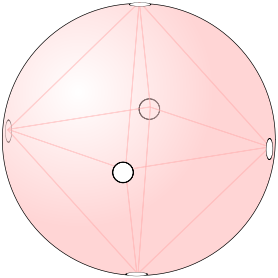
Conformal symmetries of planar regions I

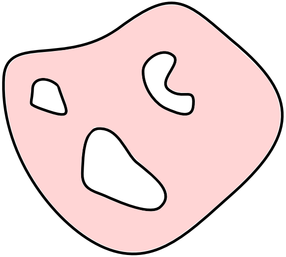
Ian Short



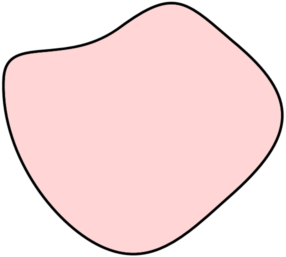
Tuesday 2 November 2010

Introduction

Regions

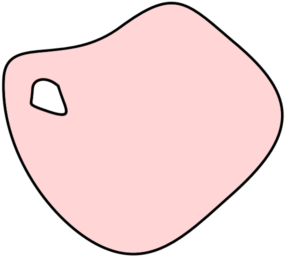


Connectivity



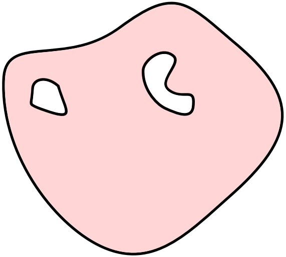
1

Connectivity



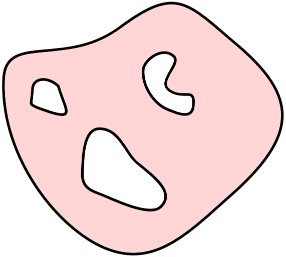
2

Connectivity



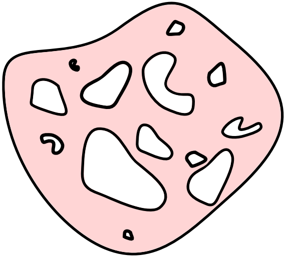
3

Connectivity



4

Connectivity



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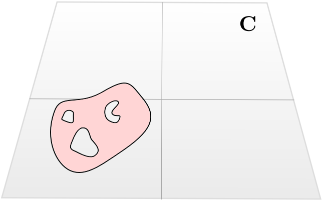
Stereographic projection



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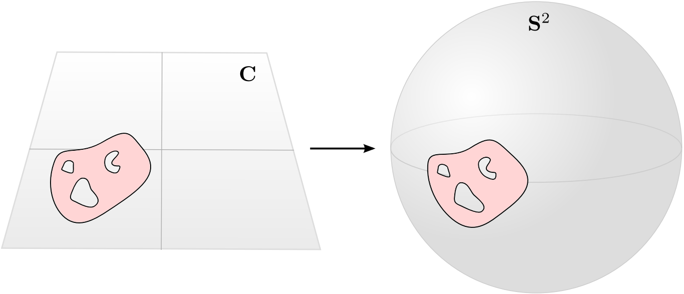
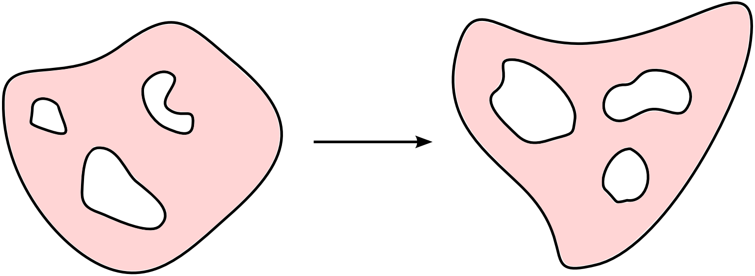
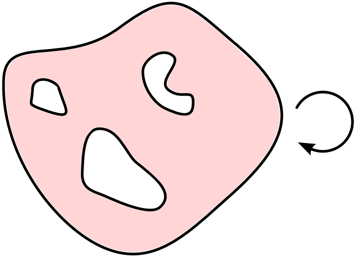


Conformal maps



Conformal symmetries



Two questions

Question 1. What are the conformal maps from one region to another?

Question 2. Which groups arise as conformal symmetry groups?

Two questions

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Question 2. Which groups arise as conformal symmetry groups?

