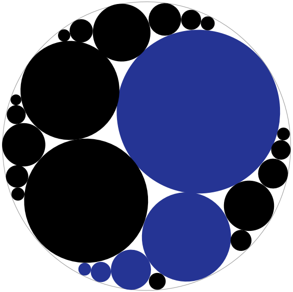
Hyperbolic geometry and continued fraction theory I

Ian Short 9 February 2010



http://maths.org/ ims25/maths/presentations.php

Collaborators

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Meira Hockman

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Meira Hockman

Alan Beardon
(University of Cambridge)

Project

The geometry of continued fractions

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The geometry of continued fractions

http://maths.org/ ims25/maths/presentations.php

Continued fractions

Continued fractions

$$\mathbf{K}(a_n|\, b_n) = \cfrac{a_1}{b_1 + \cfrac{a_2}{b_2 + \cfrac{a_3}{b_3 + \cfrac{a_4}{b_4 + \cdots}}}}$$

Continued fraction convergence

$$\frac{a_1}{b_1},$$

Continued fraction convergence

$$\frac{a_1}{b_1},\ \frac{a_1}{b_1 + \frac{a_2}{b_2}},$$

Continued fraction convergence

$$\frac{a_1}{b_1},\ \frac{a_1}{b_1+\frac{a_2}{b_2}},\ \frac{a_1}{b_1+\frac{a_2}{b_2+\frac{a_3}{b_3}}},$$

Continued fraction convergence

$$\frac{a_1}{b_1},\ \frac{a_1}{b_1+\dfrac{a_2}{b_2}},\ \frac{a_1}{b_1+\dfrac{a_2}{b_2+\dfrac{a_3}{b_3}}},\ \frac{a_1}{b_1+\dfrac{a_2}{b_2+\dfrac{a_3}{b_3+\dfrac{a_4}{b_4}}}},\ \dots$$

Expansion of $e = 2.71828182845905\ldots$ (Euler)

$$e = 2 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{4 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{6 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{8 + \cdots}}}}}}}}}}}$$

Expansion of $\pi = 3.14159265358979\ldots$ (Lange)

$$\pi = 3 + \cfrac{1^2}{6 + \cfrac{3^2}{6 + \cfrac{5^2}{6 + \cfrac{7^2}{6 + \cfrac{9^2}{6 + \cdots}}}}}$$

A problematic example

$$\cfrac{1}{2+\cfrac{1}{3+\cfrac{1}{1+\cfrac{1}{-1+\cdots}}}}$$

Möbius transformations

$$t_n(z) = \frac{a_n}{b_n + z}$$

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$$t_n(\infty) = 0$$

$$T_n(\infty) = T_{n-1}(0)$$

Convergence again

Convergence of $\mathbf{K}(a_n|\, b_n)$ equivalent to convergence of $T_1(0), T_2(0), \dots$.

Dealing with ∞

Previously

$$\frac{1}{\infty} = 0.$$

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Previously

$$\frac{1}{\infty} = 0.$$

Now

$$h(z) = \frac{1}{z} \qquad h(\infty) = 0.$$

Three more reasons for using Möbius maps

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- There is already a well developed theory of Möbius maps.

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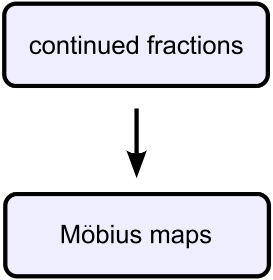
- There is already a well developed theory of Möbius maps.
- Allows us to bring in geometry.
- Simpler notation: composition of maps rather than algebraic manipulation.

Schematic diagram



Möbius maps

Reminder

$$t_n(z) = \frac{a_n}{b_n + z}$$

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$$T_n = t_1 \circ t_2 \circ \cdots \circ t_n$$

Usual integer continued fractions

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The *modular group*

$$\Gamma = \left\{ z \mapsto \frac{az+b}{cz+d} \,:\, a,b,c,d \in \mathbb{Z},\ ad-bc = 1 \right\}$$

Complex continued fractions

Complex continued fractions

$$\mathbf{K}(a_n | b_n) \qquad a_n, b_n \in \mathbb{C}$$

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$$\mathbf{K}(a_n|\, b_n) \qquad a_n, b_n \in \mathbb{C}$$

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The *Möbius group*

$$\mathcal{M} = \left\{ z \mapsto \frac{az+b}{cz+d} \;:\; a,b,c,d \in \mathbb{C},\ ad - bc \neq 0 \right\}$$

$\mathcal{M}$ generated by inversions

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Two aspects of $\mathcal{M}$

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- Three-dimensional hyperbolic isometries

