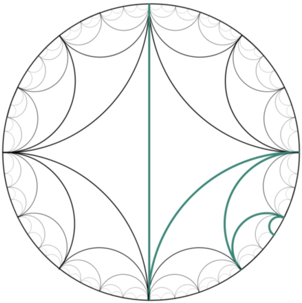
The Farey graph, continued fractions, and the modular group

Ian Short



26 November 2009

Collaborators

Alan Beardon (University of Cambridge)

Meira Hockman (University of the Witswatersrand)

Continued fractions

Euclid's algorithm

$$\frac{31}{13}$$

Euclid's algorithm

$$\frac{31}{13} = 2 + \frac{1}{\frac{13}{5}}$$

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$$\frac{31}{13} = 2 + \frac{1}{\frac{13}{5}}$$

$$= 2 + \cfrac{1}{2 + \frac{1}{\frac{5}{3}}}$$

Euclid's algorithm

$$\begin{aligned}\frac{31}{13} &= 2+\frac{1}{\frac{13}{5}}\\ &= 2+\frac{1}{2+\frac{1}{\frac{5}{3}}}\\ &= 2+\frac{1}{2+\frac{1}{1+\frac{1}{\frac{3}{2}}}}\end{aligned}$$

Euclid's algorithm

$$\frac{31}{13} = 2 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{2}}}}$$

The nearest-integer algorithm

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$$\begin{aligned}\frac{31}{13} &= 2+\frac{1}{\frac{13}{5}}\\ &= 2+\frac{1}{3+\frac{1}{-\frac{5}{2}}}\end{aligned}$$

The nearest-integer algorithm

$$
\begin{aligned}
\frac{31}{13} &= 2+\frac{1}{\frac{13}{5}} \\
&= 2+\cfrac{1}{3+\cfrac{1}{-\frac{5}{2}}} \\
&= 2+\cfrac{1}{3+\cfrac{1}{-3+\cfrac{1}{2}}}
\end{aligned}
$$

The nearest-integer algorithm

$$\frac{31}{13} = 2 + \cfrac{1}{3 + \cfrac{1}{-3 + \cfrac{1}{2}}}$$

Another expansion

$$\frac{31}{13} = 3 + \cfrac{1}{-2 + \cfrac{1}{3 + \cfrac{1}{-3}}}$$

Questions

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Is the nearest-integer continued fraction the *shortest* continued fraction with integer coefficients?

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How many shortest continued fractions are there?

Can we characterise shortest continued fractions?

Literature

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Perron O., Die Lehre von den Kettenbrüchen, *Chelsea Publishing Company, New York*, (1950).

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Srinivisan, M.S., Shortest semiregular continued fractions, *Proc. Indian Acad. Sci., Sect. A*, 35 (1952).

The modular group

Möbius transformations

Möbius transformations

Define, for an integer b,

$$S_b(z) = b + 1/z.$$

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$$S_{b_1} \circ S_{b_2} \circ \cdots \circ S_{b_n}(\infty) = b_1 + \cfrac{1}{b_2 + \cfrac{1}{b_3 + \cfrac{1}{b_4 + \cdots + \cfrac{1}{b_n}}}}$$

Note that $S_{b_i}(0) = \infty$.

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$$S_{b_1} \circ S_{b_2} \circ \cdots \circ S_{b_n}(0) = S_{b_1} \circ S_{b_2} \circ \cdots \circ S_{b_{n-1}}(\infty).$$

What is the group generated by the maps $S_{b_i}(z) = b_i + 1/z$?