Selected publications

On directly conformal maps

Maurice Heins

Bulletin of the American Mathematical Society, 1946

Conformal automorphisms of finitely connected regions

Alan Beardon and David Minda

London Mathematical Society Lecture Note Series 348, 2008

Rigidity of configuration of balls and points in the N-sphere

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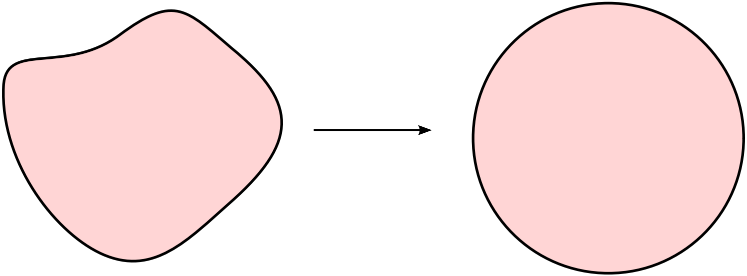
The Riemann mapping theorem



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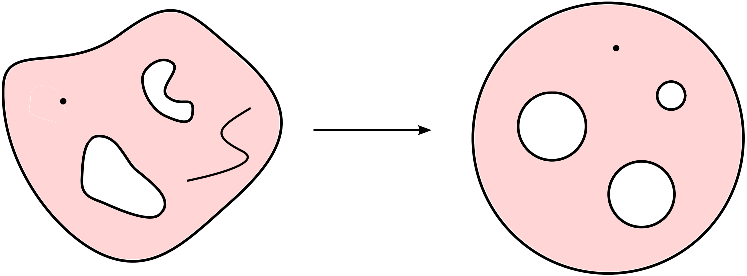


Koebe's generalisation

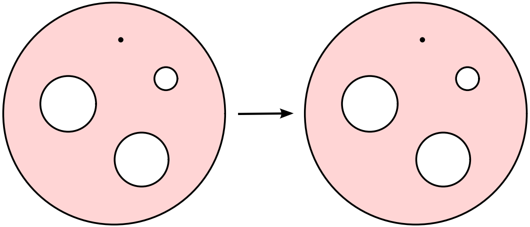
Publication

Fixed points, Koebe uniformization and circle packings

Zheng-Xu He and Oded Schramm

The Annals of Mathematics, 1993

Schwarz reflection principle and Möbius maps



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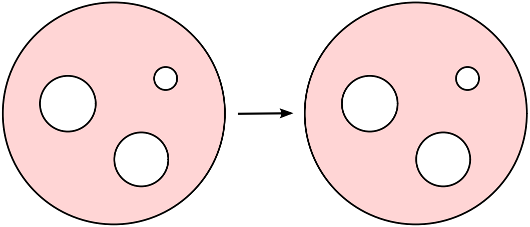

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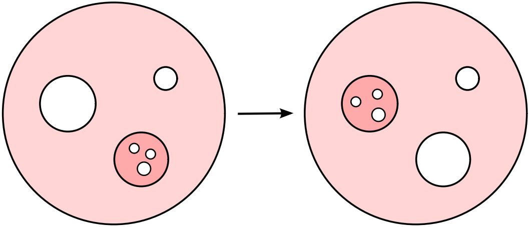

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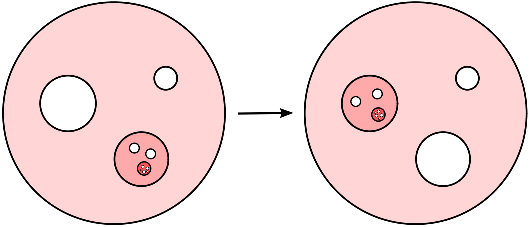

Möbius maps

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Circular regions

Two questions

Question 1. Is there a conformal map from one region to another?

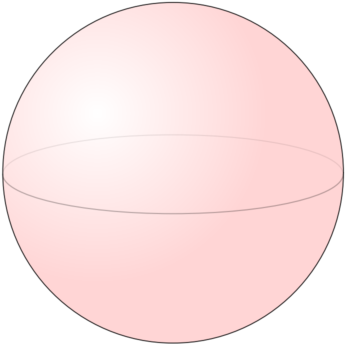
Question 2. Which groups arise as conformal symmetry groups?

Two questions

Question 1. Is there a conformal map from one region to another?

Question 2. Which groups arise as conformal symmetry groups *of finitely connected regions*?

Connectivity 0



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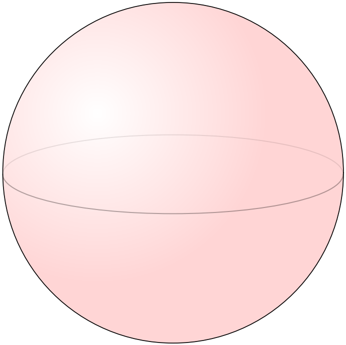
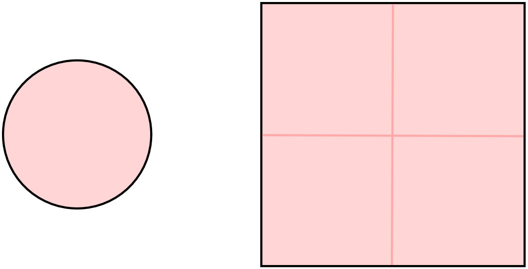
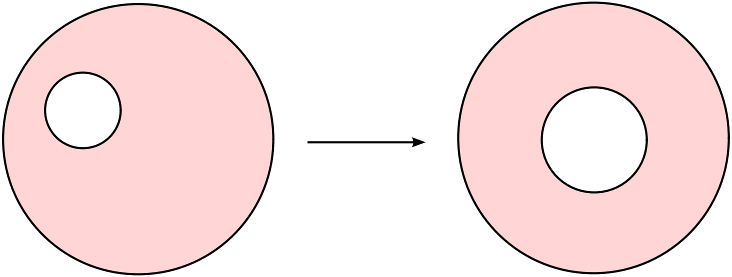


Conformal symmetry group is $\mathcal{M}$.

