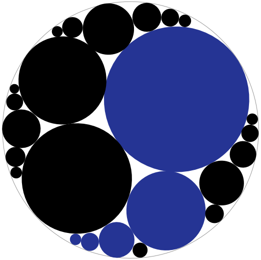
Ford circles and the convergence of continued fractions

Ian Short



4 March 2010

Introduction

Project

The geometry of continued fractions

Alan Beardon, Meira Hockman, and Ian Short

Continued fractions

$$\mathbf{K}(b_n) = \cfrac{1}{b_1 + \cfrac{1}{b_2 + \cfrac{1}{b_3 + \cfrac{1}{b_4 + \cdots}}}}$$

Criteria for convergence

The Seidel–Stern Theorem.

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The Seidel–Stern Theorem. Suppose that $b_n \geqslant 0$ for $n = 1, 2, \ldots$. If $\sum_n b_n$ diverges then $\mathbf{K}(b_n)$ converges.

Corollary. All positive integer continued fractions $\mathbf{K}(b_n)$ converge.

Publication

The Seidel, Stern, Stolz and Van Vleck Theorems on continued fractions

Alan Beardon and Ian Short

Bulletin of the London Mathematical Society

Ford circles

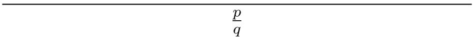
Publication

Fractions

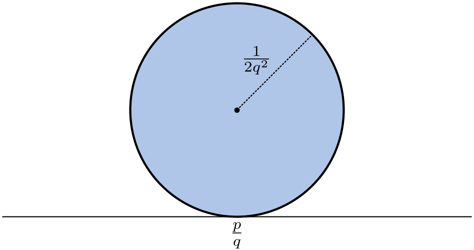
Lester Ford

The American Mathematical Monthly, 45 (1938)

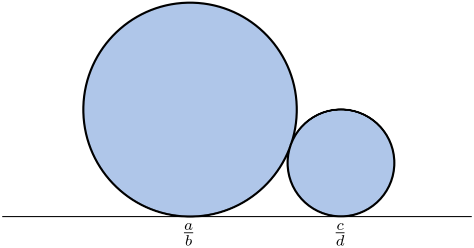
Ford circle



Ford circle

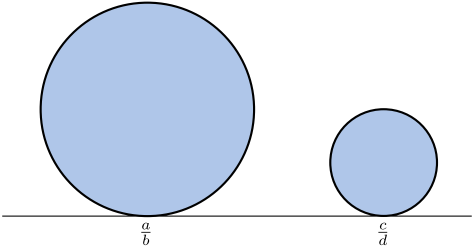


Ford circles



$$|ad - bc| = 1$$

Ford circles



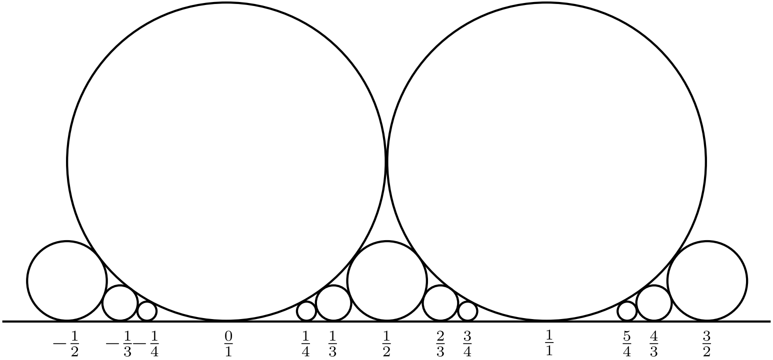
$$|ad - bc| > 1$$

Ford circles

!

$$|ad - bc| < 1$$

Ford circles



Ford circles

Continued fractions

$$\mathbf{K}(b_n) = \cfrac{1}{b_1 + \cfrac{1}{b_2 + \cfrac{1}{b_3 + \cfrac{1}{b_4 + \cdots}}}}$$

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$$b_n \in \mathbb{N}$$

Continued fraction approximants

$$\frac{A_n}{B_n} = \cfrac{1}{b_1 + \cfrac{1}{b_2 + \cfrac{1}{b_3 + \cdots + \cfrac{1}{b_n}}}}$$

Continued fraction approximants

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$$\begin{pmatrix} A_{n-1} & A_n \\ B_{n-1} & B_n \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & b_1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & b_2 \end{pmatrix} \cdots \begin{pmatrix} 0 & 1 \\ 1 & b_n \end{pmatrix}$$

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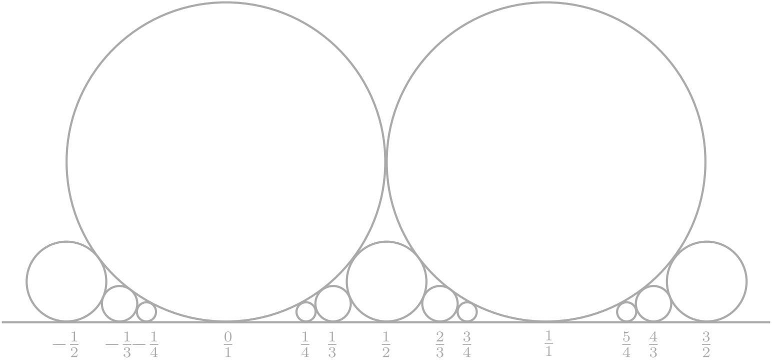
$$|A_n B_{n-1} - A_{n-1} B_n| = 1$$