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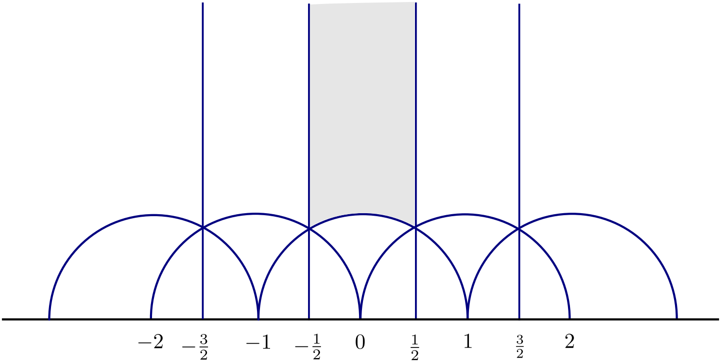
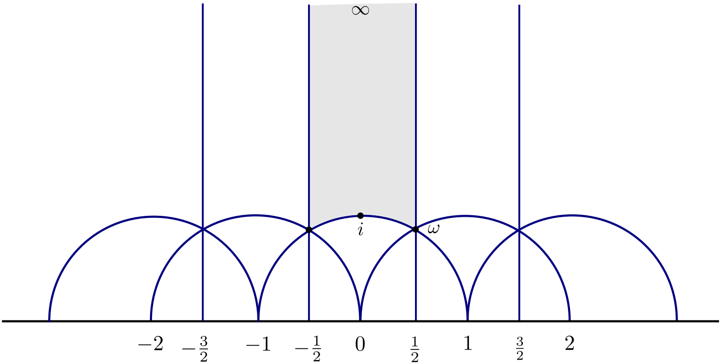
$$\begin{pmatrix} b & 1 \\ 1 & 0 \end{pmatrix} \longmapsto S_b$$

The modular group

$$\Gamma = \left\{ z \mapsto \frac{az+b}{cz+d} \;:\; a, b, c, d \in \mathbb{Z}, \quad ad - bc = 1 \right\}$$



−2
−3/2
−1
−1/2
0
1/2
1
3/2
2


∞
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0
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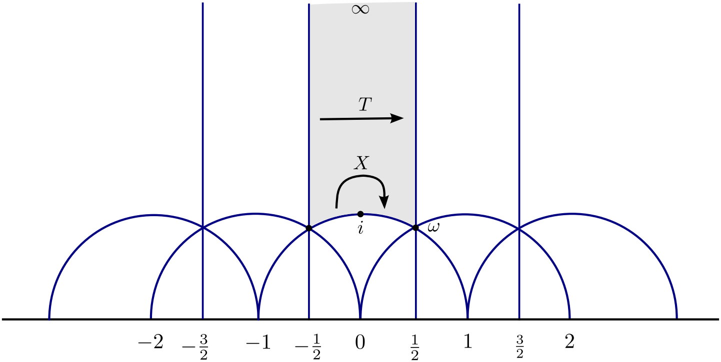
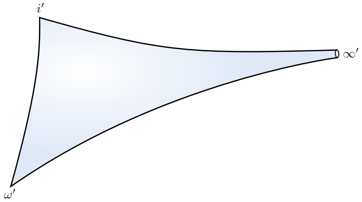
3/2
2



$$T(z) = z + 1 \qquad X(z) = -1/z$$

The modular surface



The extended modular group

$$\widetilde{\Gamma} = \left\{ z \mapsto \frac{az+b}{cz+d} \;:\; a,b,c,d \in \mathbb{Z}, \quad |ad-bc| = 1 \right\}$$

Presentation for the extended modular group

$$T(z) = z + 1 \qquad U(z) = \frac{1}{z} \qquad V(z) = -z$$

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$$V = UTUT^{-1}UT$$

$$\widetilde{\Gamma} = \left\langle T, U \;\mid U^2 = 1, \quad UV = VU, \quad VT = T^{-1}V \right\rangle$$

Words in the extended modular group

Each word in $\widetilde{\Gamma}$ has the form

$$T^{b_1}UT^{b_2}U\cdots T^{b_n}U,$$

for integers $b_1, b_2, \ldots, b_n$ ($T(z) = z + 1$ and $U(z) = 1/z$).

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$$S_b = T^bU$$

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$$S_b = T^bU$$

Each word in $\widetilde{\Gamma}$ has the form

$$S_{b_1}S_{b_2}\cdots S_{b_n},$$

for integers $b_1, b_2, \ldots, b_n$.

