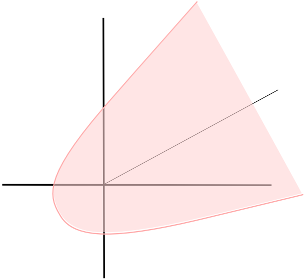
The Parabola Theorem on continued fractions

Ian Short



7 September 2009

A classical theorem

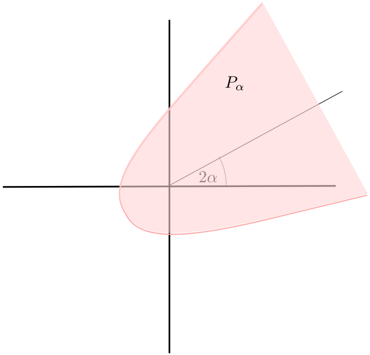
Continued fractions

$$\mathbf{K}(a_n|\,1) = \cfrac{a_1}{1+\cfrac{a_2}{1+\cfrac{a_3}{1+\cfrac{a_4}{1+\cdots}}}}$$

Continued fraction convergence

$$\frac{a_1}{1}, \quad \frac{a_1}{1+\dfrac{a_2}{1}}, \quad \frac{a_1}{1+\dfrac{a_2}{1+\dfrac{a_3}{1}}}, \ldots$$

Parabolic region



The Parabola Theorem

Suppose that $a_n \in P_\alpha$ for $n = 1, 2, \ldots$.

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Suppose that $a_n \in P_\alpha$ for $n = 1, 2, \ldots$. Then $\mathbf{K}(a_n | 1)$ converges if and only if the series

$$\left|\frac{1}{a_1}\right| + \left|\frac{a_1}{a_2}\right| + \left|\frac{a_2}{a_1 a_3}\right| + \left|\frac{a_1 a_3}{a_2 a_4}\right| + \left|\frac{a_2 a_4}{a_1 a_3 a_5}\right| + \cdots$$

diverges.

Understanding the theorem

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What is the significance of the parabolic region?

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What is the signifance of the series?

Years of confusion

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Split the theorem in two.

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Theorem involving the parabolic region.

Years of confusion

Split the theorem in two.

Theorem involving the parabolic region.

Theorem involving the Stern–Stolz series.

The parabolic region

Möbius transformations

$$t_n(z) = \frac{a_n}{1+z}$$

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$$T_n = t_1 \circ t_2 \circ \cdots \circ t_n$$

Convergence using Möbius transformations

The continued fraction $\mathbf{K}(a_n|\,1)$ converges if and only if $T_1(0), T_2(0), T_3(0), \dots$ converges.

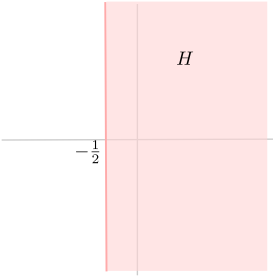
Question

What does the condition $a \in P_\alpha$ signify for the map $t(z) = a/(1+z)$?

Answer

The coefficient a belongs to P_α if and only if t maps a half-plane H_α within itself.

Proof $(\alpha = 0)$



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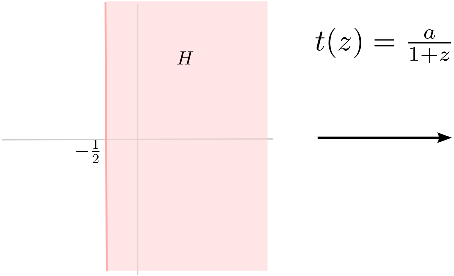

$t(z) = \frac{a}{1+z}$

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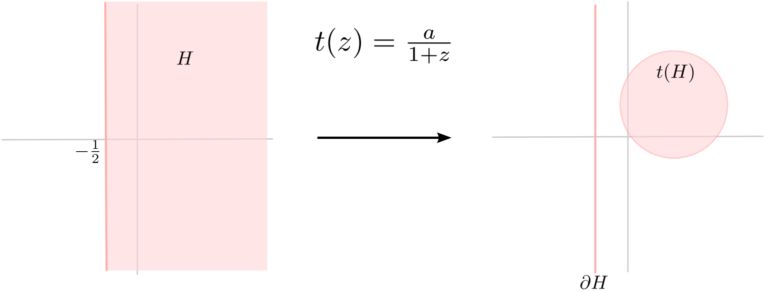

