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Hyperbolic geometry and continued fraction theory I

Ian Short 9 February 2010



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Collaborators

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Hyperbolic geometry

TOPLOGICAL GROUPS

Collaborators

Meira Hockman

Hyperbolic geometry

TOPLOGICAL GROUPS

Collaborators

Meira Hockman

Alan Beardon (University of Cambridge)

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Hyperbolic geometry

TOPLOGICAL GROUPS

Project

The geometry of continued fractions

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Hyperbolic geometry

TOPLOGICAL GROUPS

Project

The geometry of continued fractions

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Continued fractions	Möbius maps	Hyperbolic geometry	TOPLOGICAL GROUPS
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CONTINUED FRACTIONS

 $\mathbf{K}(a_n | b_n) = \frac{a_1}{b_1 + \frac{a_2}{b_2 + \frac{a_3}{b_3 + \frac{a_4}{b_4 + \dots}}}}$

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Continued fractions Möbi	us maps Hyperboli	C GEOMETRY TOPLO	GICAL GROUPS
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$$\frac{a_1}{b_1},$$

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Continued fractions	Möbius maps	Hyperbolic geometry	TOPLOGICAL GROUPS
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$$\frac{a_1}{b_1}, \frac{a_1}{b_1 + \frac{a_2}{b_2}},$$

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Continued fractions	Möbius maps	Hyperbolic geometry	TOPLOGICAL GROUPS
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$$\frac{a_1}{b_1}, \frac{a_1}{b_1 + \frac{a_2}{b_2}}, \frac{a_1}{b_1 + \frac{a_2}{b_2 + \frac{a_3}{b_3}}},$$

Continued fractions	Möbius maps	Hyperbolic geometry	TOPLOGICAL GROUP
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$$\frac{a_1}{b_1}, \frac{a_1}{b_1 + \frac{a_2}{b_2}}, \frac{a_1}{b_1 + \frac{a_2}{b_2 + \frac{a_3}{b_3}}}, \frac{a_1}{b_1 + \frac{a_2}{b_2 + \frac{a_3}{b_3}}}, \dots$$

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Continued fractions	Möbius maps	Hyperbolic geometry	Toplogical groups
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EXPANSION OF e = 2.71828182845905... (EULER)



Continued fractions	Möbius maps	Hyperbolic geometry	Toplogical groups
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EXPANSION OF $\pi = 3.14159265358979...$ (Lange)



Continued fractions	Möbius maps	Hyperbolic geometry	Toplogical groups
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A problematic example



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Continued fractions	Möbius maps	Hyperbolic geometry	TOPLOGICAL GROUPS
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$$t_n(z) = \frac{a_n}{b_n + z}$$

Continued fractions	Möbius maps	Hyperbolic geometry	TOPLOGICAL GROUPS
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$$t_n(z) = \frac{a_n}{b_n + z}$$

$$T_n = t_1 \circ t_2 \circ \cdots \circ t_n$$

Continued fractions	Möbius maps	Hyperbolic geometry	TOPLOGICAL GROUPS
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$$t_n(z) = \frac{a_n}{b_n + z}$$

$$T_n = t_1 \circ t_2 \circ \cdots \circ t_n$$

$$t_n(\infty) = 0$$

Continued fractions	Möbius maps	Hyperbolic geometry	TOPLOGICAL GROUPS
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$$t_n(z) = \frac{a_n}{b_n + z}$$

$$T_n = t_1 \circ t_2 \circ \cdots \circ t_n$$

$$t_n(\infty) = 0$$

$$T_n(\infty) = T_{n-1}(0)$$

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Convergence again

Convergence of $\mathbf{K}(a_n | b_n)$ equivalent to convergence of $T_1(0), T_2(0), \dots$



Continued fractions	Μ
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Hyperbolic geometry

TOPLOGICAL GROUPS

Dealing with ∞

Previously $\frac{1}{\infty} = 0.$

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Dealing with ∞

Previously $\frac{1}{\infty} = 0.$ Now $h(z) = \frac{1}{z} \qquad h(\infty) = 0.$

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Continued fractions	Möbius maps	Hyperbolic geometry	TOPLOGICAL GROUPS
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THREE MORE REASONS FOR USING MÖBIUS MAPS

Continued fractions	Möbius maps	Hyperbolic geometry	TOPLOGICAL GROUPS
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THREE MORE REASONS FOR USING MÖBIUS MAPS

$\circ~$ There is already a well developed theory of Möbius maps.