Chain of Ford circles



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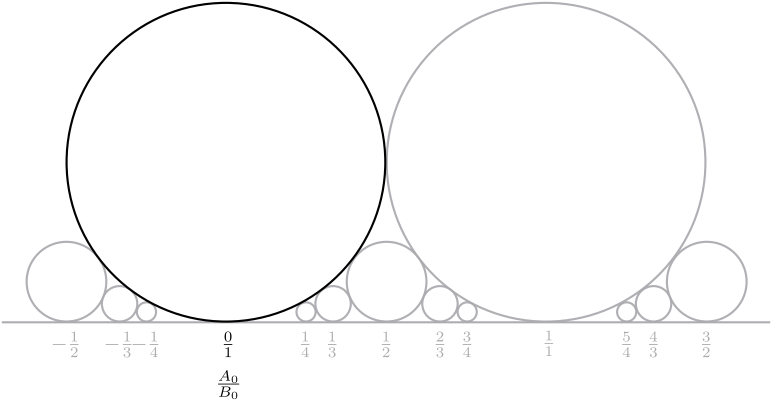
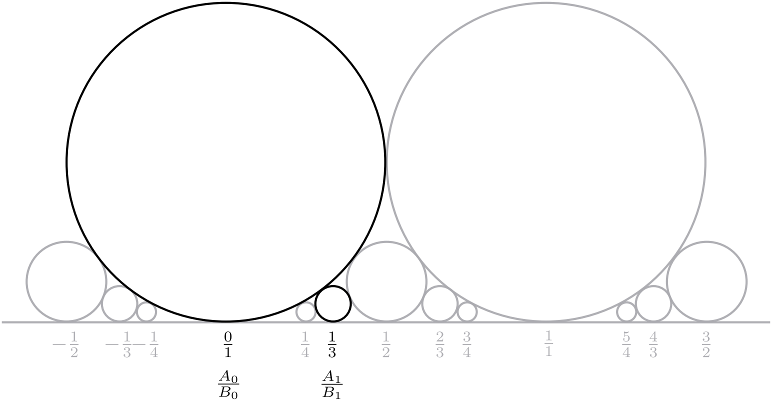


Chain of Ford circles



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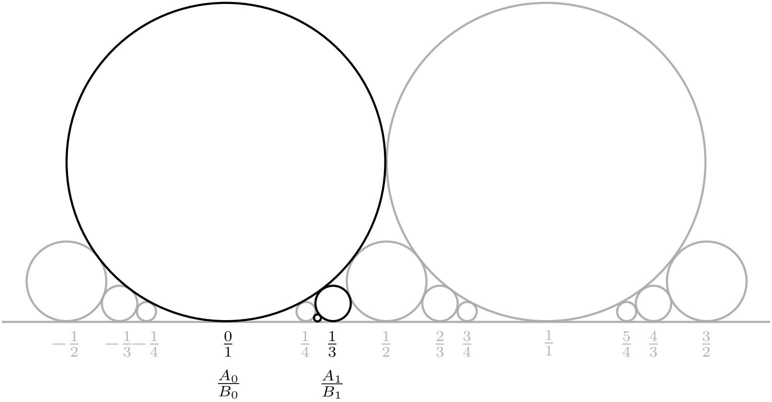

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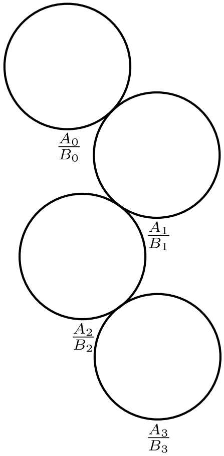

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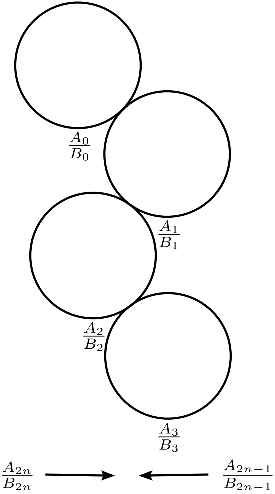

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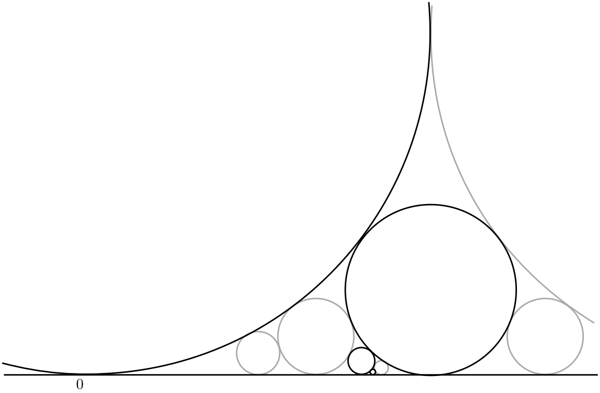
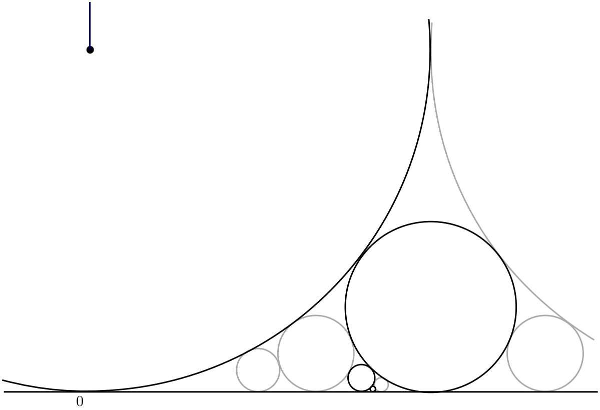


