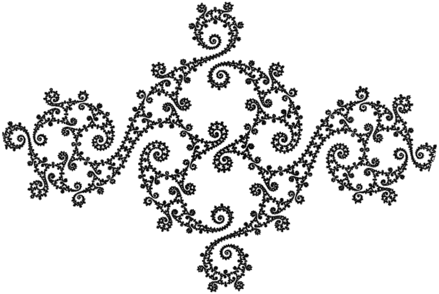
Conformal symmetries of planar regions II

Ian Short



Tuesday 2 November 2010

Recap

Two questions

Question 1. What are the conformal maps from one region to another?

Question 2. Which groups arise as conformal symmetry groups?

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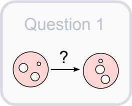

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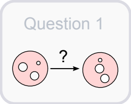

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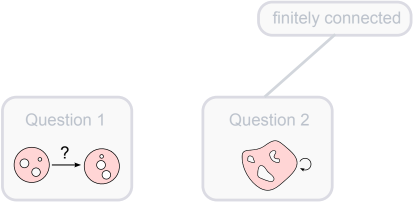

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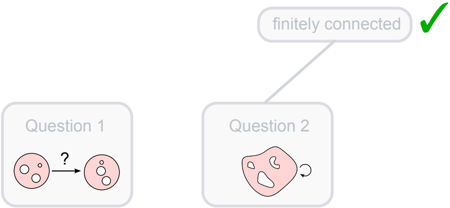

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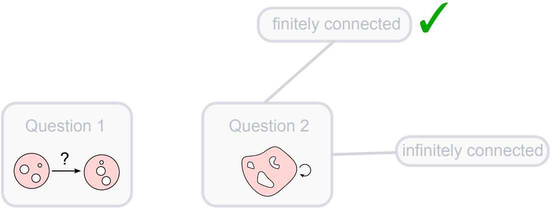
finitely connected
Question 1
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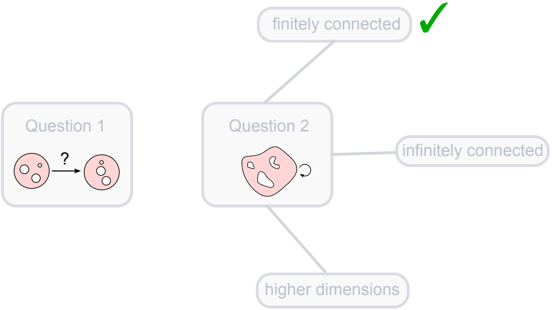
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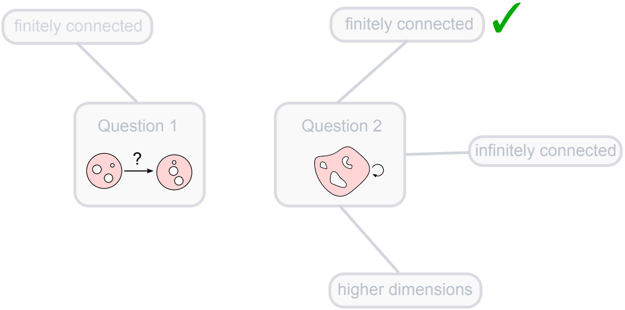
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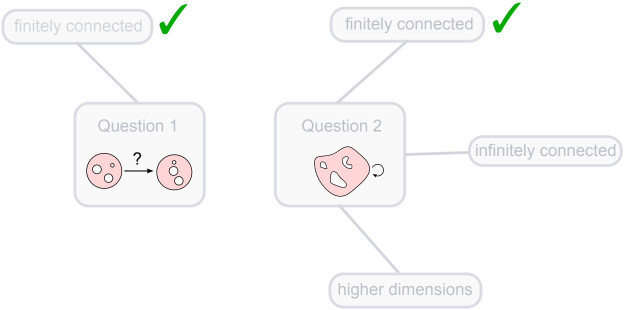
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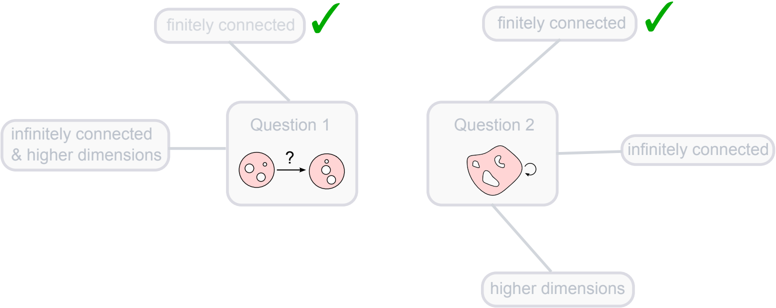
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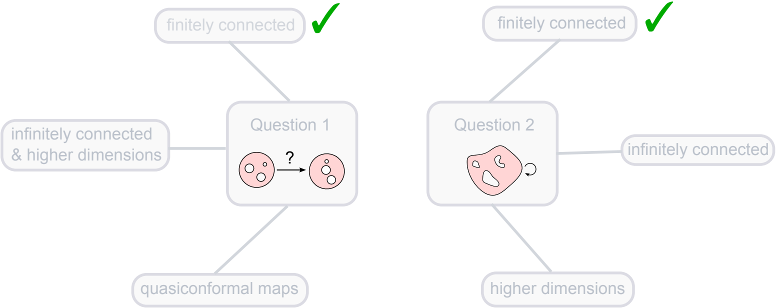
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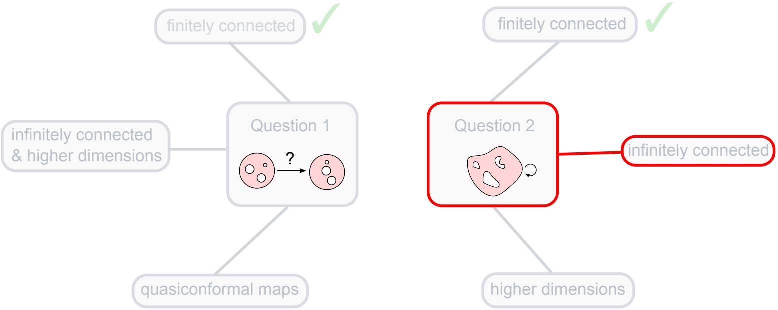
Infinite connectivity

Plan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Möbius group

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Coincides with the topology of uniform convergence.

Maskit's Theorem

Definition. Let $\text{Aut}^+(D)$ denote the group of conformal symmetries of a region D.

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Theorem (Maskit, 1968). Each region in $\mathbb{C}_\infty$ is conformally equivalent to a region D for which $\mathrm{Aut}^+(D)$ is a subgroup of $\mathcal{M}$.

Conformal symmetry groups are closed

Suppose henceforth that $\mathrm{Aut}^+(D) \leqslant \mathcal{M}$.

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Lemma. $\mathrm{Aut}^+(D)$ is closed in $\mathcal{M}$.

Conformal symmetry groups are discrete

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Theorem. If D is a region of connectivity at least three then $\mathrm{Aut}^+(D)$ is discrete.

