Hyperbolic geometry and continued fraction theory I

Ian Short 9 February 2010

http://maths.org/ims25/maths/presentations.php
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The geometry of continued fractions
The geometry of continued fractions

http://maths.org/ims25/maths/presentations.php
Continued fractions
Continued fractions

\[ K(a_n | b_n) = \frac{a_1}{b_1 + \frac{a_2}{b_2 + \frac{a_3}{b_3 + \frac{a_4}{b_4 + \cdots}}}} \]
CONTINUED FRACTION CONVERGENCE

\[
\frac{a_1}{b_1},
\]
CONTINUED FRACTION CONVERGENCE

\[ \frac{a_1}{b_1}, \frac{a_1}{b_1 + \frac{a_2}{b_2}}, \ldots \]
CONTINUED FRACTION CONVERGENCE

\[
\frac{a_1}{b_1}, \quad \frac{a_1}{b_1 + \frac{a_2}{b_2}}, \quad \frac{a_1}{b_1 + \frac{a_2}{b_2 + \frac{a_3}{b_3}}},
\]
Continued fraction convergence

\[
\frac{a_1}{b_1}, \quad \frac{a_1}{b_1 + \frac{a_2}{b_2}}, \quad \frac{a_1}{b_1 + \frac{a_2}{b_2 + \frac{a_3}{b_3}}}, \quad \frac{a_1}{b_1 + \frac{a_2}{b_2 + \frac{a_3}{b_3 + \frac{a_4}{b_4}}}}, \ldots
\]
EXPANSION OF $e = 2.71828182845905\ldots$ (EULER)

\[
e = 2 + \cfrac{1}{1 + 1 + \cfrac{1}{2 + 1 + \cfrac{1}{1 + 1 + \cfrac{1}{1 + 1 + \cfrac{1}{4 + 1 + \cfrac{1}{1 + 1 + \cfrac{1}{1 + 1 + \cfrac{1}{6 + 1 + \cfrac{1}{1 + 1 + \cfrac{1}{8 + \cdots}}}}}}}}}
\]
**Expansion of** $\pi = 3.14159265358979\ldots$ (Lange)

$$\pi = 3 + \cfrac{1^2}{6 + \cfrac{3^2}{6 + \cfrac{5^2}{6 + \cfrac{7^2}{6 + \cfrac{9^2}{6 + \cdots}}}}}$$
A problematic example

\[
\frac{1}{2 + \frac{1}{3 + \frac{1}{1 + \frac{1}{-1 + \cdots}}}}
\]
Möbius transformations

\[ t_n(z) = \frac{a_n}{b_n + z} \]
Möbius transformations

\[ t_n(z) = \frac{a_n}{b_n + z} \]

\[ T_n = t_1 \circ t_2 \circ \cdots \circ t_n \]
Möbius transformations

\[ t_n(z) = \frac{a_n}{b_n + z} \]

\[ T_n = t_1 \circ t_2 \circ \cdots \circ t_n \]

\[ t_n(\infty) = 0 \]
Möbius transformations

\[ t_n(z) = \frac{a_n}{b_n + z} \]

\[ T_n = t_1 \circ t_2 \circ \cdots \circ t_n \]

\[ t_n(\infty) = 0 \]

\[ T_n(\infty) = T_{n-1}(0) \]
Convergence again

Convergence of $\begin{pmatrix} a_n \vert b_n \end{pmatrix}$ equivalent to convergence of $T_1(0), T_2(0), \ldots$. 
Dealing with $\infty$

Previously

$$\frac{1}{\infty} = 0.$$
Dealing with $\infty$

Previously

\[
\frac{1}{\infty} = 0.
\]

Now

\[
h(z) = \frac{1}{z} \quad h(\infty) = 0.
\]
Three more reasons for using Möbius maps

◦ There is already a well developed theory of Möbius maps.
◦ Allows us to bring in geometry.
◦ Simpler notation: composition of maps rather than algebraic manipulation.
Three more reasons for using Möbius maps

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THREE MORE REASONS FOR USING MÖBIUS MAPS