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THREE MORE REASONS FOR USING MÖBIUS MAPS

There is already a well developed theory of Möbius maps. Allows us to bring in geometry.

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Three more reasons for using Möbius maps

 $\circ~$ There is already a well developed theory of Möbius maps.

- Allows us to bring in geometry.
- Simpler notation: composition of maps rather than algebraic manipulation.

Continued fractions	Möbius maps	Hyperbolic geometry	TOPLOGICAL GROUPS
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Schematic diagram



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Continued fractions	Möbius maps	Hyperbolic geometry	Toplogical groups
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Reminder

$$t_n(z) = \frac{a_n}{b_n + z}$$

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Möbius maps	Hyperbolic geometry	Toplogical groups
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Reminder

$$t_n(z) = \frac{a_n}{b_n + z}$$

$$T_n = t_1 \circ t_2 \circ \cdots \circ t_n$$

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Continued fractions	Möbius maps	Hyperbolic geometry	Toplogical groups
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Continued fractions	Möbius maps	Hyperbolic geometry	TOPLOGICAL GROUPS
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 $\mathbf{K}(1|b_n) \qquad b_n \in \mathbb{N}$

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Continued fractions	Möbius maps	Hyperbolic geometry	Toplogical groups
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$$\mathbf{K}(1|b_n) \qquad b_n \in \mathbb{N}$$
$$t_n(z) = \frac{1}{b_n + z}$$

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Continued fractions	Möbius maps	Hyperbolic geometry	TOPLOGICAL GROUPS
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$$\mathbf{K}(1|b_n) \qquad b_n \in \mathbb{N}$$
$$t_n(z) = \frac{1}{b_n + z}$$

The modular group

$$\Gamma = \left\{ z \mapsto \frac{az+b}{cz+d} : a, b, c, d \in \mathbb{Z}, ad-bc = 1 \right\}$$

Continued fractions	Möbius maps	Hyperbolic geometry	TOPLOGICAL GROUPS
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COMPLEX CONTINUED FRACTIONS

Continued fractions	Möbius maps	Hyperbolic geometry	TOPLOGICAL GROUPS
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COMPLEX CONTINUED FRACTIONS

$$\mathbf{K}(a_n | b_n) \qquad a_n, b_n \in \mathbb{C}$$

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Continued fractions	Möbius maps	Hyperbolic geometry	TOPLOGICAL GROUPS
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COMPLEX CONTINUED FRACTIONS

$$\mathbf{K}(a_n | b_n) \qquad a_n, b_n \in \mathbb{C}$$
$$t_n(z) = \frac{a_n}{b_n + z}$$

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COMPLEX CONTINUED FRACTIONS

$$\mathbf{K}(a_n | b_n) \qquad a_n, b_n \in \mathbb{C}$$

$$t_n(z) = \frac{a_n}{b_n + z}$$

The Möbius group

$$\mathcal{M} = \left\{ z \mapsto \frac{az+b}{cz+d} : a, b, c, d \in \mathbb{C}, \, ad-bc \neq 0 \right\}$$

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${\mathcal M}$ generated by inversions

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${\mathcal M}$ generated by inversions

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Two aspects of \mathcal{M}

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Two aspects of \mathcal{M}

$\circ~$ Three-dimensional hyperbolic isometries

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Two aspects of \mathcal{M}

- $\circ~$ Three-dimensional hyperbolic isometries
- Topological group (and complete metric space)

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SCHEMATIC DIAGRAM



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Schematic diagram



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THE STERN-STOLZ THEOREM

Theorem. If $\sum_{n} |b_n|$ converges then $\mathbf{K}(1|b_n)$ diverges.

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Open Problem I

Open Problem I

Interpret each result on complex continued fractions in terms of (a) hyperbolic geometry, and (b) topological group theory.