Sketch proof that $\mathrm{Aut}^+(D)$ is discrete I

Let $G = \mathrm{Aut}^+(D)$.

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If $G_I = \{I\}$ then G is discrete.

Otherwise G_I contains a one-parameter subgroup of $\mathcal{M}$.

Sketch proof that $\mathrm{Aut}^+(D)$ is discrete II

The one-parameter subgroups in $\mathrm{SL}(2,\mathbb{C})$ are $t \mapsto \exp(tA)$ for A in $\mathfrak{sl}(2,\mathbb{C})$.

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$$\begin{pmatrix} \lambda & 0 \\ 0 & -\lambda \end{pmatrix}, \qquad \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

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Hence, up to conjugation, the one-parameter subgroups in $\mathcal{M}$ are

$$z \mapsto e^{\lambda t} z, \qquad z \mapsto z + t, \qquad t \in \mathbb{R}.$$

Sketch proof that $\mathrm{Aut}^+(D)$ is discrete III

G_I contains either $z \mapsto e^{\lambda t} z$ or $z \mapsto z + t$.

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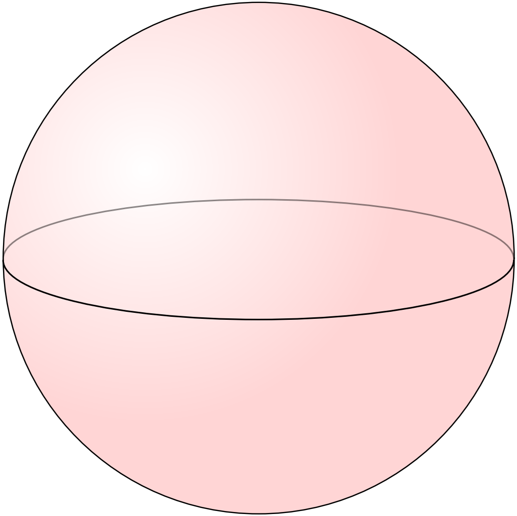
When λ purely imaginary get annuli (at most two components).

Limit sets



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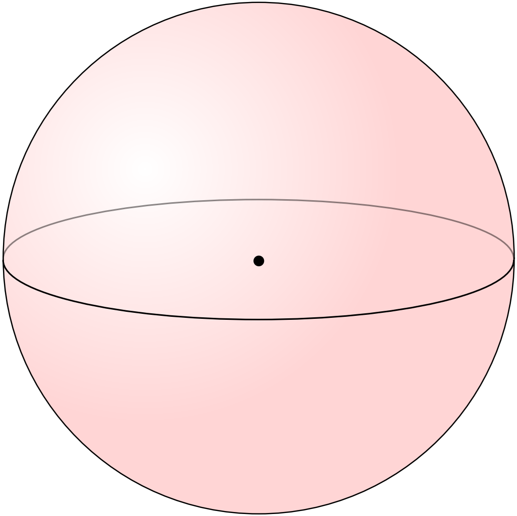

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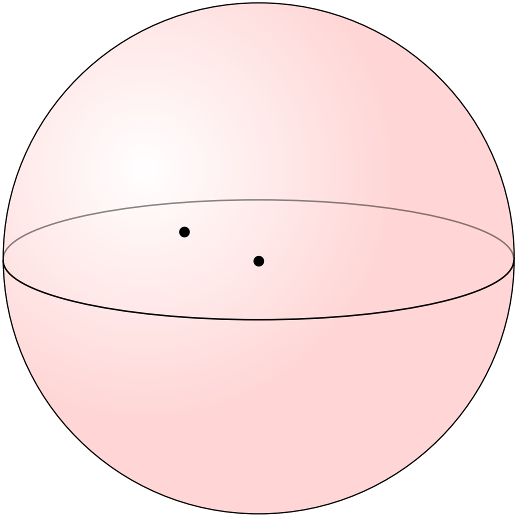

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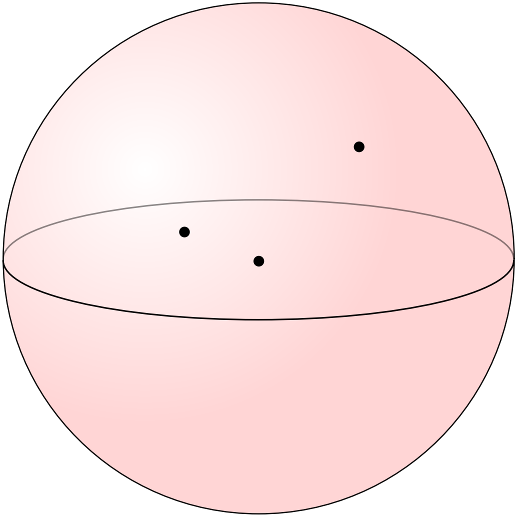

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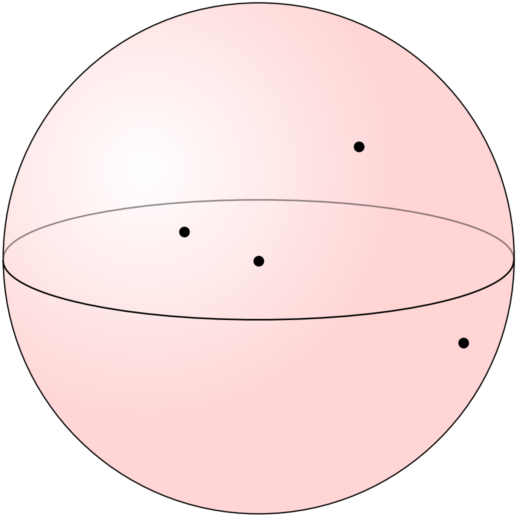

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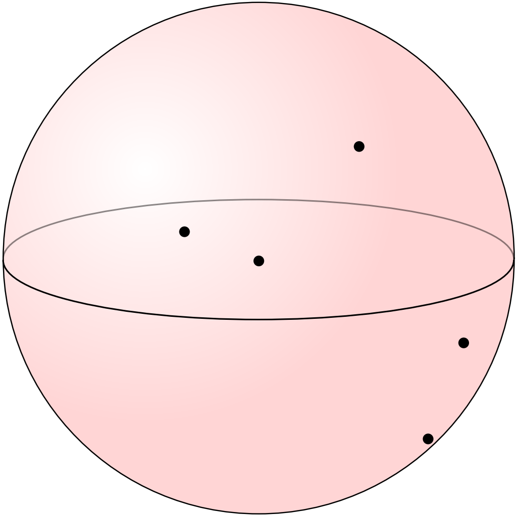

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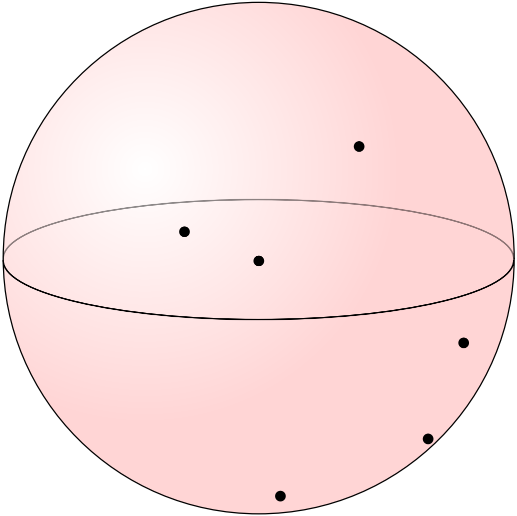