A correspondence

Finite integer continued fractions

A correspondence

Finite integer continued fractions

Words of T and U in $\widetilde{\Gamma}$

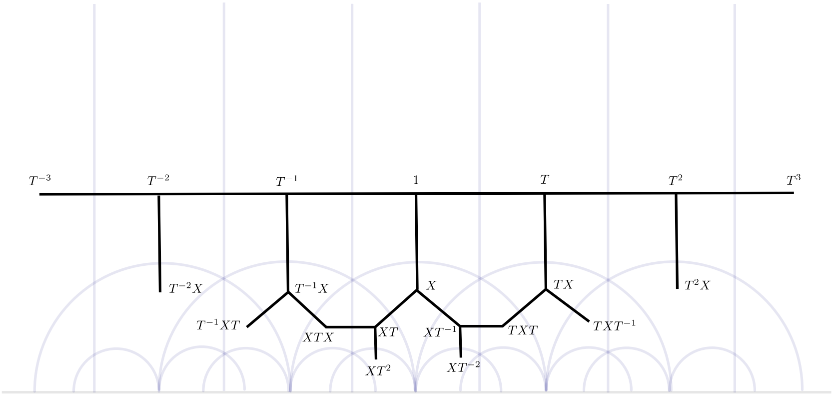
A correspondence

Finite integer continued fractions

Words of T and U in $\widetilde{\Gamma}$

$(T(z) = z + 1$ and $U(z) = 1/z)$

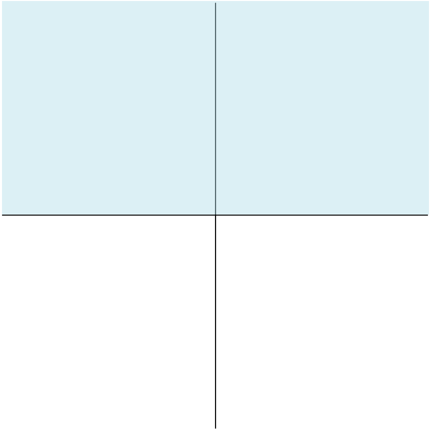
The word metric



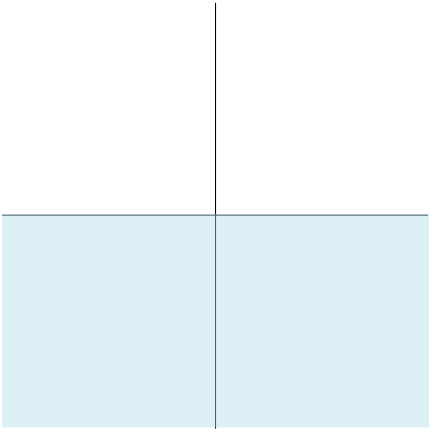
Cayley graph of Γ with respect to $T(z) = z + 1$ and $X(z) = -1/z$