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○ Simpler notation: composition of maps rather than algebraic manipulation.
SCHEMATIC DIAGRAM

continued fractions

Möbius maps
Möbius maps


**Reminder**

\[ t_n(z) = \frac{a_n}{b_n + z} \]
Reminder

\[ t_n(z) = \frac{a_n}{b_n + z} \]

\[ T_n = t_1 \circ t_2 \circ \cdots \circ t_n \]
Usual integer continued fractions
Usual integer continued fractions

\[ K(1 \mid b_n) \quad b_n \in \mathbb{N} \]
Usual integer continued fractions

\[ K(1 \| b_n) \quad b_n \in \mathbb{N} \]

\[ t_n(z) = \frac{1}{b_n + z} \]
Usual integer continued fractions

\[ K(1 \mid b_n) \quad b_n \in \mathbb{N} \]

\[ t_n(z) = \frac{1}{b_n + z} \]

The modular group

\[ \Gamma = \left\{ z \mapsto \frac{az + b}{cz + d} : a, b, c, d \in \mathbb{Z}, ad - bc = 1 \right\} \]
Complex continued fractions
**Complex continued fractions**

$$K(a_n|b_n) \quad a_n, b_n \in \mathbb{C}$$
Complex continued fractions

\[ \mathbf{K}(a_n \mid b_n) \quad a_n, b_n \in \mathbb{C} \]

\[ t_n(z) = \frac{a_n}{b_n + z} \]
Complex continued fractions

$K(a_n | b_n) \quad a_n, b_n \in \mathbb{C}$

$t_n(z) = \frac{a_n}{b_n + z}$

The Möbius group

$\mathcal{M} = \left\{ z \mapsto \frac{az + b}{cz + d} : a, b, c, d \in \mathbb{C}, ad - bc \neq 0 \right\}$
$M$ generated by inversions
\[ M \text{ generated by inversions} \]
TWO ASPECTS OF $\mathcal{M}$
Two aspects of $\mathcal{M}$

- Three-dimensional hyperbolic isometries
TWO ASPECTS OF $\mathcal{M}$

- Three-dimensional hyperbolic isometries
- Topological group (and complete metric space)
SCHEMATIC DIAGRAM

continued fractions

Möbius maps
**SCHEMATIC DIAGRAM**

- **continued fractions**
- **Möbius maps**
- **hyperbolic geometry**
- **topological groups**
The Stern–Stolz Theorem

Theorem. If $\sum_n |b_n|$ converges then $K(1| b_n)$ diverges.
Open Problem I
Open Problem I

Interpret each result on complex continued fractions in terms of (a) hyperbolic geometry, and (b) topological group theory.
HYPERBOLIC GEOMETRY
SCHEMATIC DIAGRAM

- Continued fractions
- Möbius maps
- Hyperbolic geometry
- Topological groups

Diagram:
- Continued fractions → Möbius maps
- Möbius maps
  - hyperbolic geometry
  - topological groups
**Upper half-space**

\[ \mathbb{H}^3 = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_3 > 0\} \]
**Upper half-space**

\[ \mathbb{H}^3 = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_3 > 0\} \]

\[ \mathbb{C} \leftrightarrow \{(x_1, x_2, 0) \in \mathbb{R}^3 : x_1, x_2 \in \mathbb{R}\} \]
Upper half-space

$\mathbb{H}^3$

$\mathbb{C}$
THREE-DIMENSIONAL HYPERBOLIC SPACE

\((\mathbb{H}^3, \rho)\)
Three-dimensional hyperbolic space

$$(\mathbb{H}^3, \rho)$$

$$\sinh \frac{1}{2} \rho(x, y) = \frac{|x - y|}{2 \sqrt{x_3 y_3}}$$
THREE-DIMENSIONAL HYPERBOLIC SPACE
THREE-DIMENSIONAL HYPERBOLIC SPACE
GRAPH REPRESENTATION OF HYPERBOLIC SPACE
GRAPH REPRESENTATION OF HYPERBOLIC SPACE
GRAPH REPRESENTATION OF HYPERBOLIC SPACE
Möbius action on hyperbolic space
Möbius action on hyperbolic space
Isometry group

\[ \text{Isom}(\mathbb{H}^3) = \mathcal{M} \]
Back to continued fractions

\[ t_n(\infty) = 0 \quad T_n(\infty) = T_{n-1}(0) \]
Convergence

Suppose $K(a_n | b_n)$ converges.
Suppose $\mathbf{K}(a_n \mid b_n)$ converges.