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Hyperbolic geometry

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Schematic diagram



Continued fractions	Möbius maps	Hyperbolic geometry	Toplogical groups
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UPPER HALF-SPACE

$$\mathbb{H}^3 = \{ (x_1, x_2, x_3) \in \mathbb{R}^3 : x_3 > 0 \}$$

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UPPER HALF-SPACE

$$\mathbb{H}^3 = \{ (x_1, x_2, x_3) \in \mathbb{R}^3 : x_3 > 0 \}$$
$$\mathbb{C} \longleftrightarrow \{ (x_1, x_2, 0) \in \mathbb{R}^3 : x_1, x_2 \in \mathbb{R} \}$$

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UPPER HALF-SPACE



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 \mathbf{H}^3

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 (\mathbb{H}^3,ρ)

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(\mathbb{H}^3,ρ)

$$\sinh \frac{1}{2}\rho(x,y) = \frac{|x-y|}{2\sqrt{x_3y_3}}$$

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GRAPH REPRESENTATION OF HYPERBOLIC SPACE



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GRAPH REPRESENTATION OF HYPERBOLIC SPACE



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GRAPH REPRESENTATION OF HYPERBOLIC SPACE



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MÖBIUS ACTION ON HYPERBOLIC SPACE



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TOPLOGICAL GROUPS

MÖBIUS ACTION ON HYPERBOLIC SPACE



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ISOMETRY GROUP

 $\mathrm{Isom}(\mathbb{H}^3)=\mathcal{M}$

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BACK TO CONTINUED FRACTIONS

$$t_n(\infty) = 0 \qquad T_n(\infty) = T_{n-1}(0)$$

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Convergence

Suppose $\mathbf{K}(a_n | b_n)$ converges.

Continued	FRACTIONS
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Möbius maps 000000000 Hyperbolic geometry

Toplogical groups

Convergence

Suppose $\mathbf{K}(a_n | b_n)$ converges.

In other words, suppose $T_1(0), T_2(0), \ldots$ converges.



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Convergence

If

 $T_n(0) \to p$

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Convergence

If

 $T_n(0) \to p$

then

 $T_n(j) \to p.$

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Continued fractions	Möbius maps	Hyperbolic geometry	TOPLOGICAL GROUPS
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Convergence

If

$$T_n(0) \to p$$

then

$$T_n(j) \to p.$$

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(Lorentzen, Aebischer, Beardon)

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RECALL THE STERN-STOLZ THEOREM

Theorem. If $\sum_{n} |b_n|$ converges then $\mathbf{K}(1|b_n)$ diverges.

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RECALL THE HYPERBOLIC METRIC

$$\sinh \frac{1}{2}\rho(x,y) = \frac{|x-y|}{2\sqrt{x_3y_3}}$$

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HYPERBOLIC DISTANCE CALCULATIONS

$\rho(j, t_n(j))$

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HYPERBOLIC DISTANCE CALCULATIONS

$$\rho(j, t_n(j)) = \rho(h(j), ht_n(j))$$

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$$\rho(j, t_n(j)) = \rho(h(j), ht_n(j))$$
$$= \rho(j, b_n + j)$$

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$$\rho(j, t_n(j)) = \rho(h(j), ht_n(j))$$
$$= \rho(j, b_n + j)$$

$$\sinh \frac{1}{2}\rho(j, t_n(j)) = \frac{|b_n|}{2}$$

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$$\rho(j, t_n(j)) = \rho(h(j), ht_n(j))$$
$$= \rho(j, b_n + j)$$

$$\sinh \frac{1}{2}\rho(j, t_n(j)) = \frac{|b_n|}{2}$$

$$\rho(j, t_n(j)) = \rho(T_{n-1}(j), T_n(j))$$

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$$\sinh \frac{1}{2}\rho(T_{n-1}(j), T_n(j)) = \frac{|b_n|}{2}$$

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Continued fractions	Möbius maps	Hyperbolic geometry	TOPLOGICAL GROUPS
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$$\sinh \frac{1}{2}\rho(T_{n-1}(j), T_n(j)) = \frac{|b_n|}{2}$$

Hence if $\sum_{n} |b_n| < +\infty$ then

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$$\begin{split} \sinh \frac{1}{2}\rho(T_{n-1}(j),T_n(j)) &= \frac{|b_n|}{2} \end{split}$$
 Hence if $\sum_n |b_n| < +\infty$ then

$$\sum_n \rho(T_{n-1}(j),T_n(j)) < +\infty$$

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CANNOT REACH THE BOUNDARY (BEARDON)



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Open Problem II

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Open Problem II

What is the geometric significance of the *argument* of b_n to the orbit $T_1(j), T_2(j), \ldots$?