The Farey graph

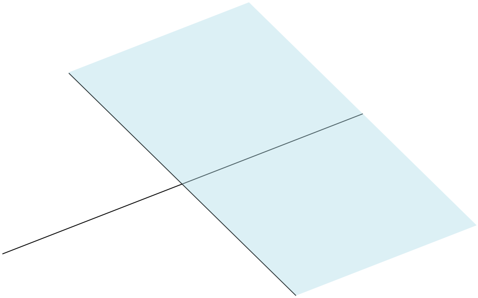
The upper half-plane



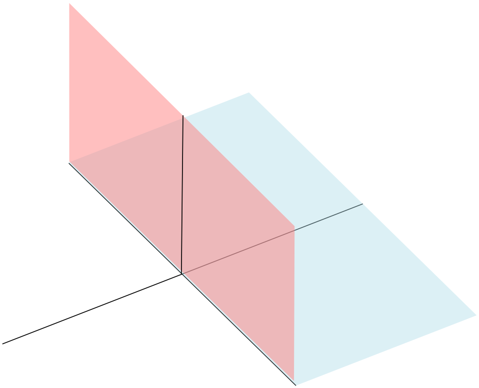
The upper half-plane



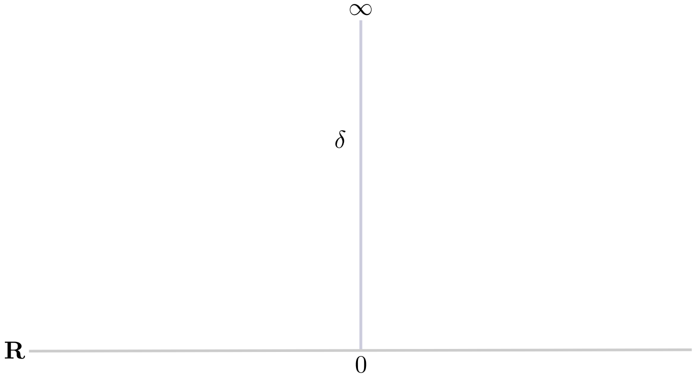
The vertical half-plane



The vertical half-plane

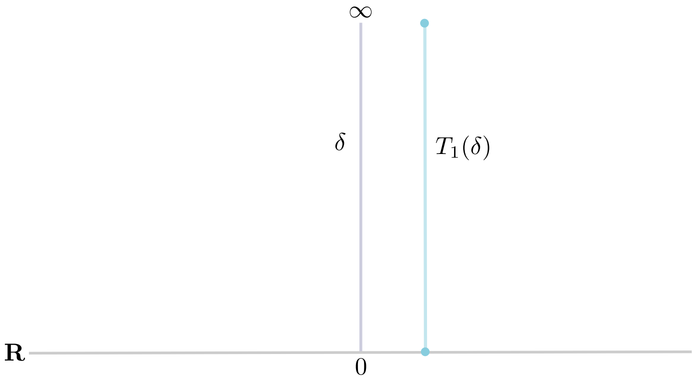


A path of geodesics



$$T_n = S_{b_1} \circ S_{b_2} \circ \cdots \circ S_{b_n}$$

A path of geodesics



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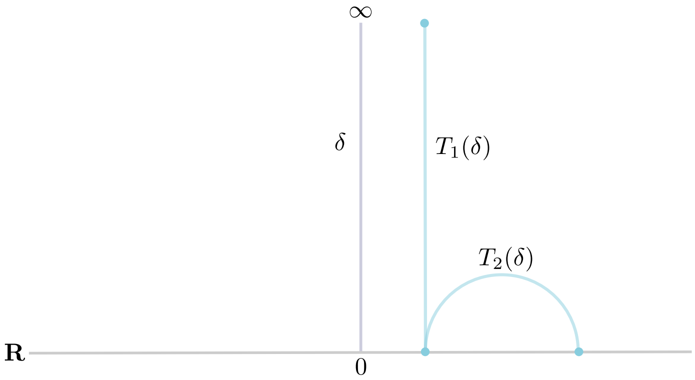

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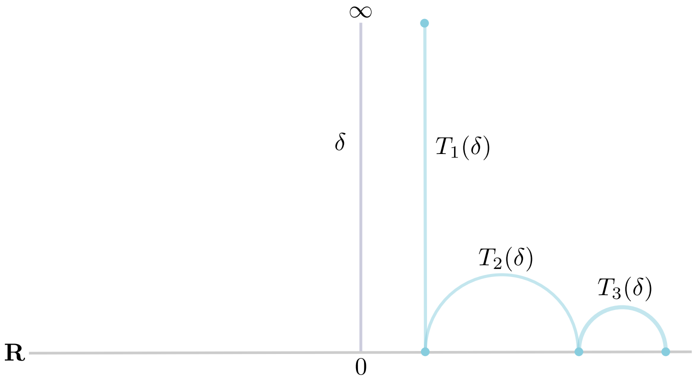
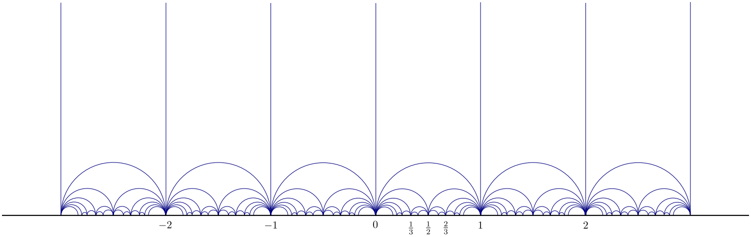


$$T_n = S_{b_1} \circ S_{b_2} \circ \cdots \circ S_{b_n}$$

The Farey graph



$\widetilde{\Gamma}(\delta)$

The Farey graph

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$$\text{Vertices} = \mathbb{Q}$$