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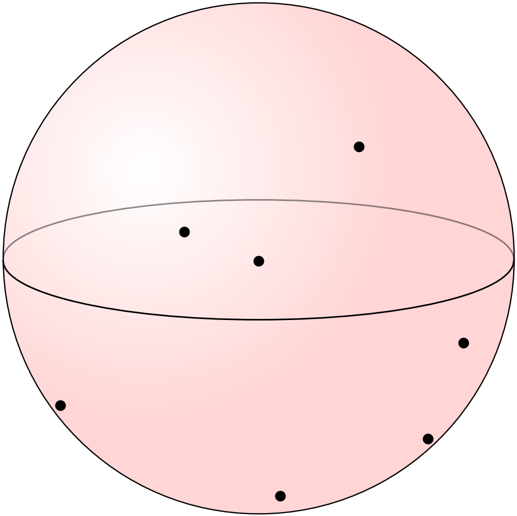

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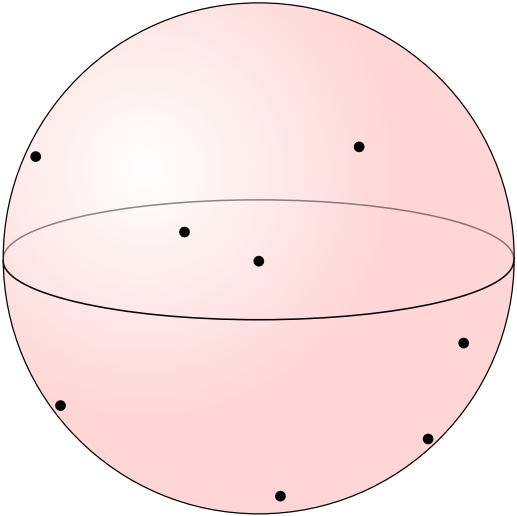

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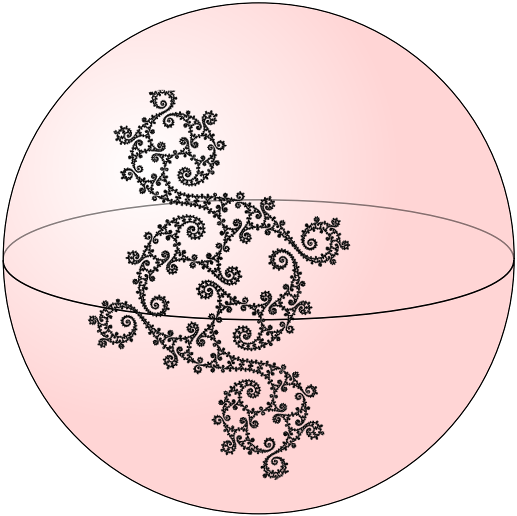

Properties of limit sets

Λ is closed

Properties of limit sets

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Λ is invariant under G

Properties of limit sets

- Λ is closed
- Λ is invariant under G
- Λ is the *smallest* closed set invariant under G

Size of limit sets

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$|\Lambda|$ uncountable

Punctured spheres of countable connectivity

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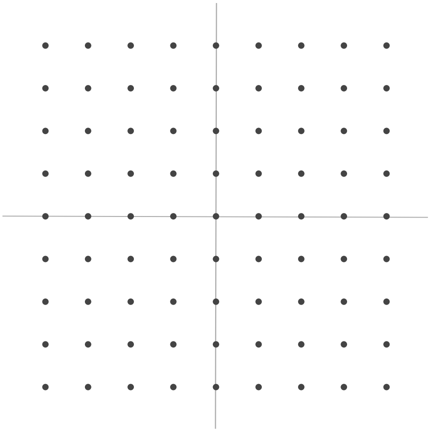
Finite punctures $\longrightarrow$ finite conformal symmetry group

Punctured spheres of countable connectivity

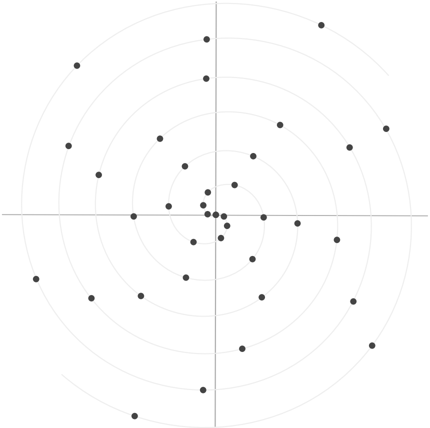
Finite punctures ⟶ finite conformal symmetry group

Countable punctures ⟶ elementary discrete conformal symmetry group

Discrete group of isometries of $\mathbb{C}$



Discrete group of isometries of $\mathbb{C}^*$



Conformal symmetry groups of countably connected regions

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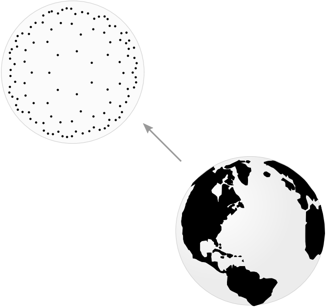

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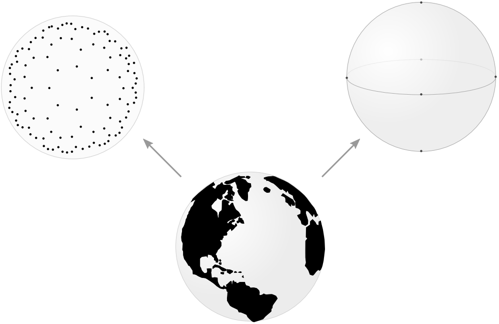

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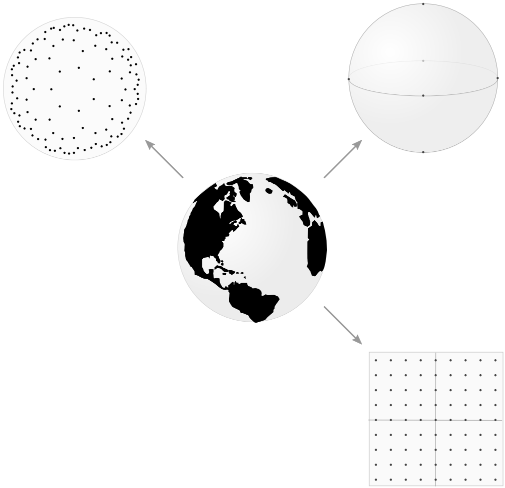