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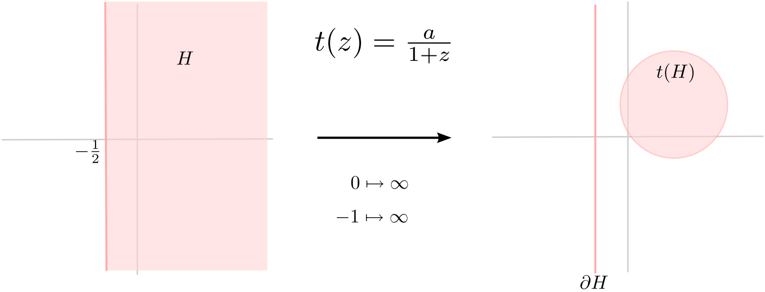

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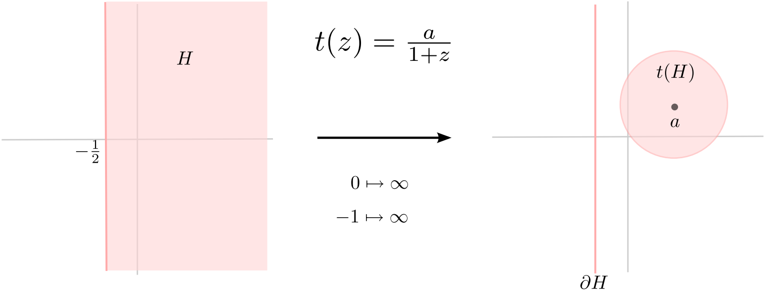

Proof ($\alpha = 0$)

$$t(H) \subset H \iff |a - 0| \leq |a - \partial H|$$

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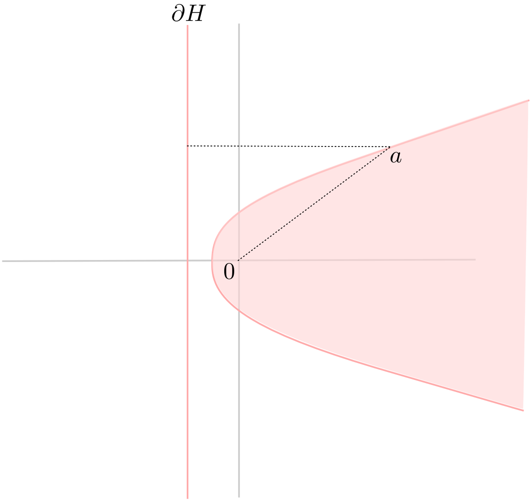
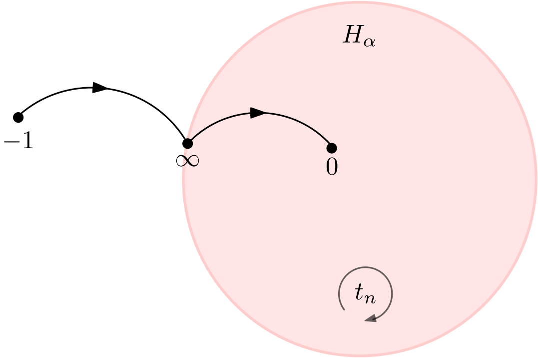


Parabola $|a - 0| = |a - \partial H|$.

Original condition

$$a_n \in P_\alpha$$

New condition



New condition

$$t_n(-1) = \infty$$

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Does $T_n = t_1 \circ t_2 \circ \cdots \circ t_n$ converge at 0?