The Farey graph

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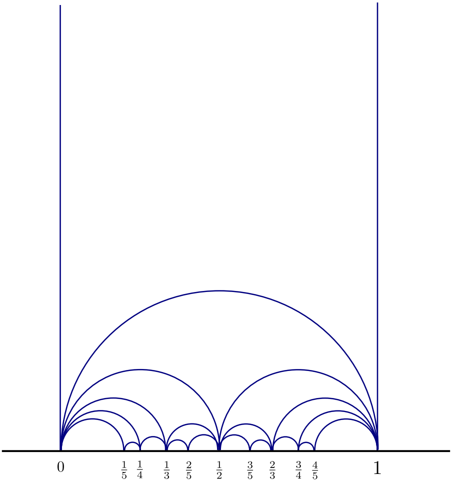
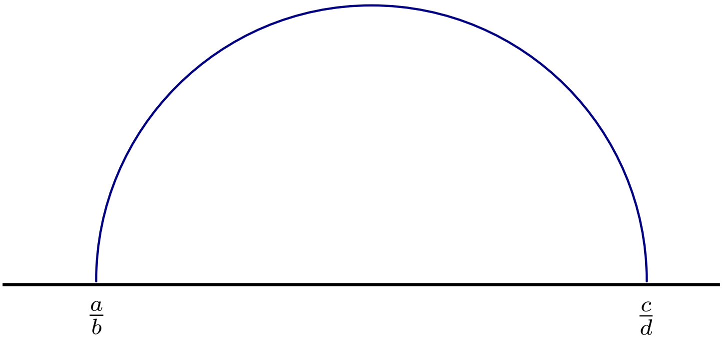
$$\text{Join } \tfrac{a}{b} \text{ to } \tfrac{c}{d} \text{ if and only if } |ad - bc| = 1.$$

The Farey graph



Mediants



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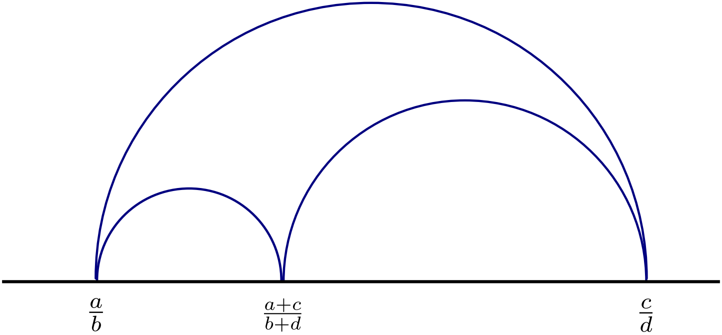
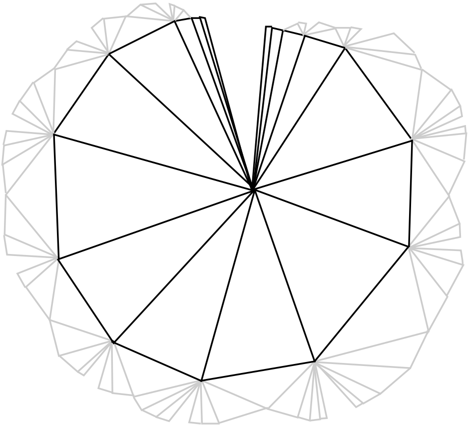
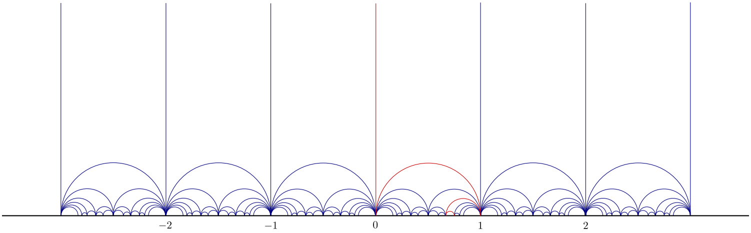