Chain of Ford circles



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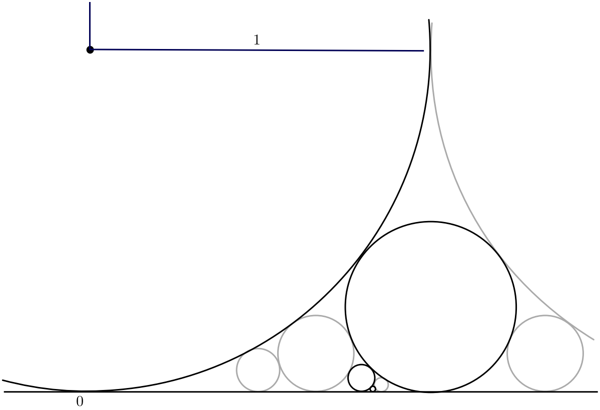

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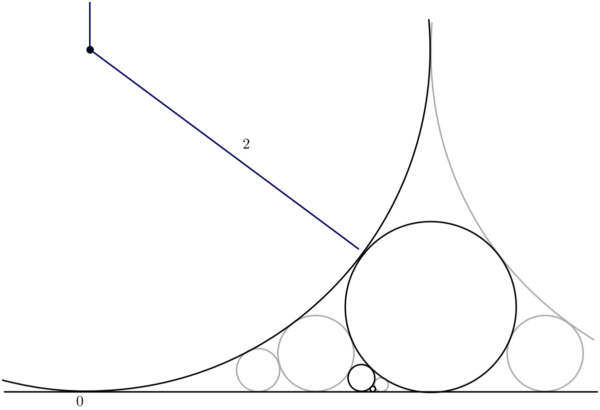

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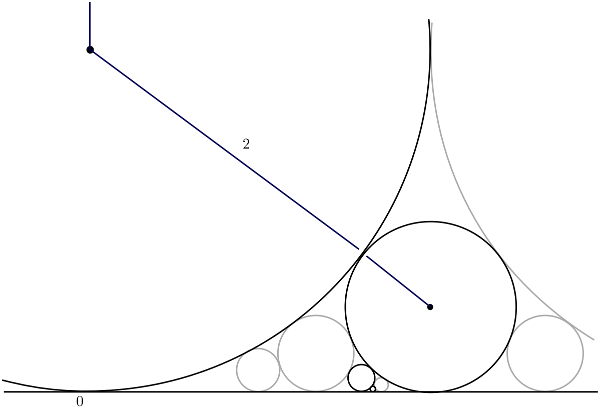

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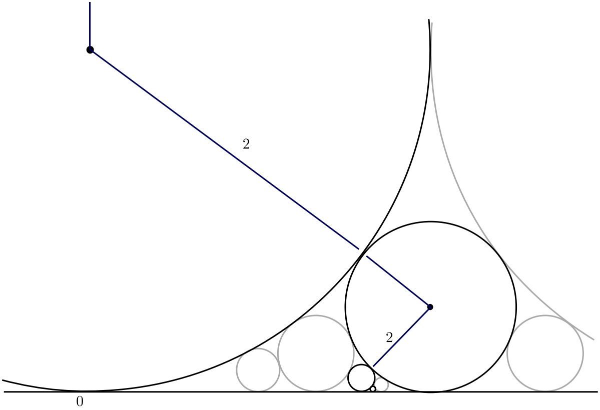
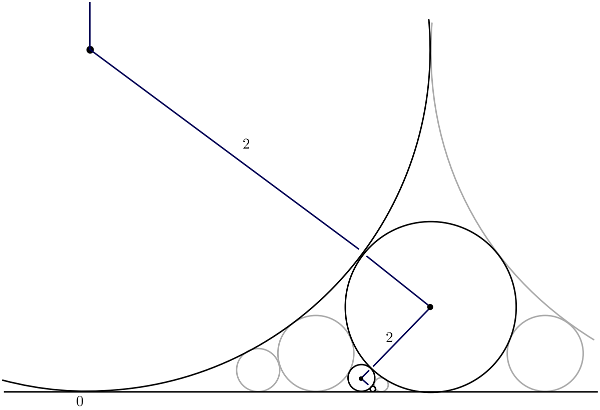
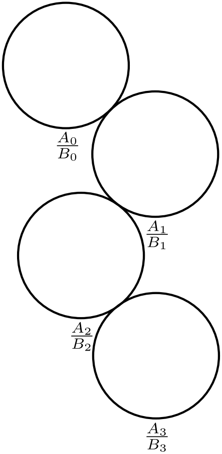


Chain of Ford circles



Chain of Ford circles



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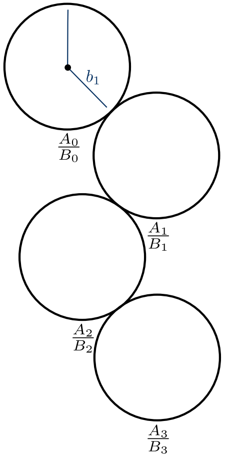

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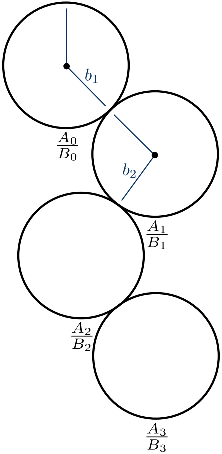

