On the trail of Reverse Divisors: 1089 and all that follow

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We determine all natural numbers that divide their reverses

This is an account of the authors' adventures in tracking down, capturing and cataloguing a rare species of whole number, known as a *reverse divisor*. So scarce are they that there are only 6 of them under a million, 16 under a billion and 38 under a trillion. As the name suggests, a reverse divisor is a decimal integer that divides the number obtained by reversing its digits. It should be emphasized here, that our whole discussion in the first two sections is based on the *decimal* expansion of a number. To avoid the commonplace, we exclude palindromic numbers, those that are unchanged by reversal of digits, in our formal definition: a *non-palindromic* natural number in decimal form $ab \ldots cd \, (a \neq 0)$ which divides its *reverse* $dc \ldots ba$ is called a *reverse divisor*. For such a number, the quotient of the reverse by the original is called the *quotient of the reverse divisor*. Reverse divisors must have at least two digits, *cannot* end in 0 and the quotient of a reverse divisor is one of the numbers $2, 3, \ldots, 9$.

The first sighting of a reverse divisor is hard to come by. Single digit numbers, being palindromic, do not qualify. A few minutes' mental arithmetic shows there are no two-digit reverse divisors, and even longer on a pocket calculator shows there are no three-digit reverse divisors. Rather than baldly announce the first reverse divisor, we invite the eagle-eyed amongst you to take stock and make an inspired stab in the dark at it, without lifting a finger. For those of you who were successful, and those who weren't, please read on and join us on a mathematical journey, from knowing nothing about reverse divisors to knowing everything! We found it exciting, we hope that you will do so too.

The *only* reverse divisors having four or fewer digits are 1089 and its double 2178, with respective quotients 9 and 4, an observation that G. H. Hardy, greatest English number theorist of the twentieth century, alludes to in his delightful '*A Mathematician's Apology*' [1, pp. 104-5] as *non-serious* mathematics. The *smallest reverse divisor* 1089 has over recent years become something of a *nombre célèbre* in recreational mathematics ([2, p. 9], [3, p. 163]) on account of its following remarkable property, which you should try out for yourself, if you have not already done so:

Subtract from any *three digit number, whose first digit exceeds the last, its reverse. Add this difference to its own reverse. Then* (*in three-digit arithmetic*) *this last sum is always* 1089.

The result generalizes to four or more digit numbers [4]. A best selling paperback by David Acheson [5] *actually* bears the title **1089 and All That**, despite not mentioning that 1089 is the *smallest reverse divisor*! Lewis Carroll entertained his child-friends with this arithmetical curiosity, and may even have been its discoverer [6, pp. 158-159]. It appears under the heading ABRACADABRA in the News Chronicle's *I-SPY Annual* for 1956, whilst Johnny Ball in his fun maths book *Think of a Number* [7, p. 48] exploits it in a *mind-boggling* conjuring trick!

0. Guidance for the first-time adventurer

The *official* account of our adventure that is presented here is both detailed and formal, which can be off-putting for the beginner. We suggest, therefore, that the reader whose only interest is in discovering the facts should study the statements of the **lemmas**, **theorems** and **corollaries**, but omit their *proofs*. Even such a cursory reading will reveal the intricacy of the sequence of arguments designed to pin down reverse divisors in Theorem 2.10. For the more adventurous amongst you, and those eager to learn what is happening behind the scenes, we encourage you to read through the proofs, attempting to understand them. They are in truth, much of a muchness, depending on basic arithmetic and argument by contradiction. For those of you, even more ambitious yet, we challenge you to state and supply proofs for all the results in Section 2, suitably modified for reverse divisors with quotient 4, and to investigate the ideas mooted in Section 3.

This article may be likened to the log of a mountain ascent. The lemmas, theorems and corollaries mark the high points on the trek upwards, the proofs record the paths traversed. It tells nothing of the days of despair when, confined to camp, no progress was possible. Nor does it tell of the thrill of an unexpected thrust forward, climbing from one perch to another higher up, closer to a summit we could not see or even know was there. The adventure began amidst uncertainty, for we knew to begin with 1089 and 2178, but had no idea of where to go from there! That is the thrill of the chase, the challenge of mathematics. Finally, the elation of reaching the summit, a fitting climax to a memorable adventure.