Connectivity 1



Connectivity 2



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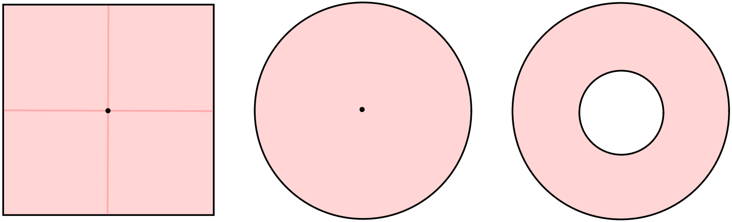
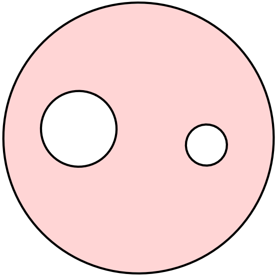


Connectivity 3+



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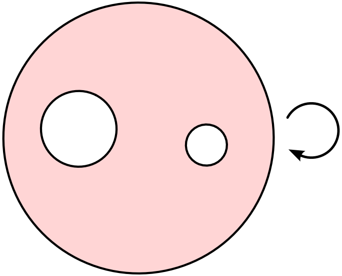
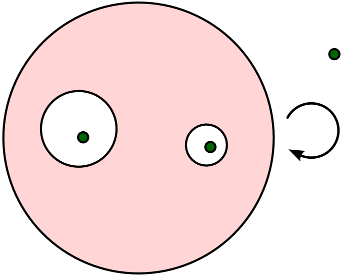


Connectivity 3+



Connectivity 3+

Conformal symmetry group is finite.

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Which finite groups arise as conformal symmetry groups?

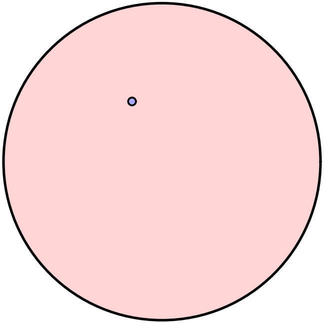
Connectivity 3+

Conformal symmetry group is finite.

Which finite groups arise as conformal symmetry groups?

Which finite groups arise as subgroups of the Möbius group?

Finite groups of Möbius transformations



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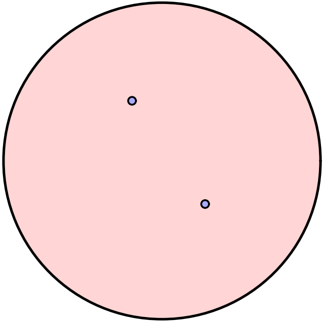

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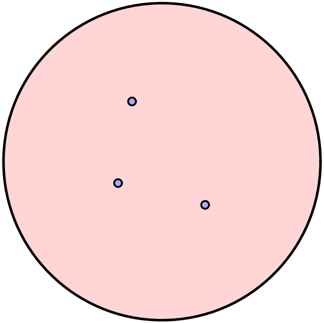

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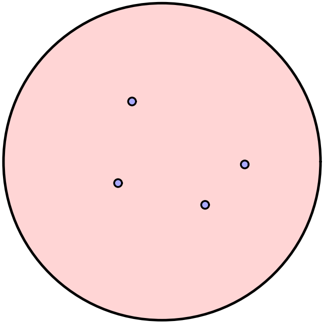

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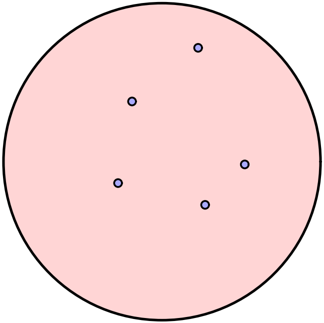

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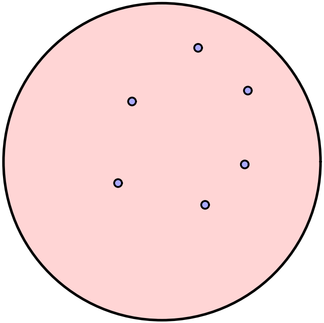

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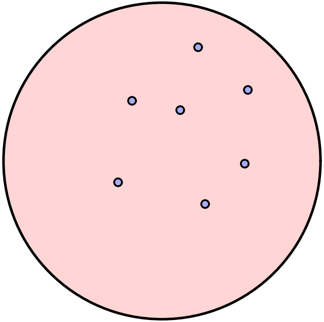

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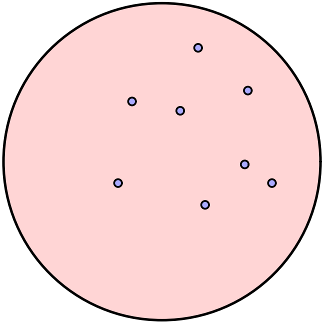