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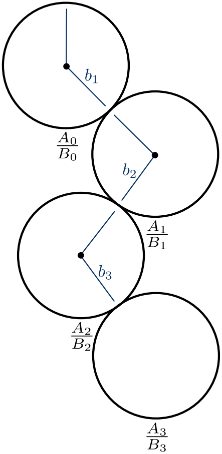
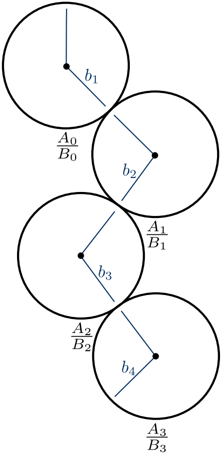


Chain of Ford circles



The Seidel–Stern Theorem

Continued fractions

$$\mathbf{K}(b_n) = \cfrac{1}{b_1 + \cfrac{1}{b_2 + \cfrac{1}{b_3 + \cfrac{1}{b_4 + \cdots}}}}$$

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$$b_n \geqslant 0$$

Continued fractions

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$$b_n \geqslant 0$$

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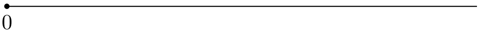
The Seidel–Stern Theorem

Theorem. Suppose that $b_n \geqslant 0$ for $n = 1, 2, \ldots$. If $\sum_n b_n$ diverges then $\mathbf{K}(b_n)$ converges.

Convergence of odd and even parts



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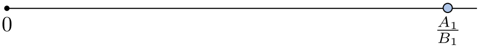

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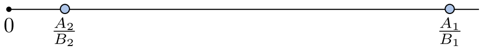

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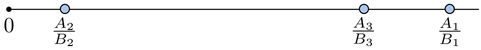

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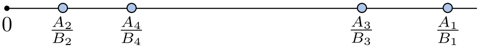

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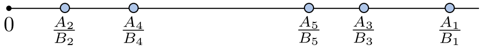

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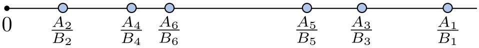

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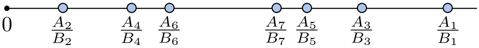

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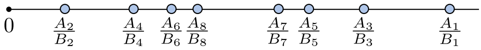
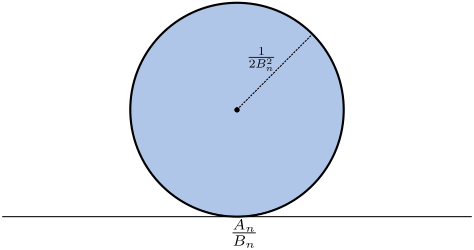


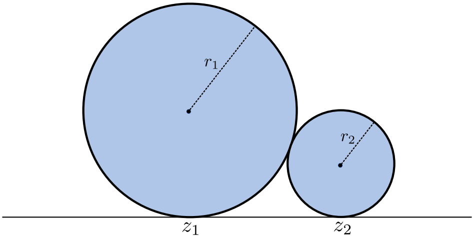
Generalised Ford circle

$$\frac{A_n}{B_n}$$

Generalised Ford circle

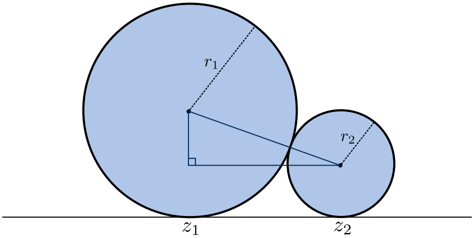


Euclidean geometry lemma



$$|z_1 - z_2| = 2\sqrt{r_1 r_2}$$

Euclidean geometry lemma



$$|z_1 - z_2| = 2\sqrt{r_1 r_2}$$

Tangent horocircles

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Tangent horocircles

$$|z_1 - z_2| = 2\sqrt{r_1 r_2}$$

$$\left|\frac{A_n}{B_n} - \frac{A_{n-1}}{B_{n-1}}\right|$$

Tangent horocircles

$$|z_1 - z_2| = 2\sqrt{r_1 r_2}$$

$$\left|\frac{A_n}{B_n} - \frac{A_{n-1}}{B_{n-1}}\right| = \left|\frac{A_n B_{n-1} - B_n A_{n-1}}{B_n B_{n-1}}\right|$$

Tangent horocircles

$$|z_1 - z_2| = 2\sqrt{r_1 r_2}$$

$$\left|\frac{A_n}{B_n} - \frac{A_{n-1}}{B_{n-1}}\right| = \left|\frac{A_n B_{n-1} - B_n A_{n-1}}{B_n B_{n-1}}\right| = \frac{1}{B_n B_{n-1}}$$

Tangent horocircles

$$|z_1 - z_2| = 2\sqrt{r_1 r_2}$$

$$\left|\frac{A_n}{B_n} - \frac{A_{n-1}}{B_{n-1}}\right| = \left|\frac{A_n B_{n-1} - B_n A_{n-1}}{B_n B_{n-1}}\right| = \frac{1}{B_n B_{n-1}} = 2\sqrt{\left(\frac{1}{2B_n^2}\right)\left(\frac{1}{2B_{n-1}^2}\right)}$$

Tangent horocircles

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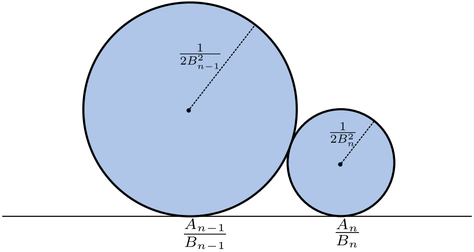
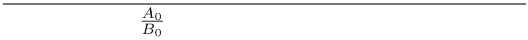