1. The fundamental theorem

Theorem 1.1 The quotient of a reverse divisor is either 4 or 9. The two n-digit ($n \geq 4$) numbers

$$11 \times (10^{n-2} - 1) \;=\; \mathbf{10}99\ldots99\mathbf{89} \quad \text{and} \quad 22 \times (10^{n-2} - 1) \;=\; \mathbf{21}99\ldots99\mathbf{78},$$

each with $n-4$ central digits 9, are reverse divisors with respective quotients 9 and 4. No reverse divisor has *fewer* than four digits.

Proof Suppose that $a\,\ldots\,d$ $(a \neq 0)$ is a reverse divisor with quotient q. Then

$$q \times (a\,\ldots\,d) = d\,\ldots\,a. \quad *$$

We show that each of the cases $q = 2,\ 3,\ 5,\ 6,\ 7,\ 8$ is untenable, drawing on the observation that $qa \leq 9$ to restrict possible choices for a. This will establish the first assertion of the theorem.

$\mathbf{q=2}$: Then a is one of $1, 2, 3, 4$. But * is even, so a is 2 or 4. For the units digit of * to be 2, d would have to be 1 or 6, but it is 4 or 5. For the units digit of * to be 4, d would have to be 2 or 7, but it is 8 or 9. Thus 2 is *never* a quotient of a reverse divisor.

$\mathbf{q=3}$: Then a is one of $1, 2, 3$. For the units digit of * to be 1, d would have to be 7, but it is one of $3, 4, 5$. For the units digit of * to be 2, d would have to be 4, but it is one of $6, 7, 8$. For the units digit of * to be 3, d would have to be 1, but it is 9. Thus 3 is *never* a quotient of a reverse divisor.

$\mathbf{q=5}$: Then a is 1, which is impossible for * is divisible by 5.

$\mathbf{q=6}$, $\mathbf{q=8}$: Then a is 1, which is impossible for * is even.

$\mathbf{q=7}$: Then a is 1. For the units digit of * to be 1, d would have to be 3, but it is one of $7, 8, 9$.

The numbers $\mathbf{10}99\ldots99\mathbf{89}$ & $\mathbf{21}99\ldots99\mathbf{78}$ are reverse divisors with respective quotients 9 and 4, since

$$9899\ldots9901 \;=\; 9 \times (\mathbf{10}99\ldots99\mathbf{89}) \quad \& \quad 8799\ldots9912 \;=\; 4 \times (\mathbf{21}99\ldots99\mathbf{78}).$$

Finally, we observe that none of the equations $9\times(ab) = ba$, $4\times(ab) = ba$, $9\times(abc) = cba$, $4\times(abc) = cba$ has a solution in which $a \neq 0$. Thus no reverse divisor has *fewer* than four digits. □

The *two* n-digit reverse divisors displayed in Theorem 1.1, from which all others can be constructed (Theorem 2.10), are called *basic* reverse divisors. For $n = 4, 5, 6, 7$, they are the *only* reverse divisors. For *all* higher values of n, there are non-basic reverse divisors – for example 10891089 and 108901089, both with quotient 9.

A *curious and interesting* cancellation property of basic reverse divisors is exhibited below:

$$\frac{1}{9} = \frac{1089}{9801} = \frac{10989}{98901} = \frac{109989}{989901} = \cdots \qquad \& \qquad \frac{1}{4} = \frac{2178}{8712} = \frac{21978}{87912} = \frac{219978}{879912} = \cdots .$$

Since the quotients 4 and 9 of reverse divisors are *squares*, reverse divisors enjoy the following property.