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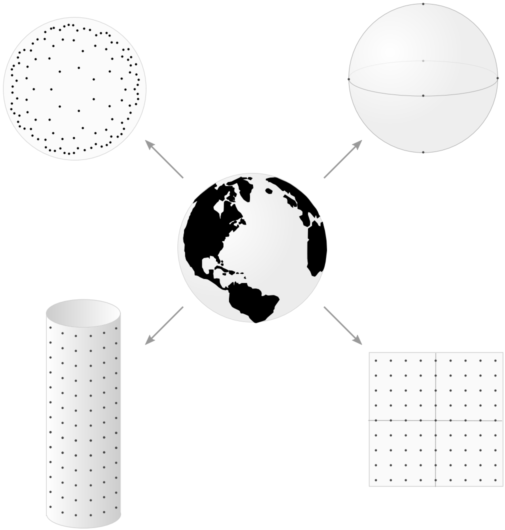
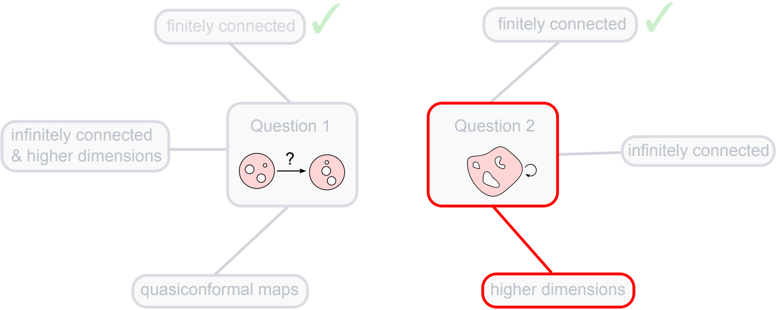


Higher dimensions

Plan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Decomposing Möbius maps

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$$\frac{az+b}{cz+d} = A\sigma(z) + B$$

Möbius maps in higher dimensions

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Liouville's theorem

Liouville's theorem

Theorem (Liouville, 1850). A smooth conformal map from one region in $\mathbb{R}^n$ to another is a Möbius transformation.

Significance

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Positive. Fewer conformal maps to worry about.

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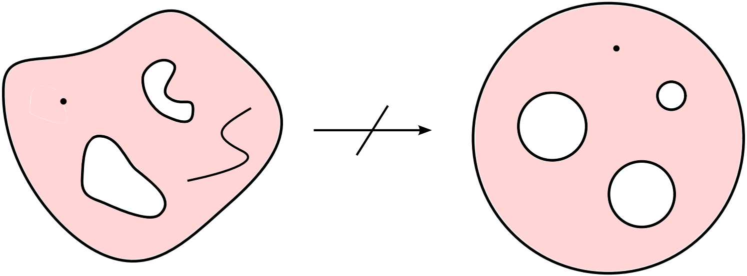
Negative. No Riemann mapping theorem.



Finitely connected regions

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Two dimensions. Groups A_4, S_4, A_5, C_n, and D_n.

Finitely connected regions

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Higher dimensions. *All finite groups arise.*

Finitely punctured spheres

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Sketch proof that every finite group arises

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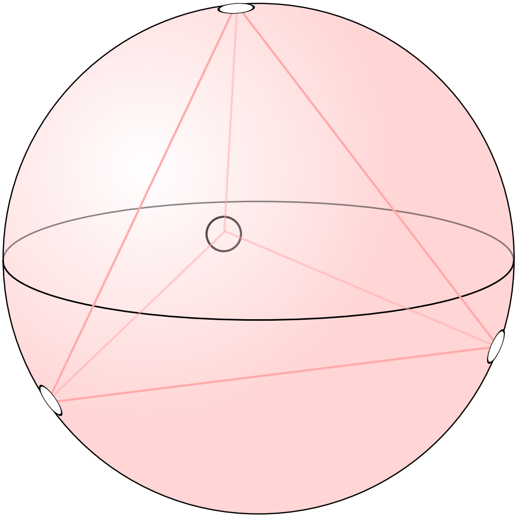

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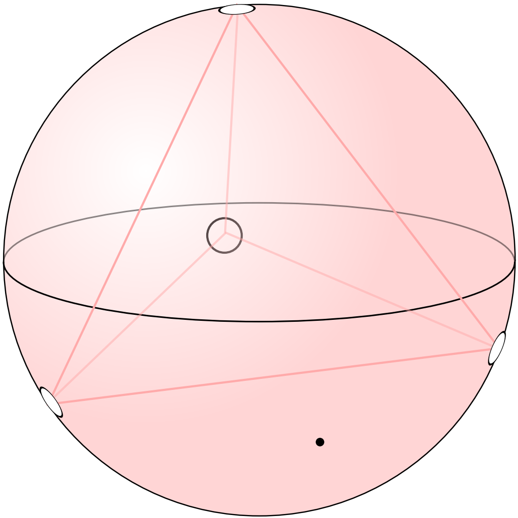

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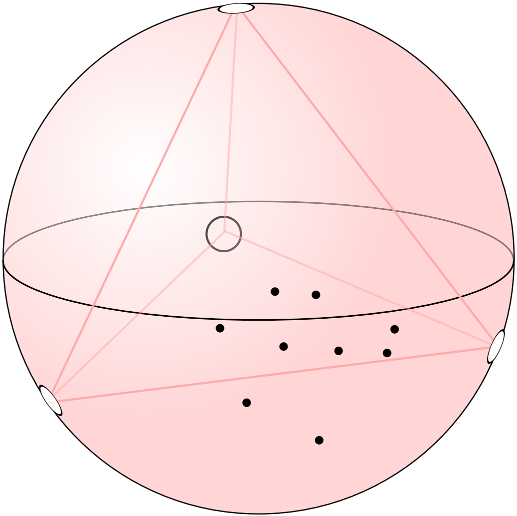
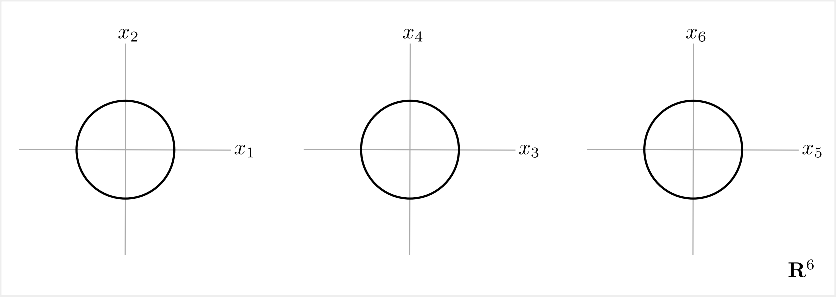