In other words, suppose $T_1(0), T_2(0), \ldots$ converges.
Convergence
Convergence
**Convergence**

\[ \infty \quad 0 \quad H^3 \quad \mathbb{C} \]
Convergence
Convergence
CONVERGENCE

\[ \infty \]

\[ \gamma \]

\[ T_1(\gamma) \]

\[ 0 \]

\[ T_1(0) \]

\[ C \]

\[ \mathbb{H}^3 \]
CONVERGENCE
Convergence
Convergence
Convergence
CONVERGENCE

If

\[ T_n(0) \to p \]
Convergence

If

\[ T_n(0) \rightarrow p \]

then

\[ T_n(j) \rightarrow p. \]
**Convergence**

If

\[ T_n(0) \rightarrow p \]

then

\[ T_n(j) \rightarrow p. \]

(Lorentzen, Aebischer, Beardon)
Recall the Stern–Stolz Theorem

**Theorem.** If $\sum_n |b_n|$ converges then $\mathbf{K}(1|b_n)$ diverges.
RECALL THE HYPERBOLIC METRIC

\[ \sinh \frac{1}{2} \rho(x, y) = \frac{|x - y|}{2 \sqrt{x^3 y^3}} \]
Hyperbolic distance calculations

$$\rho(j, t_n(j))$$
HYPERBOLIC DISTANCE CALCULATIONS

$$\rho(j, t_n(j)) = \rho(h(j), ht_n(j))$$
Hyperbolic distance calculations

\[ \rho(j, t_n(j)) = \rho(h(j), ht_n(j)) \]
\[ = \rho(j, b_n + j) \]
Hyperbolic distance calculations

\[ \rho(j, t_n(j)) = \rho(h(j), h t_n(j)) \]
\[ = \rho(j, b_n + j) \]

\[ \sinh \frac{1}{2} \rho(j, t_n(j)) = \frac{|b_n|}{2} \]
**Hyperbolic distance calculations**

\[
\rho(j, t_n(j)) = \rho(h(j), h t_n(j)) \\
= \rho(j, b_n + j)
\]

\[
\sinh \frac{1}{2} \rho(j, t_n(j)) = \frac{|b_n|}{2}
\]

\[
\rho(j, t_n(j)) = \rho(T_{n-1}(j), T_n(j))
\]
Hyperbolic distance calculations

\[ \sinh \frac{1}{2} \rho(T_{n-1}(j), T_n(j)) = \frac{|b_n|}{2} \]
**Hyperbolic distance calculations**

\[
\sinh \frac{1}{2} \rho(T_{n-1}(j), T_n(j)) = \frac{|b_n|}{2}
\]

Hence if \(\sum_n |b_n| < +\infty\) then
Hyperbolic distance calculations

\[ \sinh \frac{1}{2} \rho(T_{n-1}(j), T_n(j)) = \frac{|b_n|}{2} \]

Hence if \( \sum |b_n| < +\infty \) then

\[ \sum_n \rho(T_{n-1}(j), T_n(j)) < +\infty \]
CANNOT REACH THE BOUNDARY (Beardon)

\[ j \]

\[ T_1(j) \]

\[ T_2(j) \]

\[ T_3(j) \]
Open Problem II

What is the geometric significance of the argument of $b_n$ to the orbit $T_1(j), T_2(j), ...$?
Open Problem II

What is the geometric significance of the argument of $b_n$ to the orbit $T_1(j), T_2(j), \ldots$?
Topological groups
SCHEMATIC DIAGRAM

continued fractions

Möbius maps

hyperbolic geometry
topological groups
<table>
<thead>
<tr>
<th>CONTINUED FRACTIONS</th>
<th>MÖBIUS MAPS</th>
<th>HYPERBOLIC GEOMETRY</th>
<th>TOPOLOGICAL GROUPS</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**Möbius group**
Möbius group