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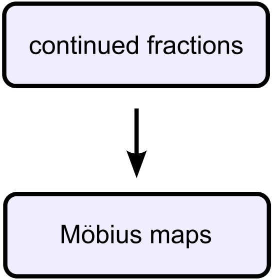
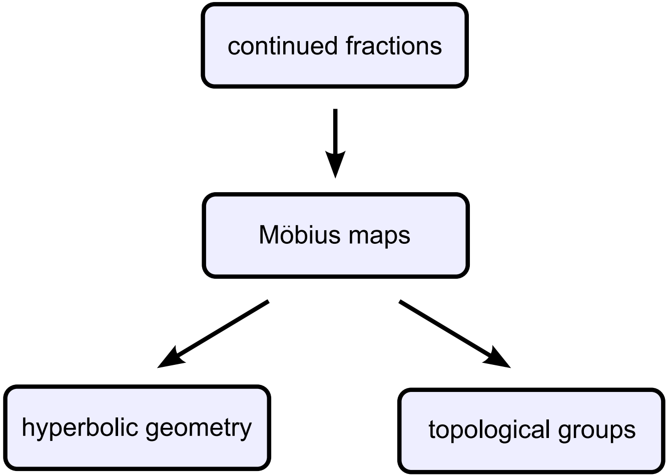
- Three-dimensional hyperbolic isometries
- Topological group (and complete metric space)

Schematic diagram



Schematic diagram



The Stern–Stolz Theorem

Theorem. If $\sum_n |b_n|$ converges then $\mathbf{K}(1|\, b_n)$ diverges.

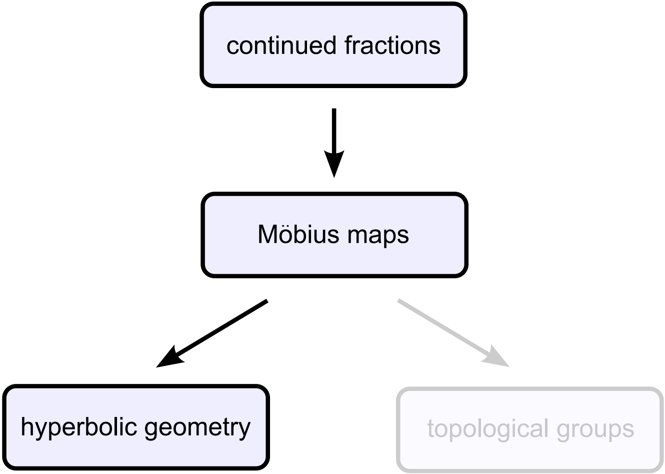
Open Problem I

Open Problem I

Interpret each result on complex continued fractions in terms of (a) hyperbolic geometry, and (b) topological group theory.

Hyperbolic geometry

Schematic diagram



Upper half-space

$$\mathbb{H}^3 = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \,:\, x_3 > 0\}$$

Upper half-space

$$\mathbb{H}^3 = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \,:\, x_3 > 0\}$$

$$\mathbb{C} \longleftrightarrow \{(x_1, x_2, 0) \in \mathbb{R}^3 \,:\, x_1, x_2 \in \mathbb{R}\}$$

Upper half-space

Three-dimensional hyperbolic space

$$(\mathbb{H}^3, \rho)$$

Three-dimensional hyperbolic space

$$(\mathbb{H}^3, \rho)$$

$$\sinh \tfrac{1}{2}\rho(x,y) = \frac{|x-y|}{2\sqrt{x_3 y_3}}$$

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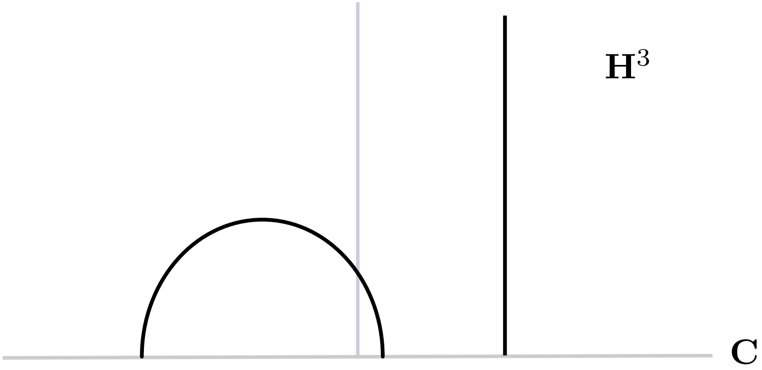