More complicated groups



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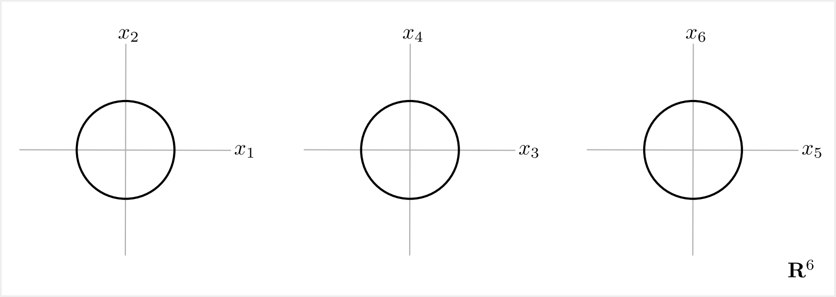
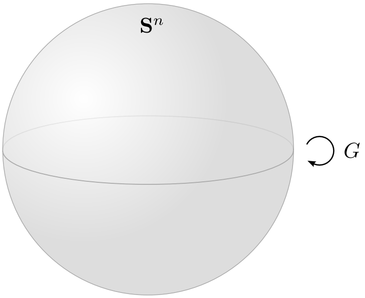


$$\mathrm{Aut}(D) \cong (\mathrm{O}_2 \times \mathrm{O}_2 \times \mathrm{O}_2) \rtimes S_3$$

Classification of closed Möbius groups



$G \leqslant \mathcal{M}_n$

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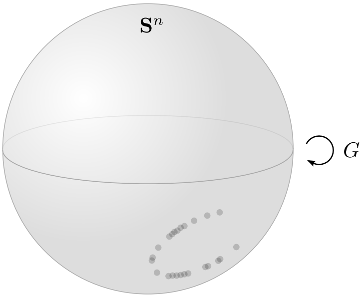

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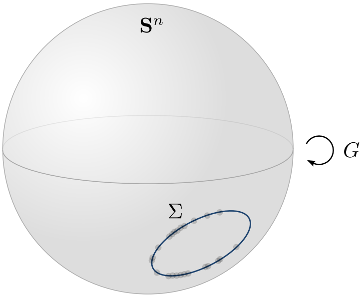

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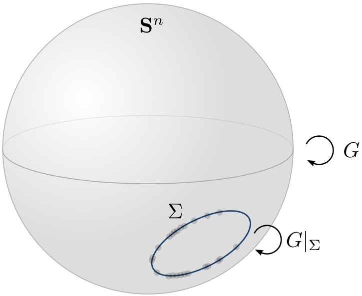

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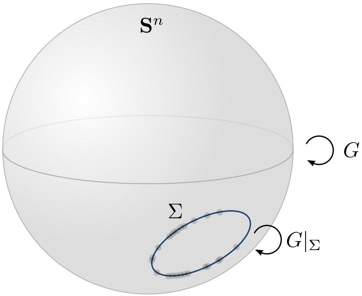
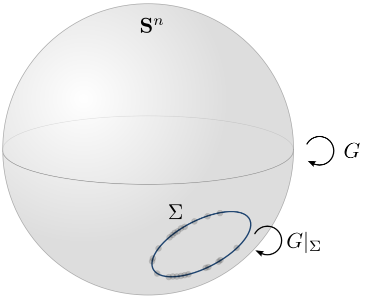


$G \leqslant \mathcal{M}_n$

$G|_\Sigma$ either discrete or $\mathcal{M}_k$

Classification of closed Möbius groups



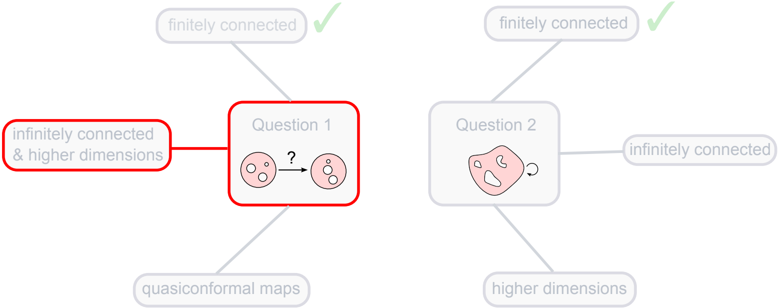
$G \leqslant \mathcal{M}_n$

$G|_\Sigma$ either discrete or $\mathcal{M}_k$

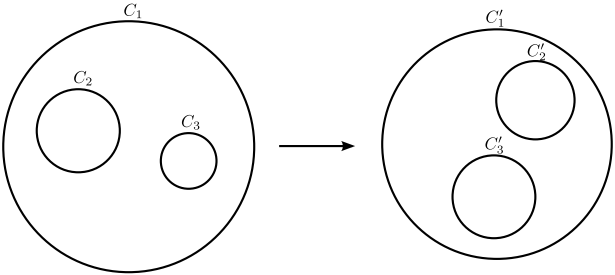
Fix(Σ) conjugate to a closed orthogonal group

Lorentz space

Plan


finitely connected
infinitely connected
& higher dimensions
Question 1
?
quasiconformal maps
finitely connected
Question 2
infinitely connected
higher dimensions

Intersecting circles



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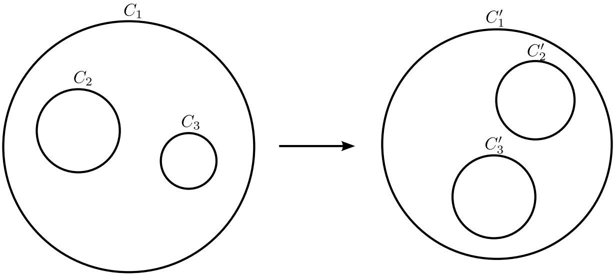
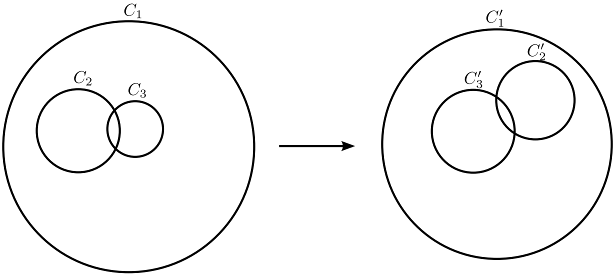
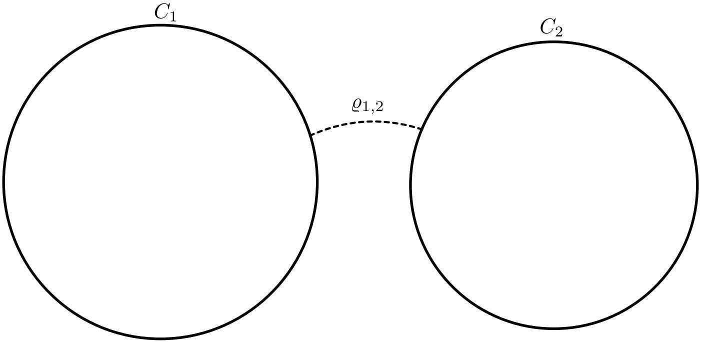


There exists a Möbius map f such that $f(C_1) = C_1'$, $f(C_2) = C_2'$, and $f(C_3) = C_3'$ if and only if $\varrho_{1,2} = \varrho_{1,2}'$, $\varrho_{2,3} = \varrho_{2,3}'$, and $\varrho_{3,1} = \varrho_{3,1}'$.