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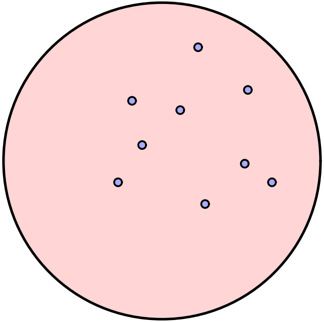

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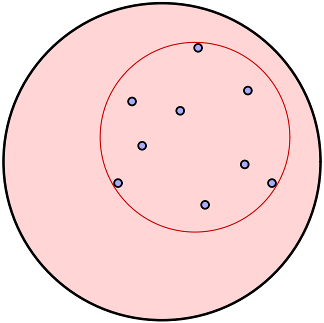

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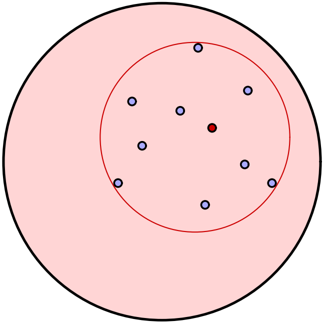

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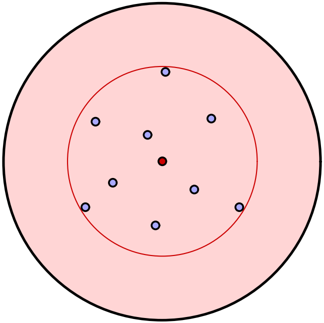
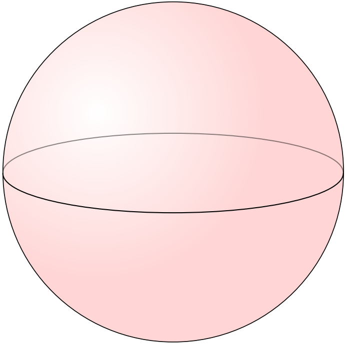


Finite orthogonal groups



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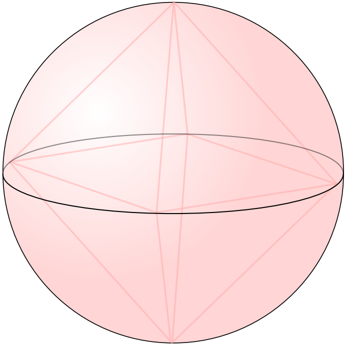

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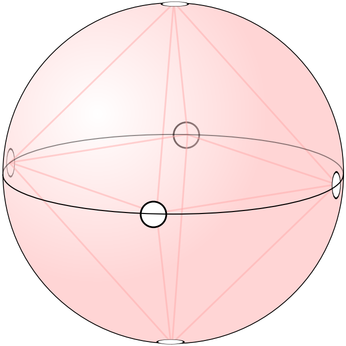

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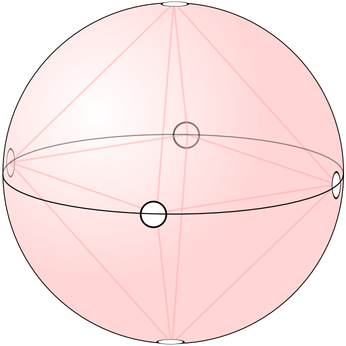

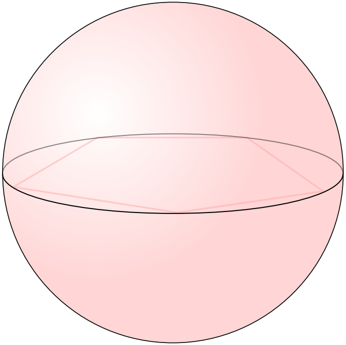

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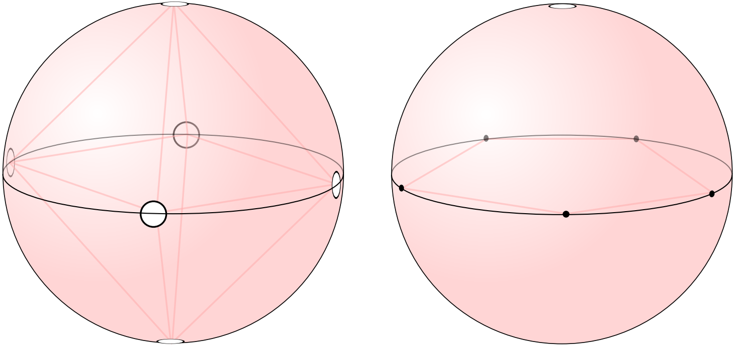

Conformal symmetry groups of finitely connected regions

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Theorem (Heins, 1946).

Conformal symmetry groups of finitely connected regions

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Conformal symmetry groups of finitely connected regions

Theorem (Heins, 1946). The only groups that arise as conformal symmetry groups of finitely connected regions of connectivity at least three are A_4, S_4, A_5, C_n, and D_n, for $n = 1, 2, \ldots$.