Corollary 1.2 The product of a reverse divisor and its reverse is a square. □

The question as to when the product of a natural number and its reverse is a *square* appears from time to time in recreational literature. Ogilvy and Anderson in their *Excursions in Number Theory* [8, pp. 88-89] mention that the product of a two-digit number and its reverse is *never* a square unless the number is palindromic, and remark that this is *not* the case for *three or more digit* numbers, citing $169 \times 961 = 403^2$ and $1089 \times 9801 = 3267^2$. Such examples led to the conjecture [9, p. 434] that, *when an integer and its reverse are unequal, their product can only be a square when both numbers are.* Corollary 1.2 provides an abundance of reverse-divisor counterexamples to the conjecture, the smallest being 2178.

2. Reverse divisors with quotient 9

We show that each reverse divisor with quotient 9 both begins and ends with the *same* basic reverse divisor with quotient 9. This leads us to a complete description of all reverse divisors with quotient 9 in terms of the basic ones.

Theorem 2.1 Let $ab\ldots cd$ $(a \neq 0)$ be a reverse divisor with quotient 9. Then it has the form $10\ldots89$.

Proof Since $9 \times (ab\ldots) = dc\ldots$, $a = 1$, $d = 9$ and b is 0 or 1. The case $b = 1$ would imply $c = 9$, and hence that $9 \times (\ldots 99) = \ldots 11$, which is false. Thus $b = 0$ and $9 \times (\ldots c9) = \ldots 01$, showing $c = 8$. □

Corollary 2.2 The only *four* digit reverse divisor with quotient 9 is 1089. □

Theorem 2.3 Let $10\,ab\ldots cd\,89$ be an n-digit ($n \geq 5$) reverse divisor with quotient 9. Then

$$9 \times (99\ldots99 \;-\; ab\ldots cd) \;=\; 99\ldots99 \;-\; dc\ldots ba,$$

where $99\ldots99$ is the $(n-4)$-digit number comprising solely of 9's.

Proof By Theorem 1.1, the n-digit ($n \geq 5$) number $1099\ldots9989$, all of whose $n-4$ central digits are 9's, is a reverse divisor with quotient 9, whence $9 \times (1099\ldots9989) = 98(99\ldots99)01$. By hypothesis, $9 \times (10ab\ldots cd89) = 98dc\ldots ba01$. Subtracting these equalities, then dividing by 100, gives the result. □

Theorem 2.3 can be used to show, even without a calculator, that the *only* 5-, 6 -, 7- digit reverse divisors with quotient 9 are the basic ones: 10989, 109989, 1099989. A routine computer search, using the theorem, shows that the only *non-basic* 8- and 9- digit reverse divisors with quotient 9 are 10891089 and 108901089. Thus the total number of reverse divisors with quotient 9 under a billion is eight.

Our study of the beginnings and endings of reverse divisors below culminates in key Theorem 2.8.

Theorem 2.4 A reverse divisor with quotient 9 ends either 089 or 989.

Proof A reverse divisor with quotient 9 that ends $b\,89\,(b > 0)$ begins $(98b\ldots) \div 9 = 109\ldots$. Hence $9 \times (\ldots b\,89) = \ldots 901$, which shows that $b = 9$. □

Theorem 2.5 A reverse divisor with quotient 9 ending 089 both begins and ends with 1089.

Proof A reverse divisor with quotient 9 that ends $b089$ begins $(980b\ldots) \div 9 = 108a\ldots$, where $a = 8$ if $b = 0$ but 9 otherwise. Also $9 \times (\ldots\, b089) = \ldots\, a801$, which shows that $9b$ has units digit a. Hence $b \neq 0$, so $a = 9$ and $b = 1$. □

For each non-negative integer r, denote by I_r resp. 0_r the string of r consecutive 9's resp. 0's and by S_r the basic reverse divisor $10I_r89$ with quotient 9. Thus S_0 is 1089 itself.

Lemma 2.6 A reverse divisor with quotient 9 ending 989 begins $10I_r8$ and ends $0I_r89$ for some $r \geq 1$.