The Farey graph



Paths in the Farey graph



A correspondence

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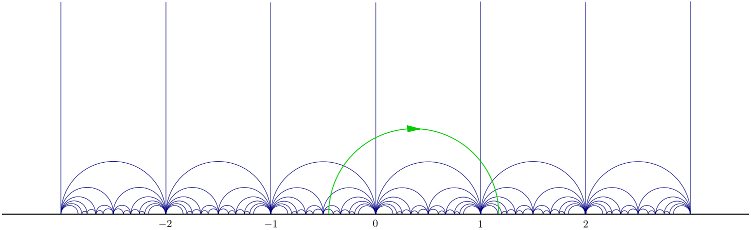
Finite integer continued fractions

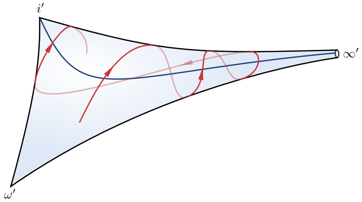
Finite integer continued fractions

Finite paths from ∞ in the Farey graph

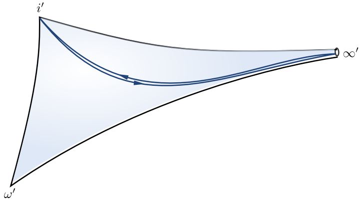
Cutting sequences



Geodesics on the modular surface



A geodesic on the modular surface



Geodesics

Questions

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Can we characterise shortest continued fractions?

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How many shortest continued fractions are there?

How many geodesics are there between two vertices in the Farey graph?

Can we characterise shortest continued fractions?

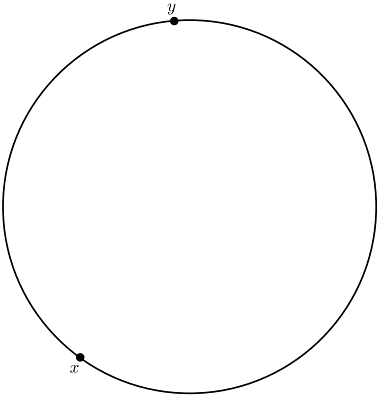
Can we characterise geodesics in terms of the coefficients of a continued fraction?

Answer 1

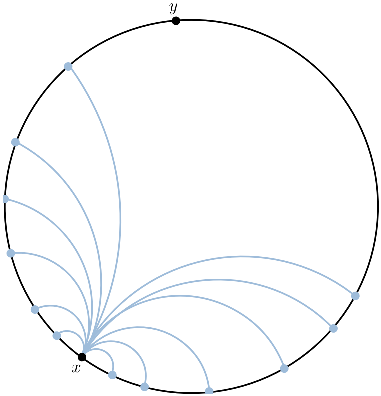
Answer 1

The nearest-integer algorithm *does* correspond to a geodesic.

The nearest-integer algorithm



The nearest-integer algorithm



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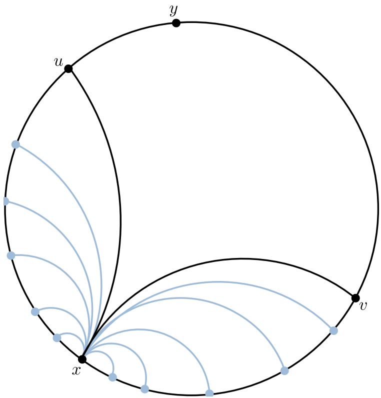

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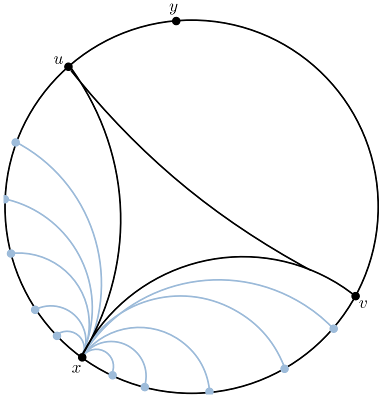
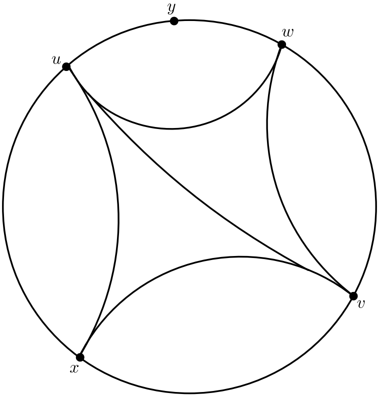
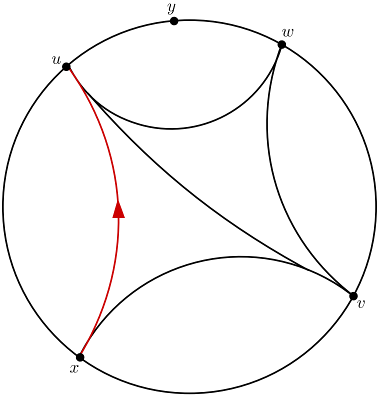