Extensive literature

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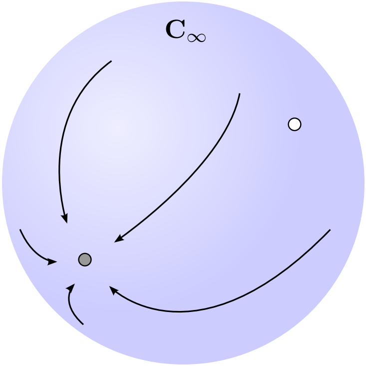
Lorentzen (*Ramanujan J.*, 2007)

Conclusion

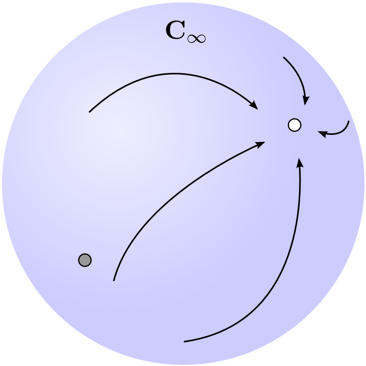
If $a_n \in P_\alpha$ then there are points p and q in H_α such that T_{2n-1} converges on H_α to p, and T_{2n} converges on H_α to q.

Divergence



Action of T_{2n-1}

Divergence



Action of T_{2n}

The Stern–Stolz series

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Suppose $a_n \in P_\alpha$.

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The Stern–Stolz series

$$\left|\frac{1}{a_1}\right| + \left|\frac{a_1}{a_2}\right| + \left|\frac{a_2}{a_1 a_3}\right| + \left|\frac{a_1 a_3}{a_2 a_4}\right| + \left|\frac{a_2 a_4}{a_1 a_3 a_5}\right| + \cdots$$

Convergence of the Stern–Stolz series

$$t_n(z) = \frac{a_n}{1+z} \qquad \sim \qquad s_n(z) = \frac{a_n}{z}$$

Möbius transformations

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$$S_n = s_1 \circ s_2 \circ \cdots \circ s_n$$

Convergence of the Stern–Stolz series

Is $S_n \sim T_n$?

Chordal metric

Let χ denote the chordal metric.

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χ_0 is right-invariant.
$(\mathcal{M}, \chi_0)$ is a topological group.
$(\mathcal{M}, \chi_0)$ is a complete metric space.

The Stern–Stolz series

$$\mu_1 = \frac{1}{a_1} \qquad \mu_2 = \frac{a_2}{a_1} \qquad \mu_3 = \frac{a_2}{a_1 a_3} \ldots$$

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$$|\mu_1| + |\mu_2| + |\mu_3| + \cdots$$

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$$S_{2n-1}(z) = \frac{1}{\mu_{2n-1} z} \qquad S_{2n}(z) = \mu_{2n} z$$

Conjugation

$$\mu z \circ (1 + z) \circ \mu^{-1} z = \mu + z$$

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$$S_{2n-1} \circ (1+z) \circ S_{2n-1}^{-1}(z) = \sigma \circ (\mu_{2n-1} + z) \circ \sigma$$

Calculation

$$\chi_0(S_nT_n^{-1}, S_{n-1}T_{n-1}^{-1})$$

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Calculation

$$\begin{aligned}\chi_0(S_nT_n^{-1}, S_{n-1}T_{n-1}^{-1}) &= \chi_0(S_nt_n^{-1}, S_{n-1})\\ &= \chi_0(I, S_{n-1}t_nS_n^{-1})\\ &= \chi_0(I, S_n \circ (1+z) \circ S_n^{-1})\end{aligned}$$

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&= \chi_0(I, \mu_n + z) \\
&\sim |\mu_n|
\end{aligned}
$$

Summary

$$\left|\frac{1}{a_1}\right| + \left|\frac{a_1}{a_2}\right| + \left|\frac{a_2}{a_1 a_3}\right| + \left|\frac{a_1 a_3}{a_2 a_4}\right| + \left|\frac{a_2 a_4}{a_1 a_3 a_5}\right| + \cdots < +\infty$$

if and only if

$$\sum_n \chi_0(S_n T_n^{-1}, S_{n-1} T_{n-1}^{-1}) < +\infty$$

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Let $h = g^{-1}$.