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TOPLOGICAL GROUPS

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Schematic diagram



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Möbius group

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Möbius group

• Group of hyperbolic isometries of \mathbb{H}^3 .

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MÖBIUS GROUP

- Group of hyperbolic isometries of \mathbb{H}^3 .
- Group of conformal automorphisms of \mathbb{C}_{∞} .

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STEREOGRAPHIC PROJECTION



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The chordal metric

$$\chi(w,z) = \frac{2|w-z|}{\sqrt{1+|w|^2}\sqrt{1+|z|^2}} \qquad \chi(w,\infty) = \frac{2}{\sqrt{1+|w|^2}}$$

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The chordal metric



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The supremum metric

$$\chi_0(f,g) = \sup_{z \in \mathbb{C}_{\infty}} \chi(f(z),g(z))$$
$$f,g \in \mathcal{M}$$

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The supremum metric

$$\chi_0(f,g) = \sup_{z \in \mathbb{C}_{\infty}} \chi(f(z),g(z))$$
$$f,g \in \mathcal{M}$$

The metric of uniform convergence.

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Möbius group

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Möbius group

•
$$(\mathcal{M}, \chi_0)$$
 is a complete metric space

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Möbius group

(M, χ₀) is a complete metric space
(M, χ₀) is a topological group
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Möbius group

- $\circ (\mathcal{M}, \chi_0)$ is a complete metric space
- $\circ (\mathcal{M}, \chi_0)$ is a topological group
- \circ right-invariant: $\chi_0(fk,gk) = \chi_0(f,g)$

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Möbius group

- $\circ (\mathcal{M}, \chi_0)$ is a complete metric space
- $\circ (\mathcal{M}, \chi_0)$ is a topological group
- $\circ \ {\rm right-invariant:} \ \chi_0(fk,gk) = \chi_0(f,g)$
- $\circ\ h(z)=1/z$ is a chordal isometry: $\chi_0(hf,hg)=\chi_0(f,g)$

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RECALL THE STERN-STOLZ THEOREM

Theorem. If $\sum_{n} |b_n|$ converges then $\mathbf{K}(1|b_n)$ diverges.

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KEY OBSERVATION

If b_n small then

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KEY OBSERVATION

If b_n small then

$$t_n(z) = \frac{1}{b_n + z} \sim h(z) = \frac{1}{z}.$$

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KEY OBSERVATION

If b_n small then

$$t_n(z) = \frac{1}{b_n + z} \sim h(z) = \frac{1}{z}.$$

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We must calculate $\chi_0(t_n, h)$.

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TOPLOGICAL GROUPS

Calculate $\chi_0(t_n, h)$

 $\chi_0(t_n,h)$



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TOPLOGICAL GROUPS

Calculate $\chi_0(t_n, h)$

$$\chi_0(t_n,h) = \chi_0(ht_n,I)$$

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TOPLOGICAL GROUPS

Calculate $\chi_0(t_n, h)$

$$\chi_0(t_n, h) = \chi_0(ht_n, I)$$
$$= \chi_0(z + b_n, z)$$

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Calculate $\chi_0(t_n, h)$

$$\begin{split} \chi_0(t_n, h) &= \chi_0(ht_n, I) \\ &= \chi_0(z + b_n, z) \\ &= \sup_{z \in \mathbb{C}_\infty} \frac{2|b_n|}{\sqrt{1 + |z|^2}\sqrt{1 + |z + b_n|^2}} \end{split}$$

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Calculate $\chi_0(t_n, h)$

$$\begin{split} \chi_0(t_n,h) &= \chi_0(ht_n,I) \\ &= \chi_0(z+b_n,z) \\ &= \sup_{z \in \mathbb{C}_{\infty}} \frac{2|b_n|}{\sqrt{1+|z|^2}\sqrt{1+|z+b_n|^2}} \\ &\leq 2|b_n| \end{split}$$