Intersecting circles

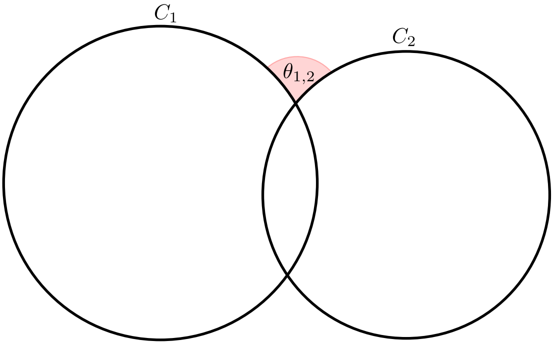


Inversive distance



$$\sigma(C_1, C_2) = \cosh \varrho_{1,2}$$

Inversive distance



$$\sigma(C_1, C_2) = \cos\theta_{1,2}$$

Extend Question 1

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Observation. Let $C_1, C_2, \ldots, C_m$ and $C_1', C_2' \ldots, C_m'$ be two sets of circles.

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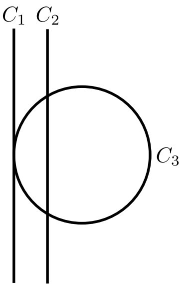
Does the converse hold?

Problematic example



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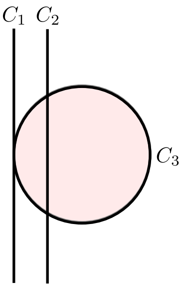

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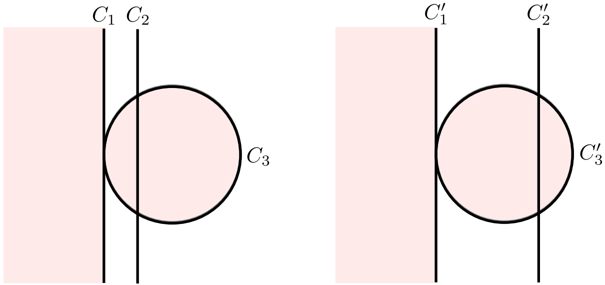
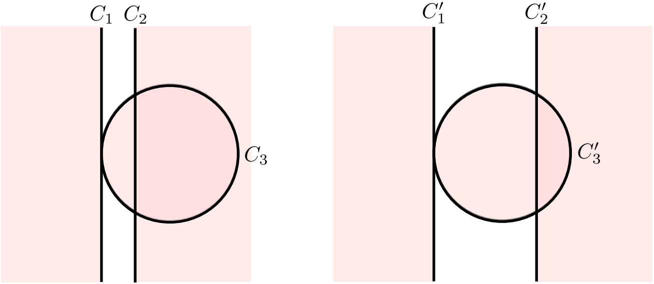
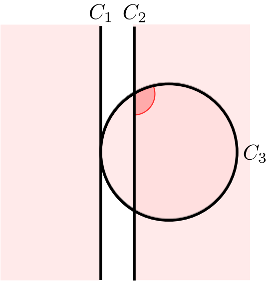
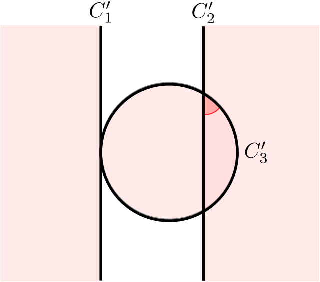


Problematic example

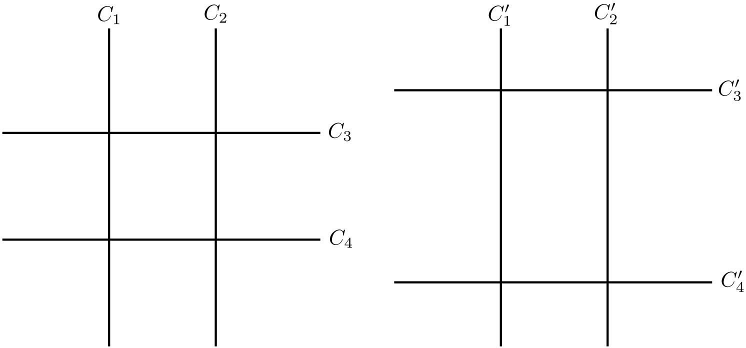


Problematic example





Another problematic example



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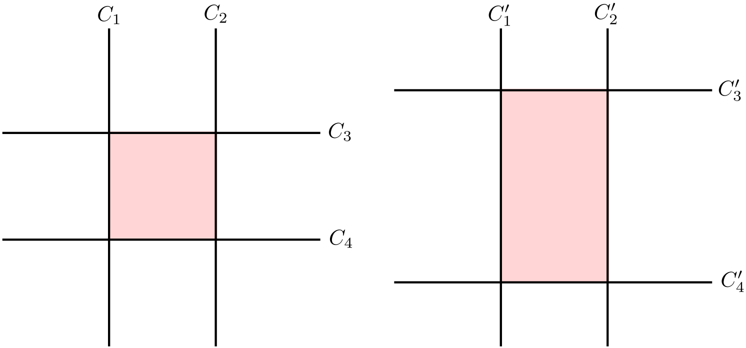

The main theorem

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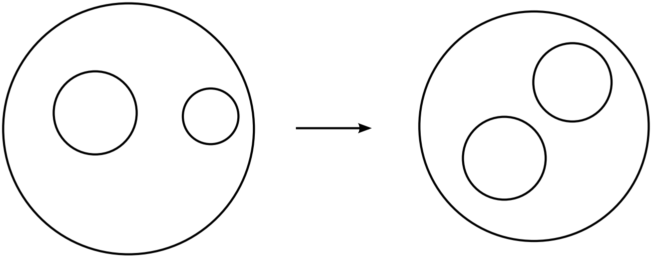
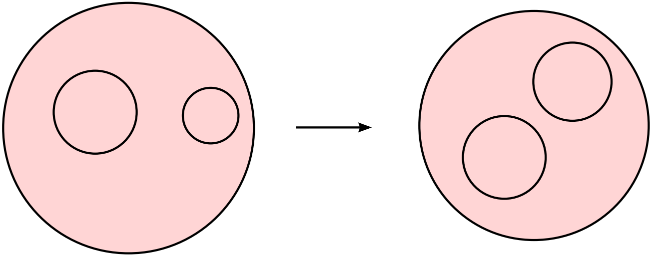
The main theorem

Theorem (Crane & Short, 2010.) Let $D_1, D_2, \ldots, D_m$ and $D_1', D_2', \ldots, D_m'$ be two collections of discs such that $\bigcap \partial D_i = \bigcap \partial D_i' = \emptyset$. Then there is a Möbius transformation f such that $f(D_i) = D_i'$ for each i if and only if $\hat{\sigma}(D_i, D_j) = \hat{\sigma}(D_i', D_j')$ for each pair i, j.