Proof A reverse divisor with quotient 9 ending 989 must end in bI_r89 for some $b \neq 9$ and $r \geq 1$. It begins $(98I_rb\ldots) \div 9 = 10I_ra\ldots$, where $a = 8$ if $b = 0$, but 9 otherwise. Since $9 \times (\ldots\, bI_r89) = \ldots\, aI_r01$, a is the units digit of $9b + 8$, so $a \neq 9$, for $a = 9$ would imply $b = 9$. Thus $b = 0$ and $a = 8$. □

Theorem 2.7 A reverse divisor with quotient 9 ending 989 both begins and ends $10I_r89$ for some $r \geq 1$.

Proof By Lemma 2.6, a reverse divisor with quotient 9 ending 989, ends $b0I_r89$ for some b and $r \geq 1$. It begins $(98I_r0b\ldots) \div 9 = 10I_r8a\ldots$, where $a = 8$ if $b = 0$, but 9 otherwise. Since $9 \times (\ldots\, b0I_r89) = \ldots\, a8I_r01$, a is the units digit of $9b$, whence $b \neq 0$. Thus $a = 9$ and $b = 1$. □

Theorems 2.4, 2.5, 2.7 together yield Theorem 2.8 below, the pivotal result in decomposing a general reverse divisor with quotient 9 into basic ones.

Theorem 2.8 Each reverse divisor with quotient 9 both begins and ends with the *same* basic reverse divisor with quotient 9. □

Below we denote by Q^R the reverse of a string of digits Q. Thus the condition for such a Q, with leading digit non-zero, to be a reverse divisor with quotient 9 becomes $9 \times Q = Q^R$.

Lemma 2.9 Let M be a *non-basic* reverse divisor with quotient 9. Then $M = S_r0_sV0_sS_r$ for some r, $s \geq 0$, where V is *either* a reverse divisor with quotient 9 *or* 0_t for some $t \geq 0$.

Proof Since M is *non-basic*, Theorem 2.8 shows that $M = S_rPS_r$ for some $r \geq 0$ and some string of digits P. If P is empty or consists only of zeros, then M is in the desired form. Suppose, then, that P contains some non-zero digits.

Since S_rPS_r is a reverse divisor with quotient 9,

$$9 \times (S_rPS_r) = (S_rPS_r)^R = (S_r)^R P^R (S_r)^R = (9 \times S_r)P^R(9 \times S_r).$$

Comparing this result with that of the direct multiplication of $S_r P S_r$ by 9 shows that $9 \times P = P^R$. If P's first digit is non-zero, then P is a reverse divisor with quotient 9, and M is in the desired form. Suppose, then, that P has the form $0_s a \ldots$ for some $a, s > 0$. Then

$$9 \times (0_s a \ldots) = (0_s a \ldots)^R = \ldots a 0_s,$$

which shows that P has the form $0_s a \ldots\ b 0_s$ for some b, and that $9 \times (a \ldots\ b) = b \ldots\ a$. Thus $a \ldots b$ is a reverse divisor with quotient 9. Finally, we note that $M = S_r 0_s (a \ldots b)\, 0_s S_r$. ☐

We now arrive at our main result. It states, in essence, that *all* reverse divisors with quotient 9 can be formed from *basic* ones by concatenating these with strings of zeros in a symmetric and alternating way.

Theorem 2.10 (Structure theorem for reverse divisors with quotient 9) Non-basic reverse divisors with quotient 9 are precisely those natural numbers of the form

$$S_{a_1} 0_{b_1} S_{a_2} 0_{b_2} \ldots S_{a_n} 0_{b_n} V\, 0_{b_n} S_{a_n} \ldots 0_{b_2} S_{a_2} 0_{b_1} S_{a_1},$$

where *either* $V = S_{a_0}$ *or* 0_{b_0}, for some non-negative integers $a_0, a_1, \ldots, a_n$ and $b_0, b_1, \ldots, b_n$.