- Group of hyperbolic isometries of $\mathbb{H}^3$. 
Möbius group

- Group of hyperbolic isometries of $\mathbb{H}^3$.
- Group of conformal automorphisms of $\mathbb{C}_\infty$. 
STEREOGRAPHIC PROJECTION
STEREOGRAPHIC PROJECTION
STEREOGRAPHIC PROJECTION
STEREOGRAPHIC PROJECTION
The chordal metric

\[ \chi(w, z) = \frac{2|w - z|}{\sqrt{1 + |w|^2} \sqrt{1 + |z|^2}} \quad \chi(w, \infty) = \frac{2}{\sqrt{1 + |w|^2}} \]
The chordal metric

\((C_\infty, \chi)\)
**The supremum metric**

\[ \chi_0(f, g) = \sup_{z \in \mathbb{C}_\infty} \chi(f(z), g(z)) \]

\[ f, g \in \mathcal{M} \]
The supremum metric

\[ \chi_0(f, g) = \sup_{z \in \mathbb{C}_\infty} \chi(f(z), g(z)) \]

\[ f, g \in \mathcal{M} \]

The metric of uniform convergence.
Möbius group
M"obius group

- $(\mathcal{M}, \chi_0)$ is a complete metric space
Möbius group

- $(\mathcal{M}, \chi_0)$ is a complete metric space
- $(\mathcal{M}, \chi_0)$ is a topological group
MöBIUS GROUP

- $(\mathcal{M}, \chi_0)$ is a complete metric space
- $(\mathcal{M}, \chi_0)$ is a topological group
- right-invariant: $\chi_0(fk, gk) = \chi_0(f, g)$
MöBIUS GROUP

- $(\mathcal{M}, \chi_0)$ is a complete metric space
- $(\mathcal{M}, \chi_0)$ is a topological group
- right-invariant: $\chi_0(fk, gk) = \chi_0(f, g)$
- $h(z) = 1/z$ is a chordal isometry: $\chi_0(hf, hg) = \chi_0(f, g)$
Recall the Stern–Stolz Theorem

**Theorem.** If $\sum_n |b_n|$ converges then $K(1| b_n)$ diverges.
Key observation

If $b_n$ small then

$$t_n(z) = 1 \frac{b_n + z}{z} \sim h(z) = 1 \frac{z}{z}.$$
**Key Observation**

If $b_n$ small then

$$ t_n(z) = \frac{1}{b_n + z} \sim h(z) = \frac{1}{z}. $$
**Key Observation**

If $b_n$ small then

$$t_n(z) = \frac{1}{b_n + z} \sim h(z) = \frac{1}{z}. $$

We must calculate $\chi_0(t_n, h)$. 
Calculate $\chi_0(t_n, h)$
Calculate $\chi_0(t_n, h)$

\[ \chi_0(t_n, h) = \chi_0(ht_n, I) \]
**CALCULATE** $\chi_0(t_n, h)$

\[
\chi_0(t_n, h) = \chi_0(ht_n, I) = \chi_0(z + b_n, z)
\]
**Calculate** $\chi_0(t_n, h)$

\[
\chi_0(t_n, h) = \chi_0(ht_n, I) \\
= \chi_0(z + b_n, z) \\
= \sup_{z \in \mathbb{C}_\infty} \frac{2|b_n|}{\sqrt{1 + |z|^2} \sqrt{1 + |z + b_n|^2}}
\]
Calculate $\chi_0(t_n, h)$

\[
\chi_0(t_n, h) = \chi_0(ht_n, I) \\
= \chi_0(z + b_n, z) \\
= \sup_{z \in \mathbb{C}_\infty} \frac{2|b_n|}{\sqrt{1 + |z|^2} \sqrt{1 + |z + b_n|^2}} \\
\leq 2|b_n|
\]
Calculate $\chi_0(T_n^{-1}, T_{n+2}^{-1})$
Calculate $\chi_0(T_{n-1}, T_{n-1}^{-1})$