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There exists Möbius g such that $\chi_0(S_n T_n^{-1}, g) \to 0$.

Let $h = g^{-1}$.

$$\chi_0(T_n, hS_n) \to 0$$

Oscillation

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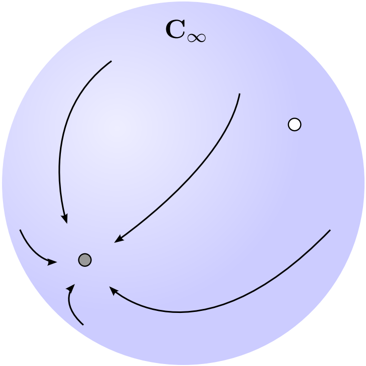
$$S_{2n-1}(z) = \frac{1}{\mu_{2n-1} z} \qquad S_{2n}(z) = \mu_{2n} z$$

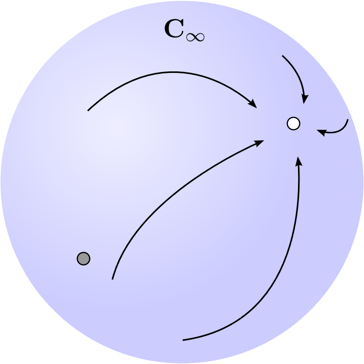
So if $T_n \sim h S_n$ then $T_{2n-1} \to h(\infty)$ and $T_{2n} \to h(0)$.

Oscillation



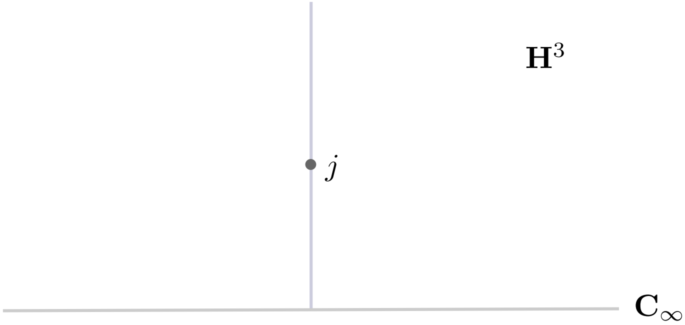
Action of T_{2n-1}

Oscillation



Action of T_{2n}

Hyperbolic space



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If $|\mu_n| < 1$ then

$$\exp\left[-\rho(j, S_n^{-1}(j))\right]$$

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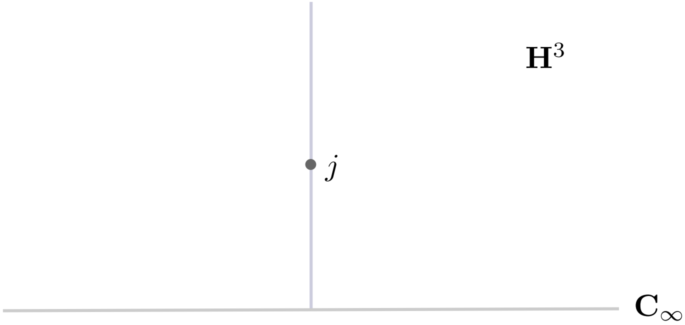
Dynamics of S_n in hyperbolic space



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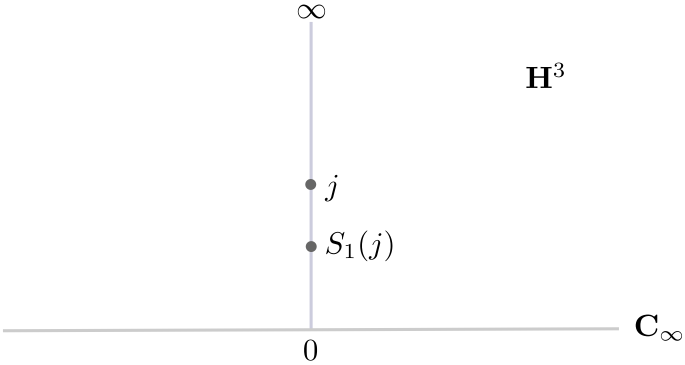