Hyperbolic geometry

The real Möbius group

$$x \mapsto \frac{ax+b}{cx+d} \qquad a, b, c, d \in \mathbb{R} \qquad ad - bc = 1$$

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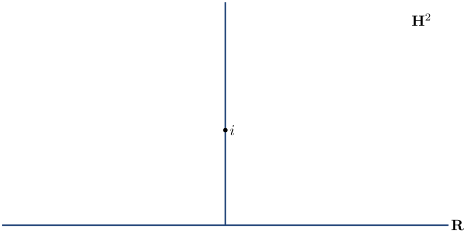

Two-dimensional hyperbolic geometry

$$z = x + iy \qquad ds = \frac{|dz|}{y}$$

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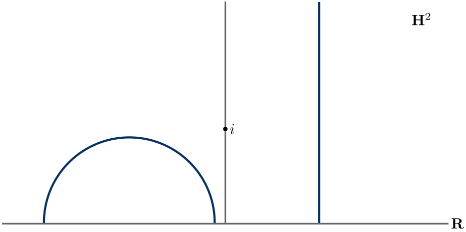

The complex Möbius group

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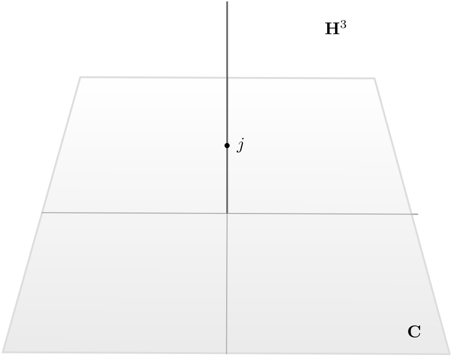

Action on $\mathbb{H}^3$

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$$\zeta = z + tj \qquad t > 0$$

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inversion in unit sphere: $\zeta \mapsto \bar{\zeta}^{-1}$

Three-dimensional hyperbolic geometry

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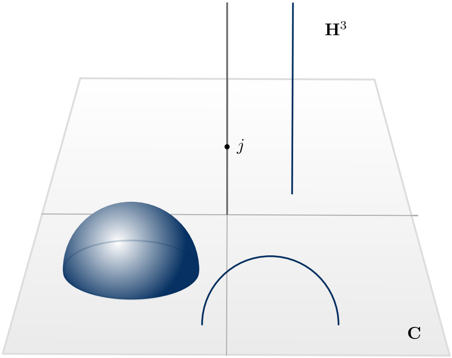

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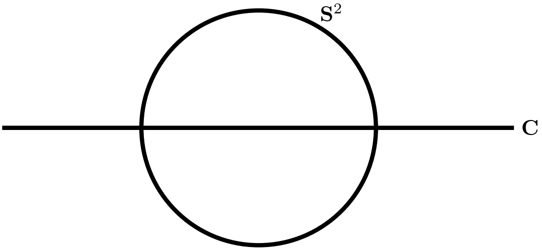

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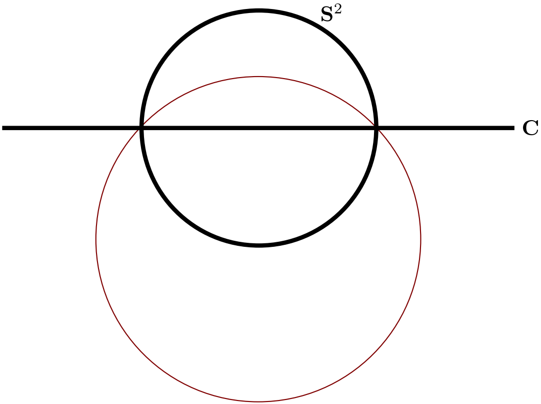

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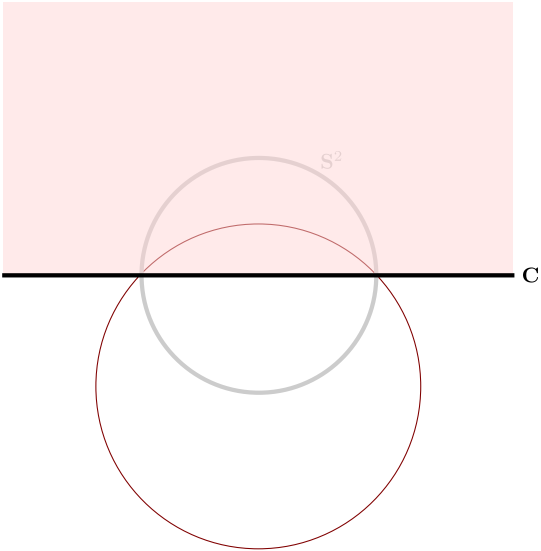