Recover the region



Recover the region



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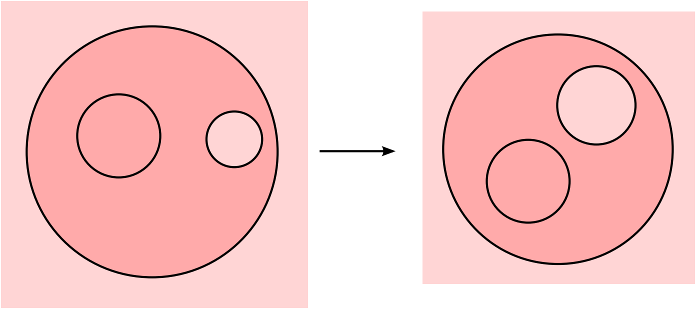
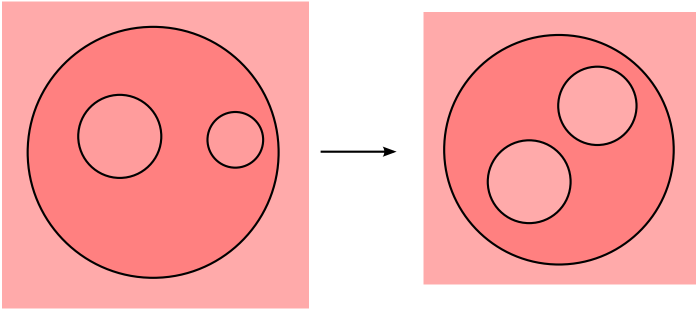


Recover the region



Proof

Use the *hyperboloid model* of hyperbolic space.

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Equip $\mathbb{R}^4$ with the *Lorentz inner product*

$$\langle (x_1, x_2, x_3, x_4), (y_1, y_2, y_3, y_4) \rangle = x_1 y_1 + x_2 y_2 + x_3 y_3 - x_4 y_4.$$

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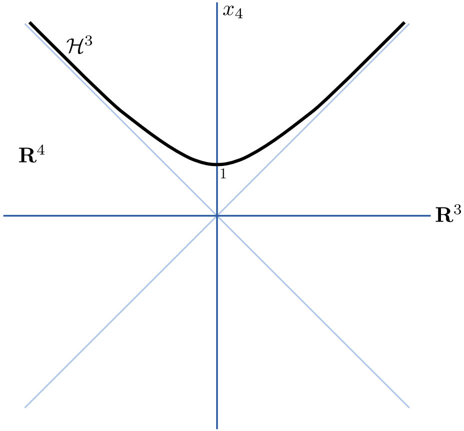
$$\cosh \varrho(x, y) = -\langle x, y \rangle$$

Hyperboloid model



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Hyperbolic isometries

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Lorentz transformations : linear maps that preserve the Lorentz inner product.

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Positive Lorentz transformations : Lorentz transformations that preserve $\mathcal{H}^3$.

Normals

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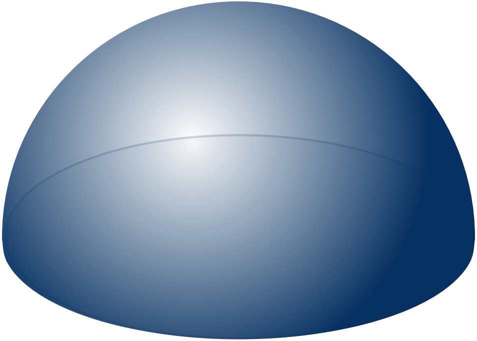
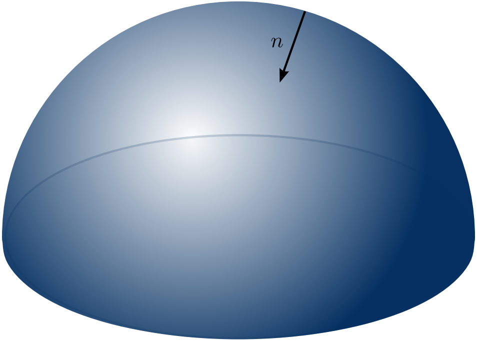


Normals



Inversive distance

Given discs D_1 and D_2 with associated space-like normals n_1 and n_2 in $\mathbb{R}^4$ we have

$$\hat{\sigma}(D_1, D_2) = \langle n_1, n_2 \rangle.$$

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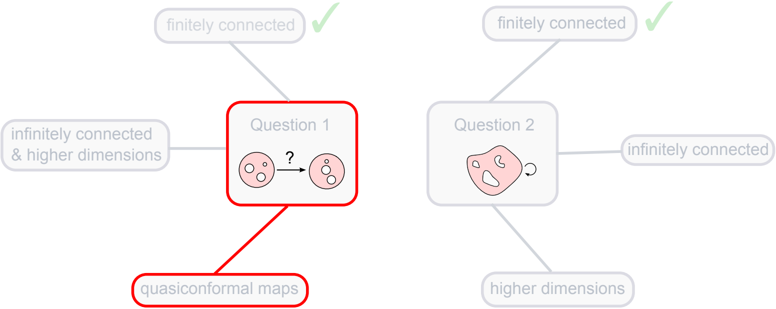
The rest is linear algebra...

Quasiconformal maps

Plan


finitely connected
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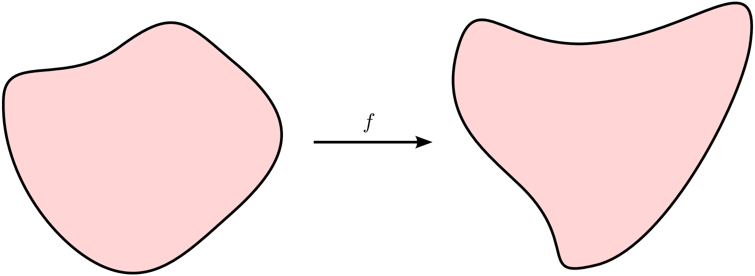
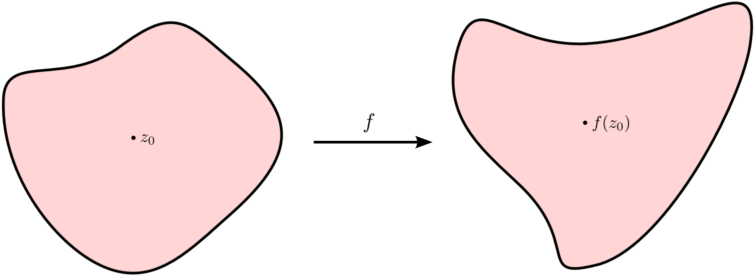
Conformal map



$$f(z_0 + z) = f(z_0) + az + \varepsilon(z)$$

Conformal map



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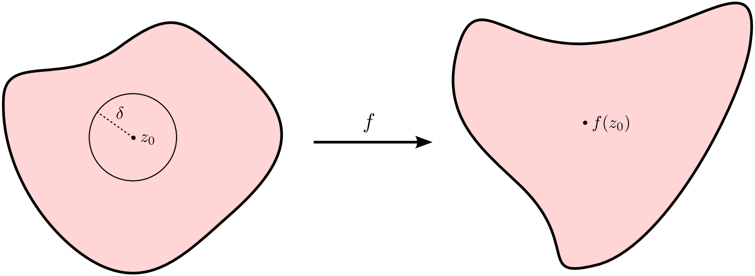