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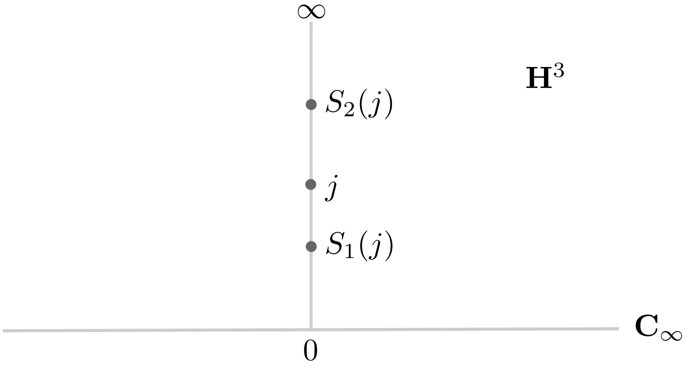

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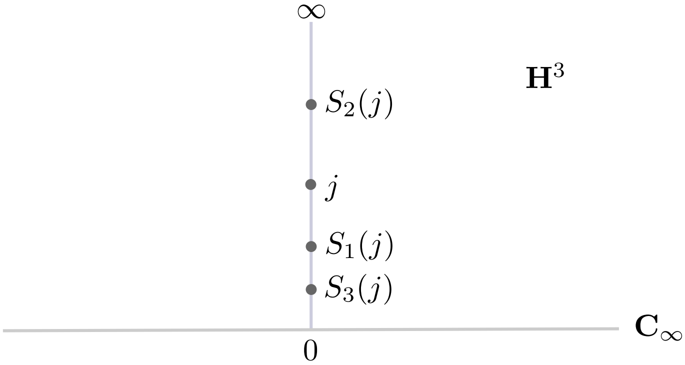

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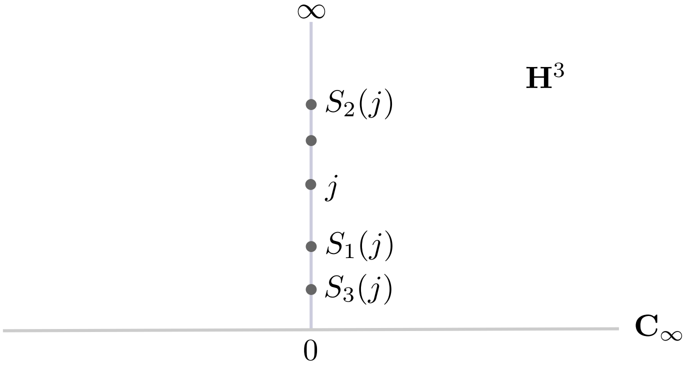

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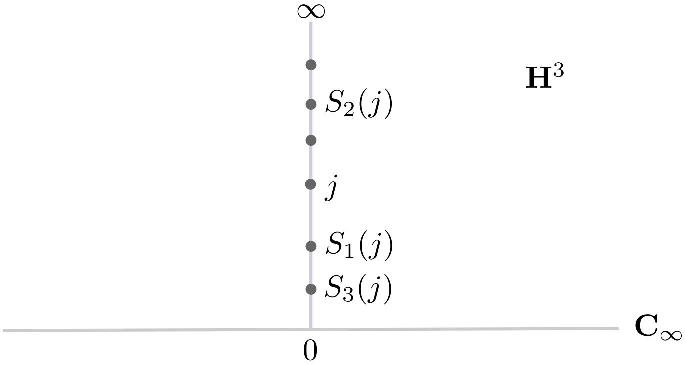

Equivalent conditions

$$\left|\frac{1}{a_1}\right| + \left|\frac{a_1}{a_2}\right| + \left|\frac{a_2}{a_1a_3}\right| + \left|\frac{a_1a_3}{a_2a_4}\right| + \left|\frac{a_2a_4}{a_1a_3a_5}\right| + \cdots < +\infty$$

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Thank you!