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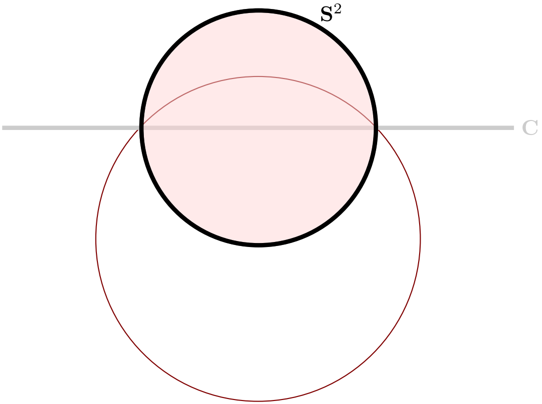
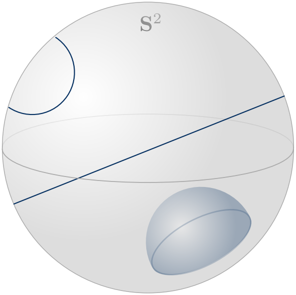


Ball model of hyperbolic space



Conformal symmetries

Two questions

Question 1. What are the conformal maps from one region to another?

Question 2. Which groups arise as conformal symmetry groups?

Punctured spheres

Cross-ratios

$$[a, b, c, d] = \left| \frac{(a-b)(c-d)}{(a-c)(b-d)} \right|$$

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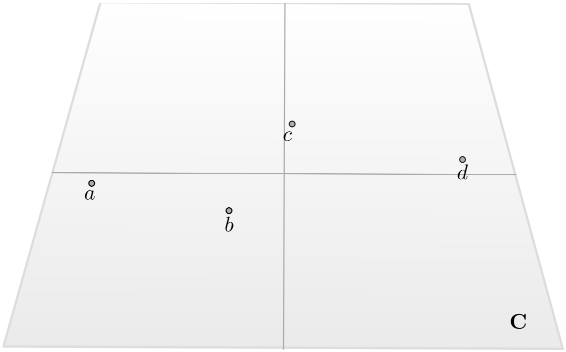

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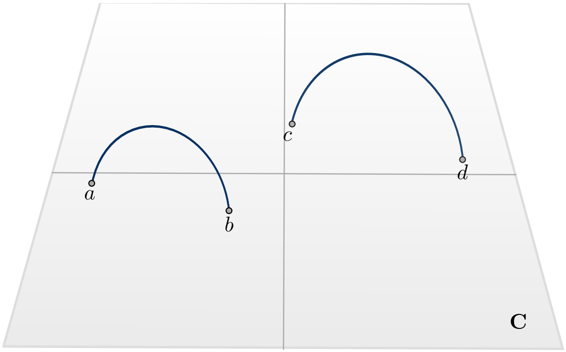

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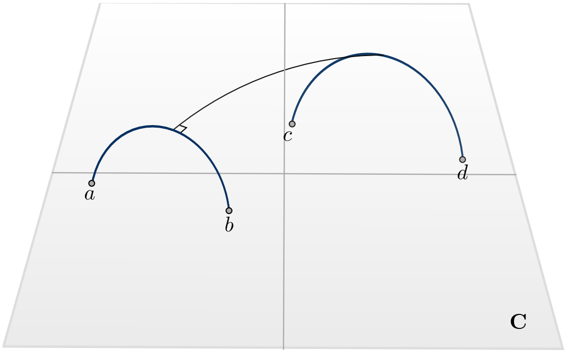
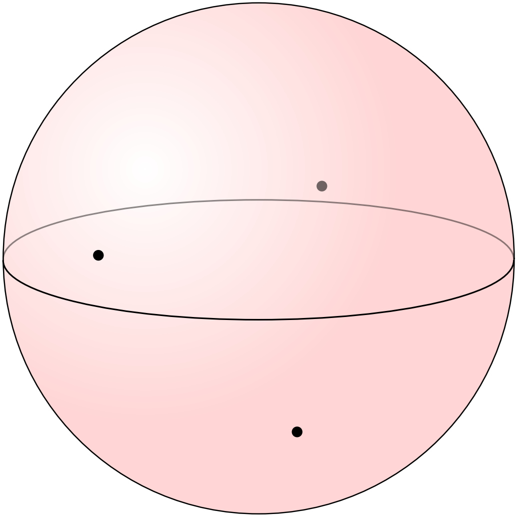
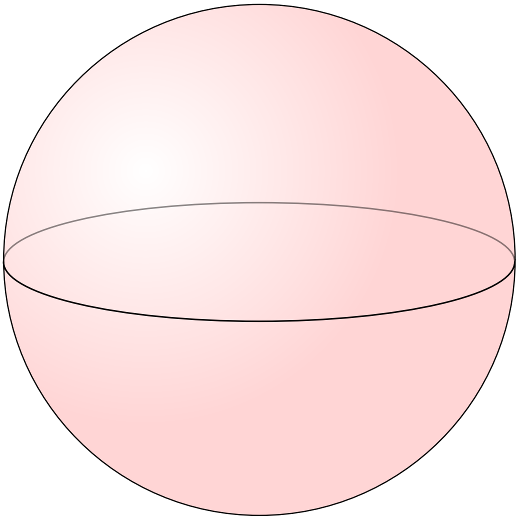