Shrinking horocircles



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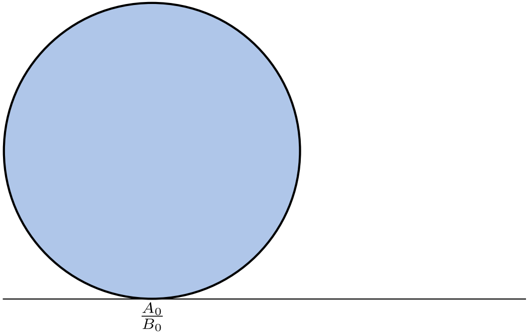

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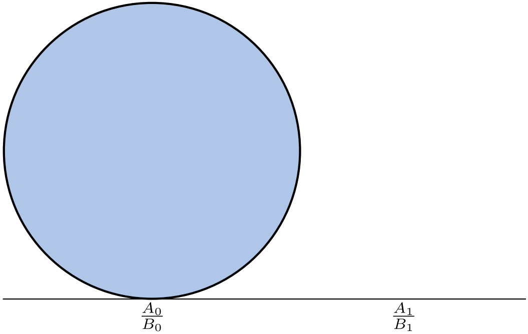

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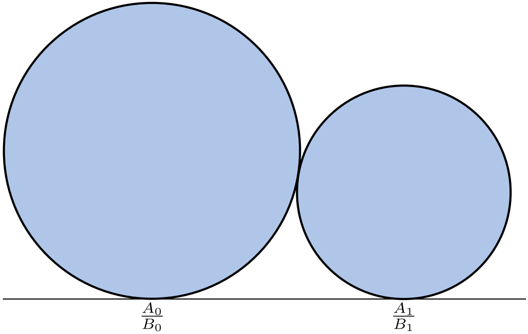

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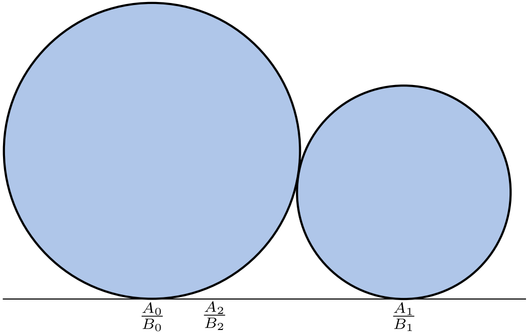

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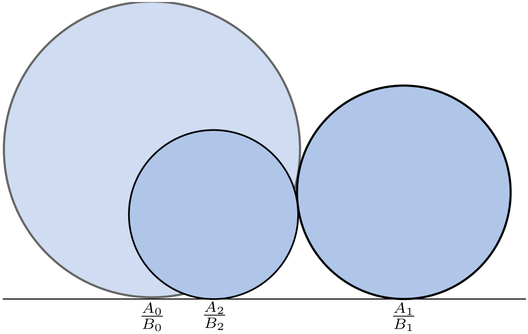

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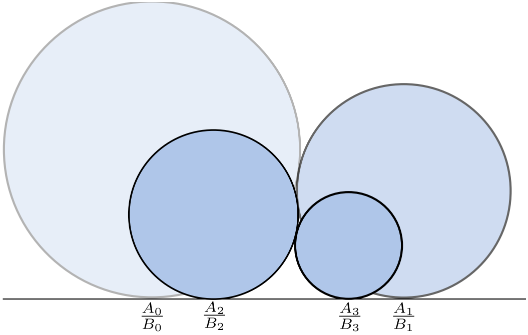

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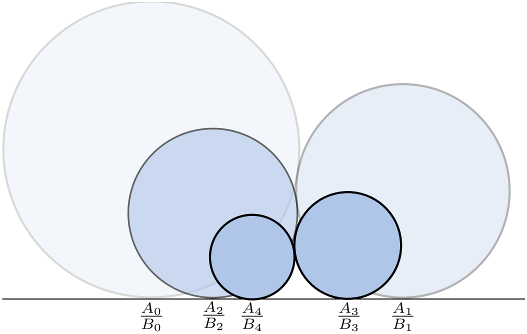

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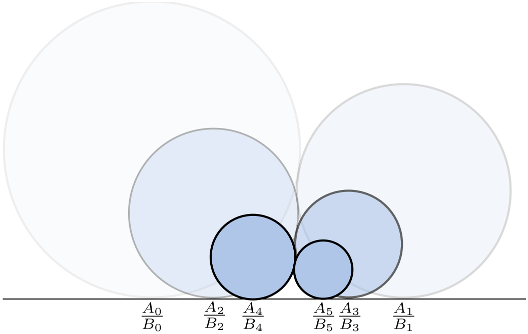

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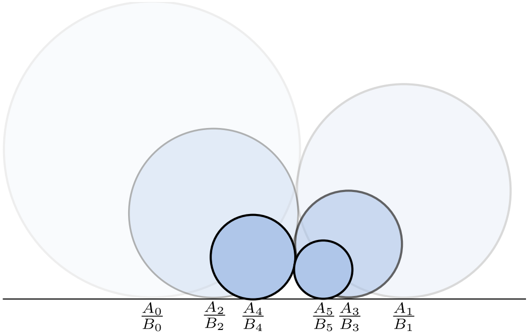
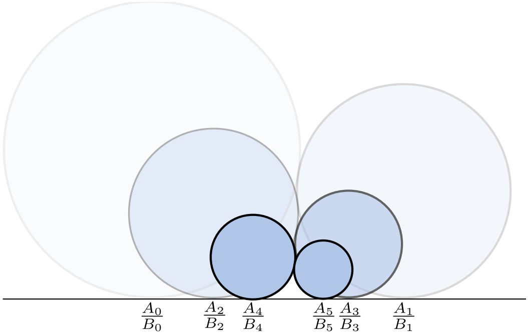


Radii $1/(2B_n^2)$ *decreasing*

Shrinking horocircles



Radii $1/(2B_n^2)$ *decreasing*, B_n *increasing*.