\[
\chi_0(T_{n-1}, T_{n-1}^{-1}) = \chi_0(t_{n+1}t_{n+2}, I)
\]
**CALCULATE** \( \chi_0(T_{n-1}^{-1}, T_{n+2}^{-1}) \)

\[
\chi_0(T_{n-1}^{-1}, T_{n+2}^{-1}) = \chi_0(t_{n+1}t_{n+2}, I)
= \chi_0(ht_{n+1}t_{n+2}, h)
\]
Calculate $\chi_0(T_n^{-1}, T_{n+2}^{-1})$

\[
\chi_0(T_n^{-1}, T_{n+2}^{-1}) = \chi_0(t_{n+1}t_{n+2}, I) \\
= \chi_0(ht_{n+1}t_{n+2}, h) \\
\leq \chi_0(ht_{n+1}t_{n+2}, t_{n+2}) + \chi_0(t_{n+2}, h)
\]
CALCULATE $\chi_0(T_{n}^{-1}, T_{n+2}^{-1})$

$$
\chi_0(T_{n}^{-1}, T_{n+2}^{-1}) = \chi_0(t_{n+1}t_{n+2}, I)
= \chi_0(ht_{n+1}t_{n+2}, h)
\leq \chi_0(ht_{n+1}t_{n+2}, t_{n+2}) + \chi_0(t_{n+2}, h)
= \chi_0(t_{n+1}, h) + \chi_0(t_{n+2}, h)
$$
Calculate $\chi_0(T_n^{-1}, T_{n+2}^{-1})$

\[
\chi_0(T_n^{-1}, T_{n+2}^{-1}) = \chi_0(t_{n+1}t_{n+2}, I) \\
= \chi_0(ht_{n+1}t_{n+2}, h) \\
\leq \chi_0(ht_{n+1}t_{n+2}, t_{n+2}) + \chi_0(t_{n+2}, h) \\
= \chi_0(t_{n+1}, h) + \chi_0(t_{n+2}, h) \\
\leq 2|b_{n+1}| + 2|b_{n+2}|
\]
CONCLUSION

If \[ \sum_{n} |b_n| < +\infty \] then
**Conclusion**

If $\sum_n |b_n| < +\infty$ then

$$\sum_n \chi_0(T_n^{-1}, T_{n+2}^{-1}) < +\infty.$$
**Conclusion**

If $\sum_n |b_n| < +\infty$ then

$$\sum_n \chi_0(T_{n-1}^{-1}, T_{n+2}^{-1}) < +\infty.$$ 

Hence $T_{2n-1}^{-1}$ converges uniformly to a Möbius map $f$. 
If $\sum_n |b_n| < +\infty$ then

$$\sum_n \chi_0(T_{n-1}^{-1}, T_{n+2}^{-1}) < +\infty.$$ 

Hence $T_{2n-1}^{-1}$ converges uniformly to a Möbius map $f$.

Hence $T_{2n-1}$ converges uniformly to a Möbius map $g$. 
If \( \sum_n |b_n| < +\infty \) then

\[
\sum_n \chi_0(T_n^{-1}, T_{n+2}^{-1}) < +\infty.
\]

Hence \( T_{2n-1}^{-1} \) converges uniformly to a Möbius map \( f \).

Hence \( T_{2n-1} \) converges uniformly to a Möbius map \( g \).

Hence \( T_{2n} = T_{2n-1}t_{2n} \) converges uniformly to \( gh \), where \( h(z) = 1/z \).
Conclusion

In summary, if $\sum_n |b_n| < +\infty$ then there is a Möbius map $g$ such that

$$T_{2n-1} \rightarrow g \quad T_{2n} \rightarrow gh.$$
In summary, if $\sum_n |b_n| < +\infty$ then there is a Möbius map $g$ such that

$$T_{2n-1} \to g \quad T_{2n} \to gh.$$ 

Hence

$$T_{2n-1}(0) \to g(0) \quad T_{2n}(0) \to gh(0) = g(\infty).$$
### Table of Contents

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Thank you!