T_r$ for some $r \geq 0$ and some string of digits P. If P is empty or consists only of zeros, then M is in the desired form. Suppose, then, that P contains some non-zero digits.
Since $T_r P T_r$ is a reverse divisor with quotient 4,

$$4 \times (T_r P T_r) = (T_r P T_r)^R = (T_r)^R P^R (T_r)^R = (4 \times T_r) P^R (4 \times T_r).$$

Comparing this with the direct multiplication of $T_r P T_r$ by 4 shows that $4 \times P = P^R$. If P's first digit is non-zero, then P is a reverse divisor with quotient 4, and we are done. Suppose, then, that P has the form $0_s a \ldots$ for some $a, s > 0$. Then

$$4 \times (0_s a \ldots) = (0_s a \ldots)^R = \ldots a 0_s,$$

which shows that P has the form $0_s a \ldots b 0_s$ for some b, and that $4 \times (a \ldots b) = b \ldots a$. Thus $a \ldots b$ is a reverse divisor with quotient 4, and we note that $M = T_r 0_s a \ldots b 0_s T_r$. □

We now reach our main result, Theorem 10. It states, in essence, that all reverse divisors with quotient 4 are constructed by concatenating basic ones with strings of zeros, in a symmetric and alternating way.

Theorem 10 (Structure theorem for reverse divisors with quotient 4) Non-basic reverse divisors with quotient 4 are precisely those natural numbers of the form

$$T_{a_1} 0_{b_1} T_{a_2} 0_{b_2} \ldots T_{a_n} 0_{b_n} V 0_{b_n} T_{a_n} \ldots 0_{b_2} T_{a_2} 0_{b_1} T_{a_1},$$

where either $V = T_{a_0}$ or $V = 0_{b_0}$, for non-negative integers $a_0, a_1, \ldots a_n$ and $b_0, b_1, \ldots, b_n$.

Proof Every number of the given form is certainly a non-basic reverse divisor with quotient 4. Repeated application of Lemma 9 shows that every non-basic reverse divisor with quotient 4 has this form. □

Theorem 10 enables us to write down all reverse divisors with quotient 4 having a given number of digits. We list all 19 that are under a trillion, boldfacing the basic ones.

2178, **21978**, **219978**, **2199978**, 21782178, **21999978**, 217802178, **219999978**, 2178002178, 2197821978, **2199999978**, 21780002178, 21978021978, **21999999978**, 217800002178, 217821782178, 219780021978, 219978219978, **219999999978**.

Reference

[1] Roger Webster and Gareth Williams, On the trail of Reverse Divisors: 1089 and all that follow. *Math. Spectrum* **45** (2012/2013) Number 3, pp. 96-102.

Teaser 2529

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Victor Bryant's *Teaser 2529* in the Sunday Times (London) of March 13, 2011 reads:

> "A letter to The Times concerning the inflated costs of projects read: 'When I was a financial controller, I found that multiplying original cost estimates by pi used to give an excellent indication of the final outcome.' Interestingly, I used this process (using $22/7$ as a good approximation for pi). On one occasion, the original estimate was a whole number of pounds (under $100,000$), and this method for the probable final outcome gave a number of pounds consisting of the same digits, but in reverse order. What was the original estimate?"

Write X for the original estimate and Y for its reverse, so that $2 \times 11 \times X = 7 \times Y$. Hence 11 divides Y. But 11 has the *special* property that, if it divides a number, then it also divides its reverse. Thus 11 divides X. Further, 7 divides X, so $X = 77 \times n$ for some positive integer n, from which $Y = 242 \times n$. It follows that both X and Y have at least three digits, say $X = a \dots e$. Since $\frac{22}{7} \times (a \dots e) = e \dots a$, a is 1, 2 or 3. But $Y = e \dots a$ is even. Thus $\boldsymbol{a = 2}$, and e is 6, 7, 8 or 9. Since $22 \times (\dots e) = 7 \times (\dots 2)$, the units digit of $22 \times e$ is 4. Thus $\boldsymbol{e = 7}$.

The *only* three digit multiple of 77 whose first digit is 2 is 231, whose final digit is *not* 7. The *only* four digit multiple of 242 having first digit 7 and final digit 2 is 7502, but 2057 is *not* a multiple of 77. Thus X, which is given as being under $100,000$, must have five digits, say $X = 2bcd7$.

Let the positive integer n be such that $7dcb2 = 242 \times n$ and $2bcd7 = 77 \times n$. The first equation shows that $290 \leq n \leq 330$, and the second that the units digit of n is 1. Thus n is 291, 301, 311 or 321. Of these, only $n = 291$ gives a solution: $X = 22407$ ($Y = 70422$). Thus the original estimate was **£22,407**.

Reference

[1] Roger Webster and Gareth Williams, On the trail of Reverse Divisors: 1089 and all that follow. *Math. Spectrum* **45** (2012/2013) Number 3, pp. 96-102.