Reminder

$$\begin{pmatrix} A_{n-1} & A_n \\ B_{n-1} & B_n \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & b_1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & b_2 \end{pmatrix} \cdots \begin{pmatrix} 0 & 1 \\ 1 & b_n \end{pmatrix}$$

Reminder

$$\begin{pmatrix} A_{n-1} & A_n \\ B_{n-1} & B_n \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & b_1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & b_2 \end{pmatrix} \cdots \begin{pmatrix} 0 & 1 \\ 1 & b_n \end{pmatrix}$$

$$|A_n B_{n-1} - A_{n-1} B_n| = 1$$

Reminder

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$$|A_n B_{n-1} - A_{n-1} B_n| = 1$$

$$A_n = b_n A_{n-1} + A_{n-2}$$

Reminder

$$\begin{pmatrix} A_{n-1} & A_n \\ B_{n-1} & B_n \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & b_1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & b_2 \end{pmatrix} \cdots \begin{pmatrix} 0 & 1 \\ 1 & b_n \end{pmatrix}$$

$$|A_n B_{n-1} - A_{n-1} B_n| = 1$$

$$A_n = b_n A_{n-1} + A_{n-2} \qquad B_n = b_n B_{n-1} + B_{n-2}$$

Reminder

$$\begin{pmatrix} A_{n-1} & A_n \\ B_{n-1} & B_n \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & b_1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & b_2 \end{pmatrix} \cdots \begin{pmatrix} 0 & 1 \\ 1 & b_n \end{pmatrix}$$

$$|A_n B_{n-1} - A_{n-1} B_n| = 1$$

$$A_n = b_n A_{n-1} + A_{n-2} \qquad B_n = b_n B_{n-1} + B_{n-2}$$

$$\left| \frac{A_n}{B_n} - \frac{A_{n-2}}{B_{n-2}} \right|$$

Reminder

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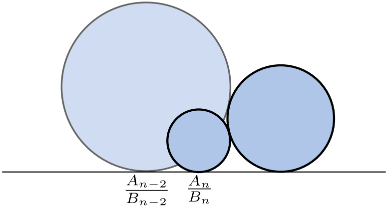
Proof of Seidel–Stern Theorem



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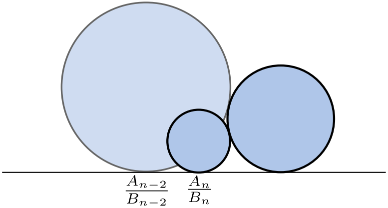

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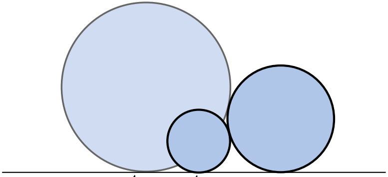

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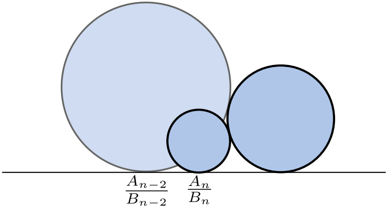

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If $\sum_n b_n$ diverges then $B_n \nearrow \infty$, so $\mathbf{K}(b_n)$ converges.

Hyperbolic geometry

Möbius transformations

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$$\mathbb{C} \longleftrightarrow \mathbb{R}^2 \times \{0\}$$

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Horosphere maps

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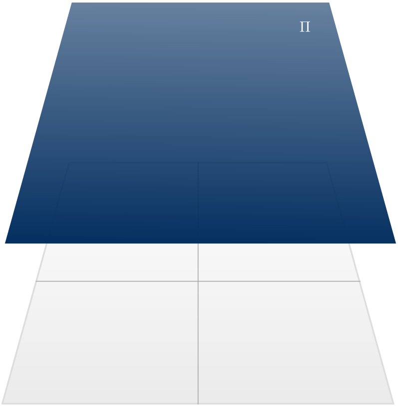

$$\Pi = \{(x_1, x_2, x_3)\ :\ x_3 = 1\}$$

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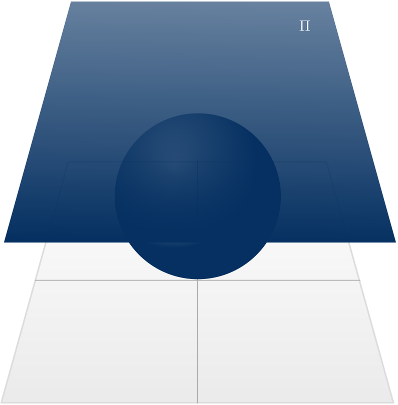