The nearest-integer algorithm



The nearest-integer algorithm

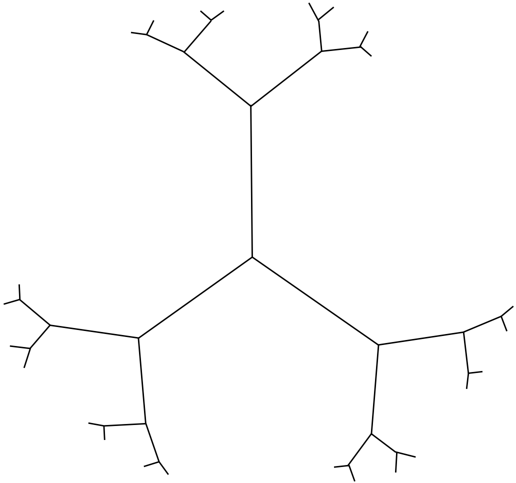


Answer 2

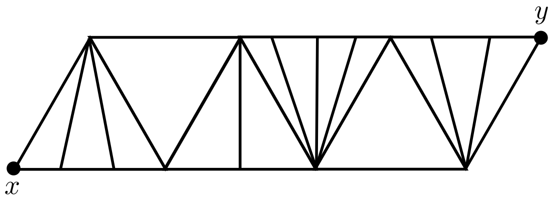
Answer 2

Count geodesics using the dual graph.

The dual graph



Unique path of triangles

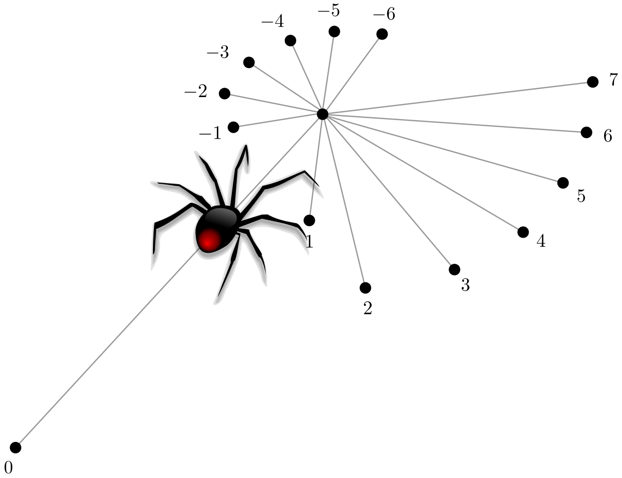


Answer 3

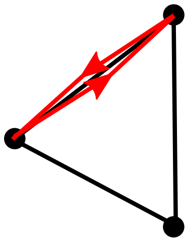
Answer 3

Theorem. A continued fraction with coefficients $b_1, b_2, \dots, b_n$ corresponds to a geodesic in the Farey graph if and only if $|b_i| \geqslant 2$ for each $i \geqslant 2$, and there is no substring of $b_2, b_3, \dots, b_n$ of the form $2, -3, 3, -3, 3, -3, \dots, 3, -2$.

Paths and coefficients

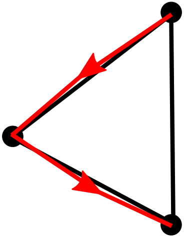


Inefficient paths



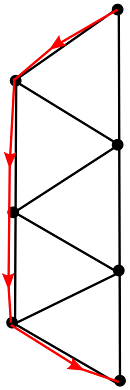
$b_i = 0$

Inefficient paths



$$b_i = \pm 1$$

Inefficient paths



$$b_i, b_{i+1}, b_{i+2}, b_{i+3} = 2, -3, 3, -2$$

Thank you!