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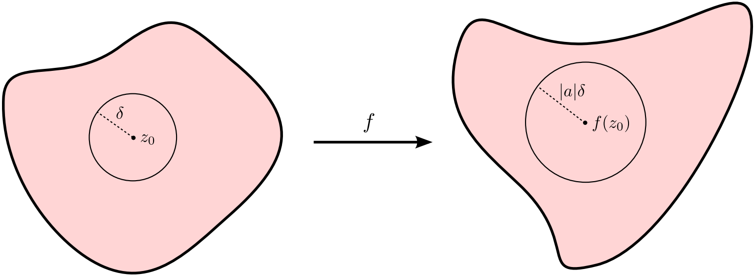

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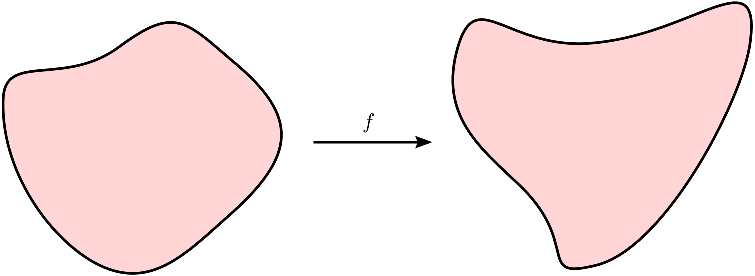
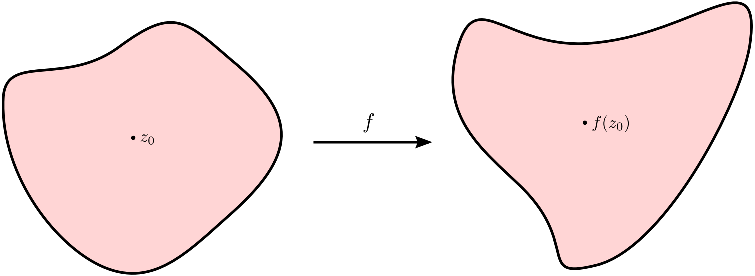


$$f(z_0 + z) = f(z_0) + az + b\bar{z} + \varepsilon(z)$$

Quasiconformal map



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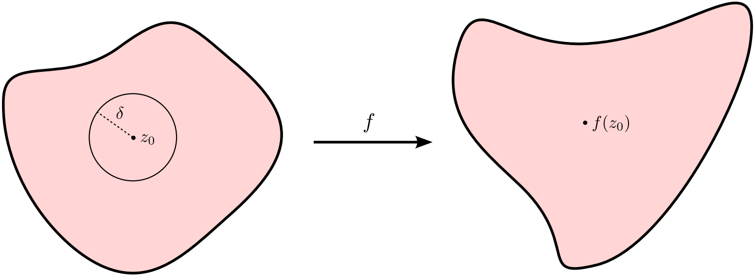

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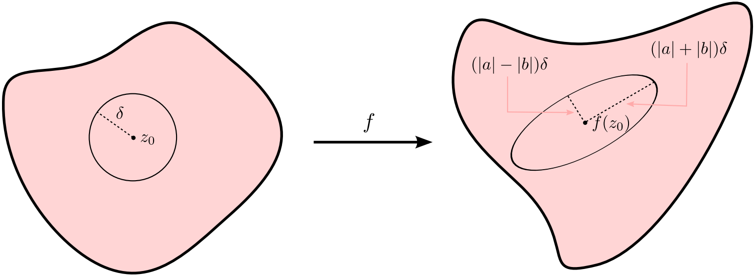

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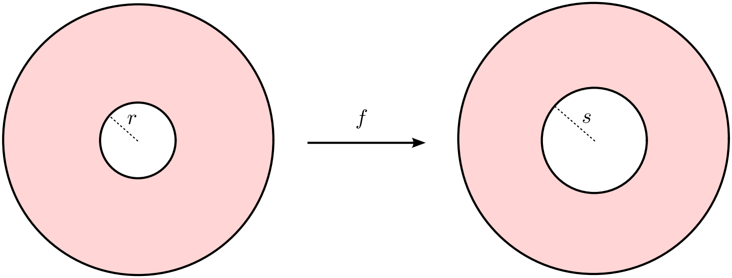
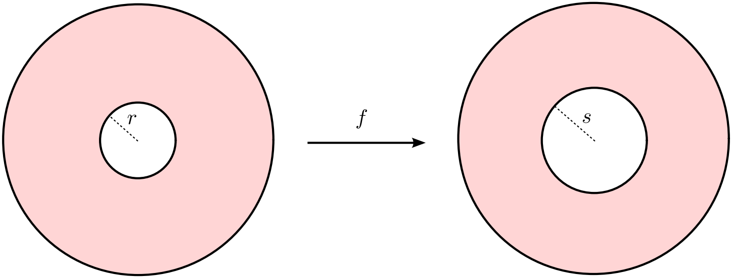
K-quasiconformal map of annuli



$$\frac{1}{K} \leqslant \frac{r}{s} \leqslant K$$

K-quasiconformal map of annuli



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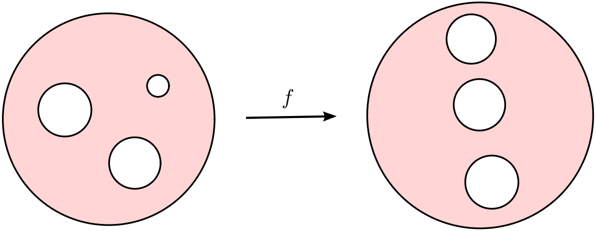
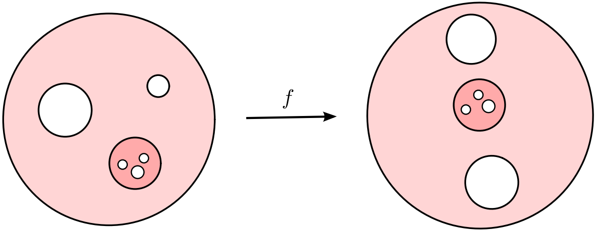
Quasiconformal map of circular regions



Suppose there is a K-quasiconformal map f.

Quasiconformal map of circular regions



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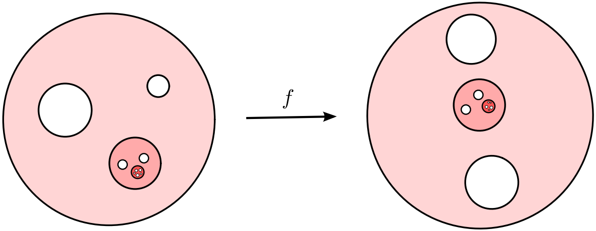

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Quasiconformal map of circular regions

Theorem.

Quasiconformal map of circular regions

Theorem. Let Ω and Ω' be two circular regions in the complex plane bounded by circles $C_1, C_2, \ldots, C_m$ and $C'_1, C'_2, \ldots, C'_m$, respectively.

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$$\frac{1}{K} \leqslant \frac{\exp \varrho_{i,j}}{\exp \varrho'_{i,j}} \leqslant K$$

for each pair i, j.

Open Problem

Open Problem

Does the converse hold?

Thank you!