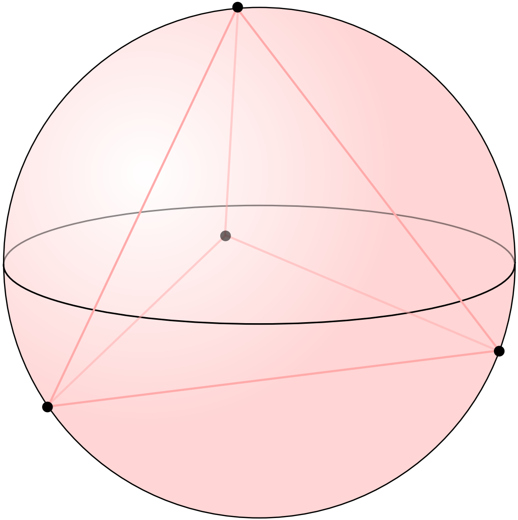
Three punctured sphere



Four punctured sphere



Four punctured sphere



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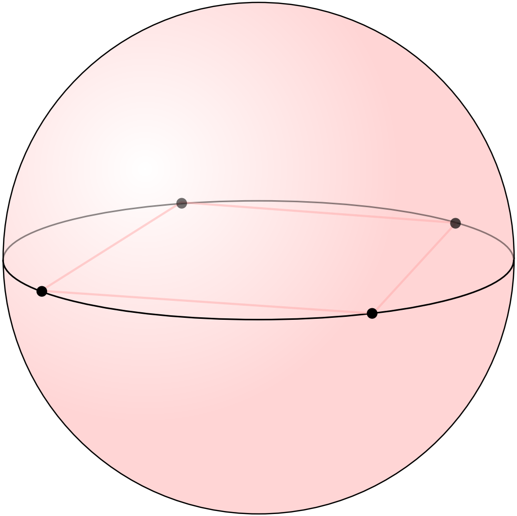

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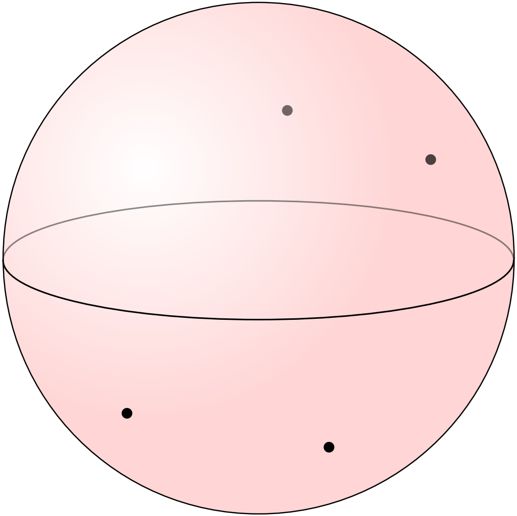

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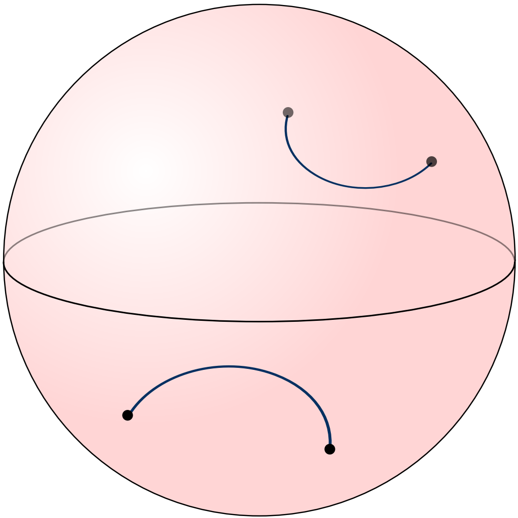

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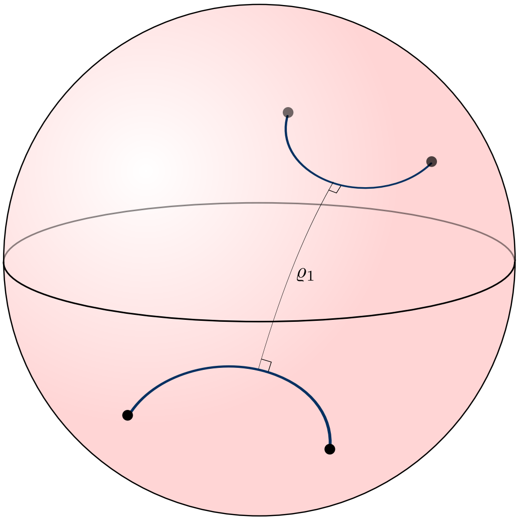