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TOPLOGICAL GROUPS

CALCULATE $\chi_0(T_n^{-1}, T_{n+2}^{-1})$

$\chi_0(T_n^{-1}, T_{n+2}^{-1})$

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TOPLOGICAL GROUPS

Calculate $\chi_0(T_n^{-1}, T_{n+2}^{-1})$

$$\chi_0(T_n^{-1}, T_{n+2}^{-1}) = \chi_0(t_{n+1}t_{n+2}, I)$$

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TOPLOGICAL GROUPS

Calculate $\chi_0(T_n^{-1}, T_{n+2}^{-1})$

$$\chi_0(T_n^{-1}, T_{n+2}^{-1}) = \chi_0(t_{n+1}t_{n+2}, I)$$
$$= \chi_0(ht_{n+1}t_{n+2}, h)$$

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Calculate $\chi_0(T_n^{-1}, T_{n+2}^{-1})$

$$\chi_0(T_n^{-1}, T_{n+2}^{-1}) = \chi_0(t_{n+1}t_{n+2}, I)$$

= $\chi_0(ht_{n+1}t_{n+2}, h)$
 $\leq \chi_0(ht_{n+1}t_{n+2}, t_{n+2}) + \chi_0(t_{n+2}, h)$

Calculate $\chi_0(T_n^{-1}, T_{n+2}^{-1})$

$$\begin{split} \chi_0(T_n^{-1}, T_{n+2}^{-1}) &= \chi_0(t_{n+1}t_{n+2}, I) \\ &= \chi_0(ht_{n+1}t_{n+2}, h) \\ &\leq \chi_0(ht_{n+1}t_{n+2}, t_{n+2}) + \chi_0(t_{n+2}, h) \\ &= \chi_0(t_{n+1}, h) + \chi_0(t_{n+2}, h) \end{split}$$

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Calculate $\chi_0(T_n^{-1}, T_{n+2}^{-1})$

$$\chi_0(T_n^{-1}, T_{n+2}^{-1}) = \chi_0(t_{n+1}t_{n+2}, I)$$

= $\chi_0(ht_{n+1}t_{n+2}, h)$
 $\leq \chi_0(ht_{n+1}t_{n+2}, t_{n+2}) + \chi_0(t_{n+2}, h)$
= $\chi_0(t_{n+1}, h) + \chi_0(t_{n+2}, h)$
 $\leq 2|b_{n+1}| + 2|b_{n+2}|$

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If $\sum_{n} |b_n| < +\infty$ then

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If $\sum_{n} |b_n| < +\infty$ then

$$\sum_{n} \chi_0(T_n^{-1}, T_{n+2}^{-1}) < +\infty.$$

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If $\sum_{n} |b_n| < +\infty$ then

$$\sum_{n} \chi_0(T_n^{-1}, T_{n+2}^{-1}) < +\infty.$$

Hence T_{2n-1}^{-1} converges uniformly to a Möbius map f.

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If $\sum_{n} |b_n| < +\infty$ then

$$\sum_{n} \chi_0(T_n^{-1}, T_{n+2}^{-1}) < +\infty.$$

Hence T_{2n-1}^{-1} converges uniformly to a Möbius map f.

Hence T_{2n-1} converges uniformly to a Möbius map g.

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If $\sum_{n} |b_n| < +\infty$ then

$$\sum_{n} \chi_0(T_n^{-1}, T_{n+2}^{-1}) < +\infty.$$

Hence T_{2n-1}^{-1} converges uniformly to a Möbius map f.

Hence T_{2n-1} converges uniformly to a Möbius map g.

Hence $T_{2n} = T_{2n-1}t_{2n}$ converges uniformly to gh, where h(z) = 1/z.

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In summary, if $\sum_n |b_n| < +\infty$ then there is a Möbius map g such that

$$T_{2n-1} \to g \qquad T_{2n} \to gh.$$

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In summary, if $\sum_n |b_n| < +\infty$ then there is a Möbius map g such that

$$T_{2n-1} \to g \qquad T_{2n} \to gh.$$

Hence

$$T_{2n-1}(0) \to g(0)$$
 $T_{2n}(0) \to gh(0) = g(\infty).$

Thank you!

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