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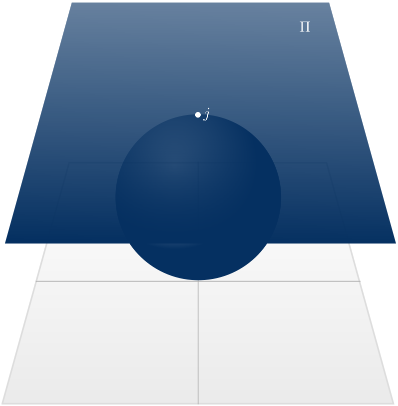

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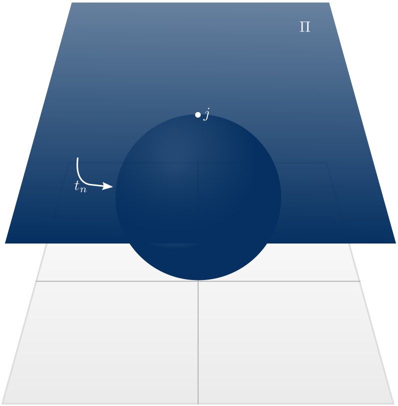

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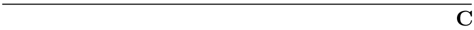

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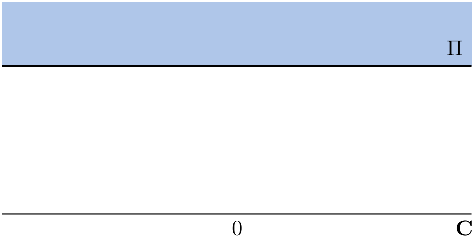

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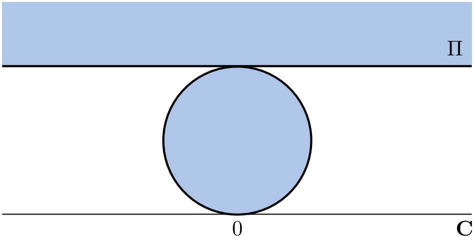

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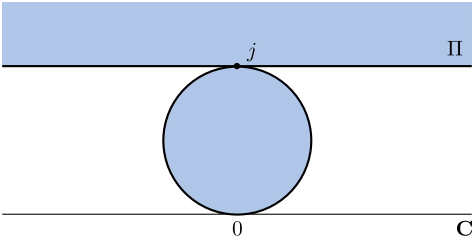

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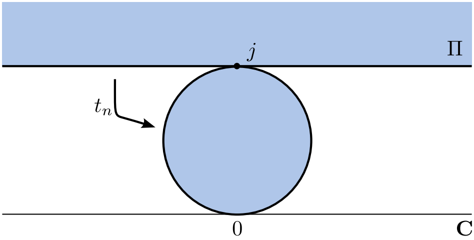

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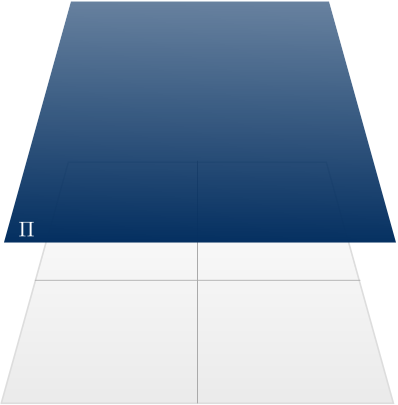

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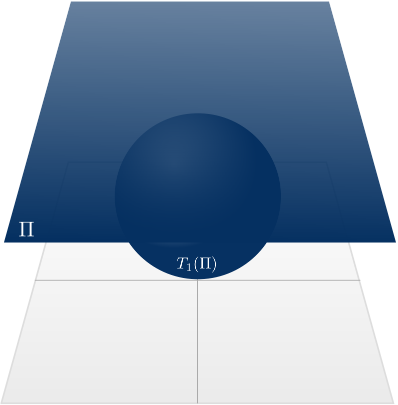

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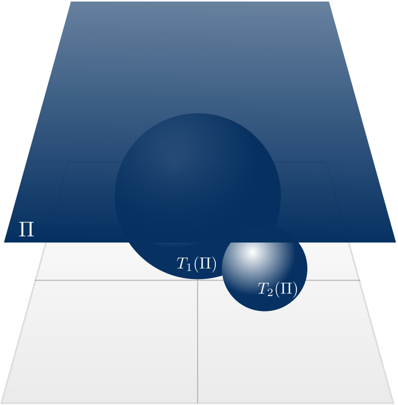

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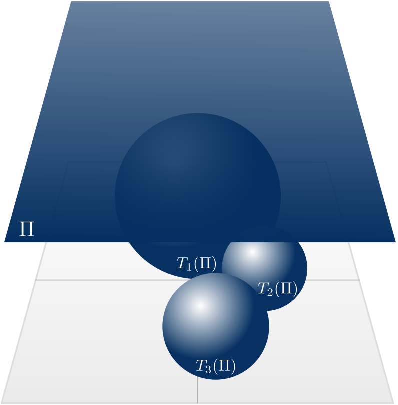

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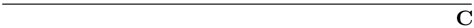

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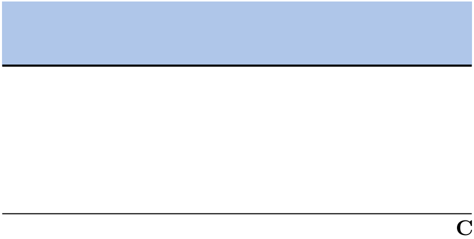