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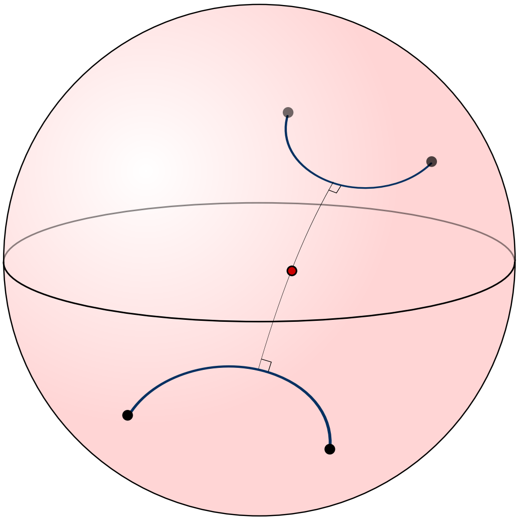

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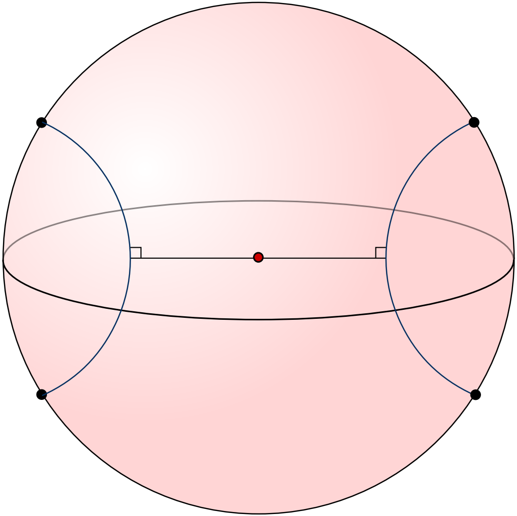

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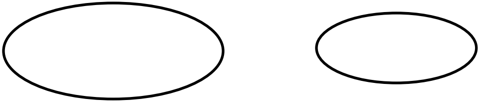
Circular regions

Cross-ratio for circles?



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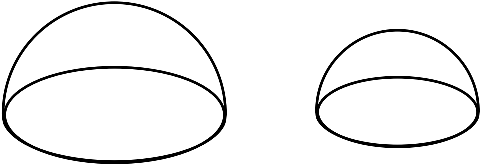

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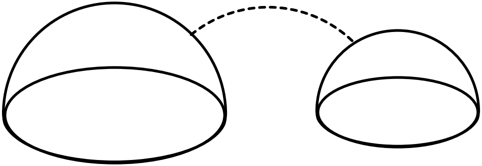
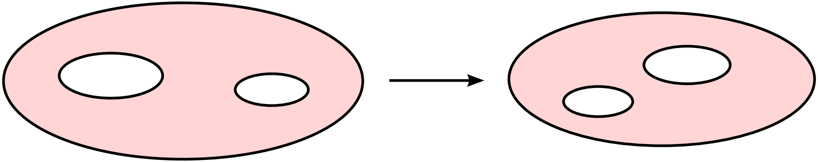


Connectivity 3



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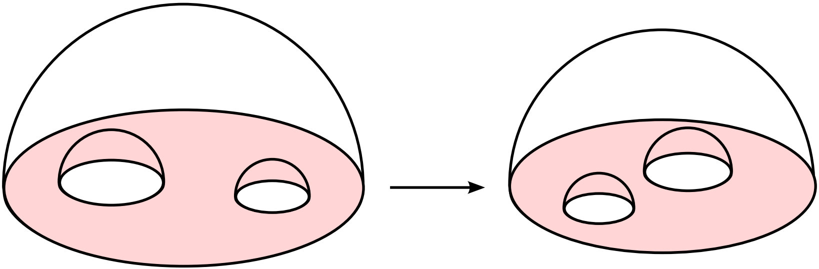

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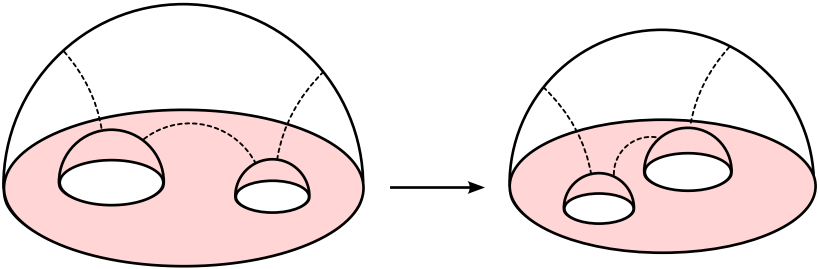

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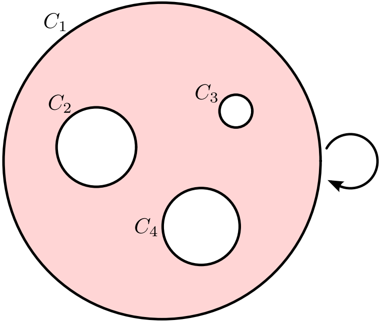

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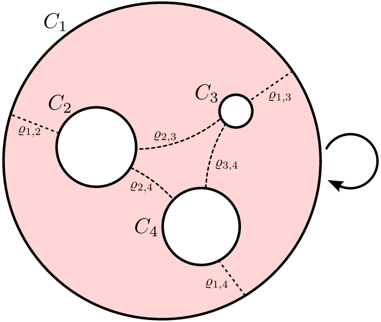

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