Proof Every number of the given form is certainly a non-basic reverse divisor with quotient 9. That every non-basic reverse divisor with quotient 9 has this form follows by repeated application of Lemma 2.9. ☐

The structure theorem enables us to write down at will all reverse divisors with quotient 9 having a given number of digits. Just for fun, we list all 19 that are under a trillion, **boldfacing** the basic ones.

1089 : **10989** : **109989** : **1099989** : 10891089 : **10999989** : 108901089 : **109999989**
1089001089 : 1098910989 : **1099999989** : 10890001089 : 10989010989 : **10999999989**
108900001089 : 108910891089 : 109890010989 : 109989109989 : **109999999989**

Postscript All the results and proofs in this section generalize to the quotient 4 case [10].

CHALLENGE: What is the thirty-ninth reverse divisor?

(retsbeW regoR) **sgnittoJ .3**

These jottings were intended for my co-author, *not* for publication. The editor, however, chanced upon them and was moved to comment ... '*I would like to see the extra, more informal section included. It helps tone down the formal presentation and stimulate readers to think for themselves.*' Who am I to disagree?

1. My interest in *reverse divisors* stems from 11's remarkable property that whenever it divides a number, it divides its reverse. This helped me solve Victor Bryant's *Sunday Times* (13 March 2011) TEASER 2529:

A letter to *The Times* concerning the inflated costs of projects read: '*When I was a financial controller, I found that multiplying original cost estimates by pi used to give an excellent indication of the final outcome*.' Interestingly, I used this process (using 22/7 as an approximation for pi). On one occasion, the original estimate was a whole number of pounds (under 100,000), and this method for the probable final outcome gave a number of pounds consisting of the same digits, but in reverse order. What was the original estimate?

The problem is to express 22/7 as the quotient of a number and its reverse, both under 100,000. See [10]. This led to the question: What numbers can be written as the ratio of a natural number to its reverse?

2. The only numbers *I* know with the property that, if they divide a number, they divide its reverse are $1, 3, 9, 11, 33, 99$. Are there others? If so, they must be palindromic or reverse divisors. Hence this study!

3. The fractions (of David Huitson) below, in which the denominator is the reverse of the numerator, exhibit the same *curious and interesting* cancellation property that we saw earlier for basic reverse divisors.

$$\frac{2}{3} = \frac{4356}{6534} = \frac{43956}{65934} = \frac{439956}{659934} = \cdots \quad \& \quad \frac{3}{8} = \frac{27}{72} = \frac{297}{792} = \frac{2997}{7992} = \cdots .$$

Scope for pleasurable investigations! The second example is a special case of the rule: if $a + b = 9$, then

$$\frac{ab}{ba} = \frac{a9b}{b9a} = \frac{a99b}{b99a} = \frac{a999b}{b999a} = \cdots .$$

4. Is it *obvious* why the basic reverse divisor 2178 should be twice the basic reverse divisor 1089? Does this occur in other bases?

5. What of reverse divisors in non-decimal bases? There are *none* in binary. In base b ($b \geq 3$), the four digit number $1\,0\,(b-2)\,(b-1)$ is a reverse divisor, quotient $b-1$. Thus reverse divisors in ternary (quaternary, quinary) are 1012 (1023, 1034), quotients 2 (3, 4). In quinary, 143 is a reverse divisor, quotient 2.

Hot off the Press: 1089 trick holds in base b with denouement **reverse divisor** $1\,0\,(b-2)\,(b-1)$!

6. The links below (found by David Cuddington) touch on reverse divisors, but are meagrely presented.

http://mathworld.wolfram.com/Reversal.html and **http://oeis.org/A008919**.

7. Students seem much taken with reverse divisors. They eagerly track them down on calculators and computers, make wild guesses, build large ones from small ones, and research them online. Suitable topic for the classroom and undergraduate projects, with conjectures, discoveries, calculations, and wall charts!

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Roger Webster *lectures on the history of mathematics at Sheffield University. As a twelve-year old, his party piece was to perform an £ s d version of the 1089 trick!* Even *today, he remembers well that final answer: £12 18s 11d.*

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