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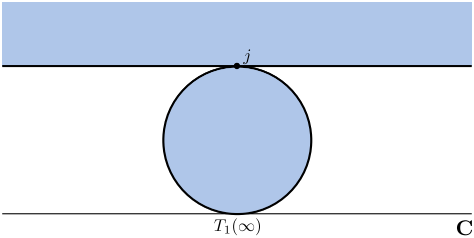

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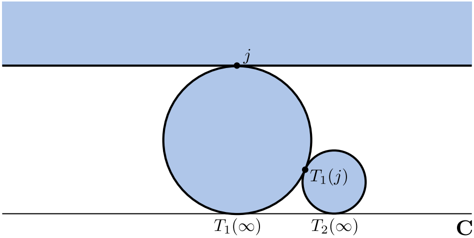

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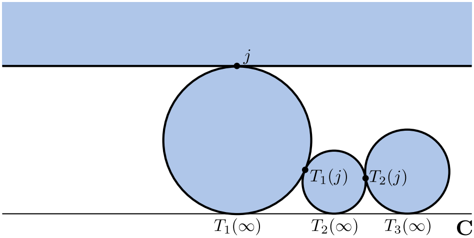

Horosphere $T_n(\Pi)$

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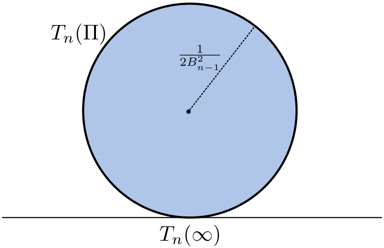

Back to the Seidel–Stern Theorem

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Publication

Continued fractions, discrete groups and complex dynamics

Alan Beardon

Computational Methods and Function Theory, 1 (2001)

Proof (Beardon)

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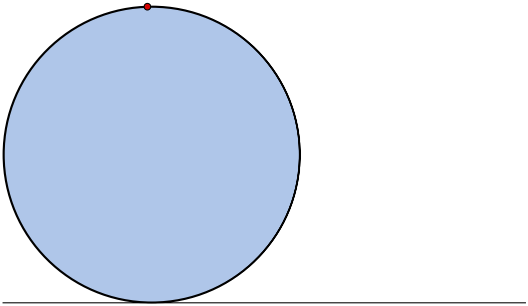
Never reach the boundary



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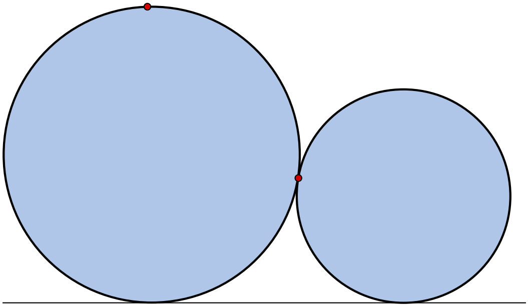

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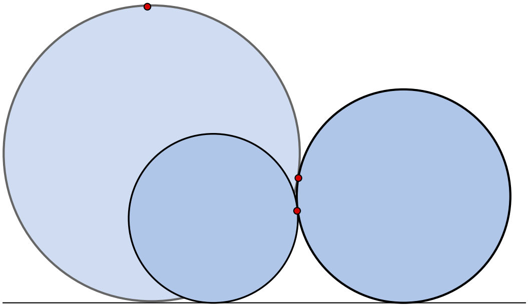

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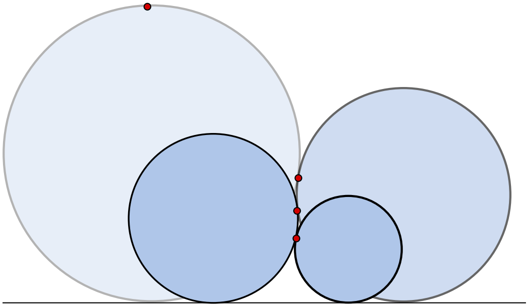

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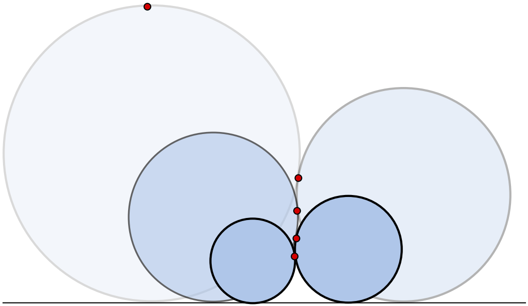

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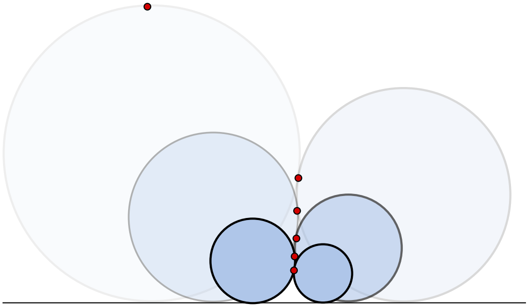

Concluding remarks

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Hence

$$T_{2n}(\infty) \to g(\infty), \qquad T_{2n+1}(\infty) \to gs(\infty) = g(0).$$

Publication

The hyperbolic geometry of continued fractions $\mathbf{K}(1|\,b_n)$

Ian Short

Annales Academiæ Scientiarum Fennicæ Mathematica, 31 (2006)

$$\cfrac{1}{T + \cfrac{1}{H + \cfrac{1}{A + \cfrac{1}{N + \cfrac{1}{K + \cfrac{1}{Y + \cfrac{1}{O + \cfrac{1}{U + \cfrac{1}{!}}}}}}}}}$$