Graph representation of hyperbolic space

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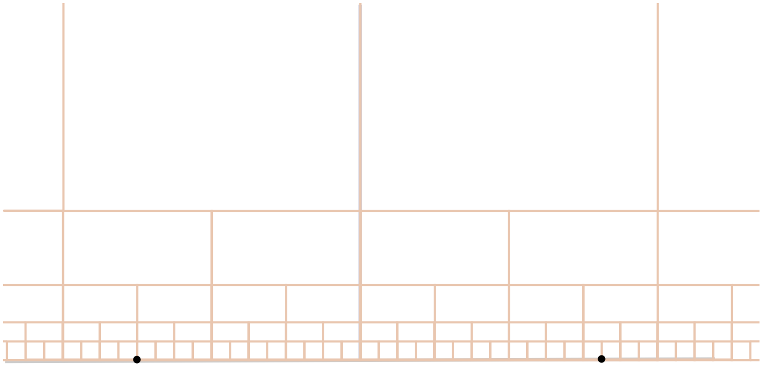
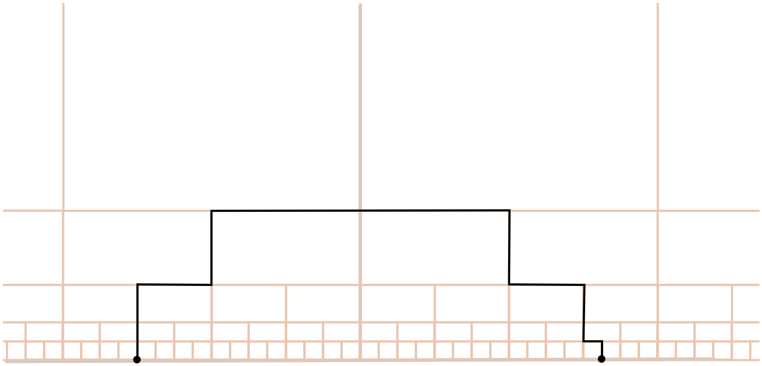
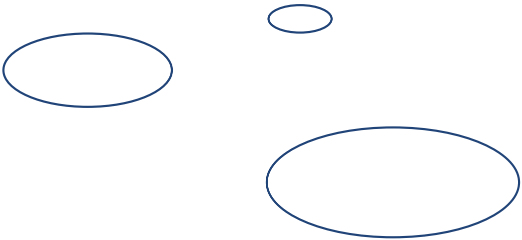


Graph representation of hyperbolic space



Möbius action on hyperbolic space



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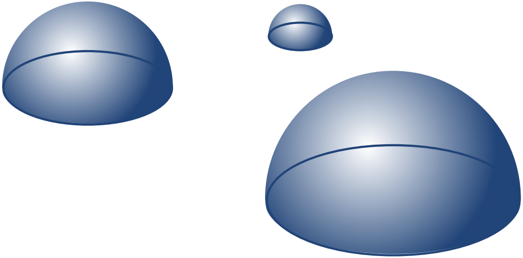


Isometry group

$$\mathrm{Isom}(\mathbb{H}^3) = \mathcal{M}$$

Back to continued fractions

$$t_n(\infty) = 0 \qquad T_n(\infty) = T_{n-1}(0)$$

Convergence

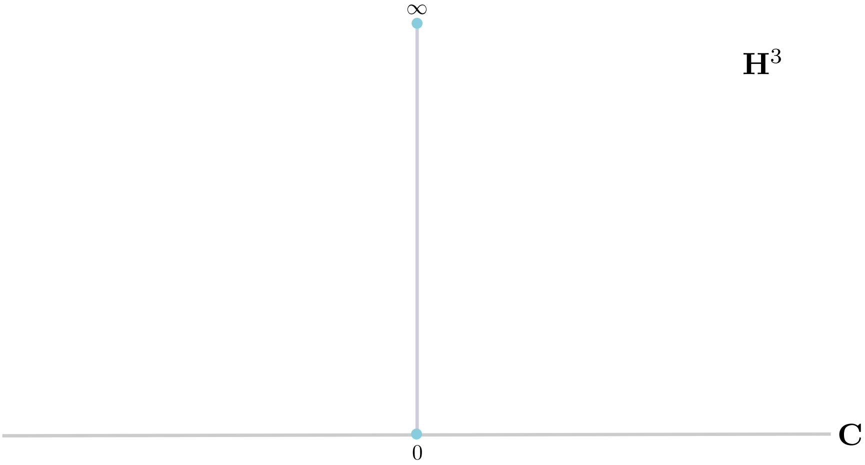
Suppose $\mathbf{K}(a_n|\, b_n)$ converges.

Convergence

Suppose $\mathbf{K}(a_n|\,b_n)$ converges.

In other words, suppose $T_1(0), T_2(0), \dots$ converges.

Convergence



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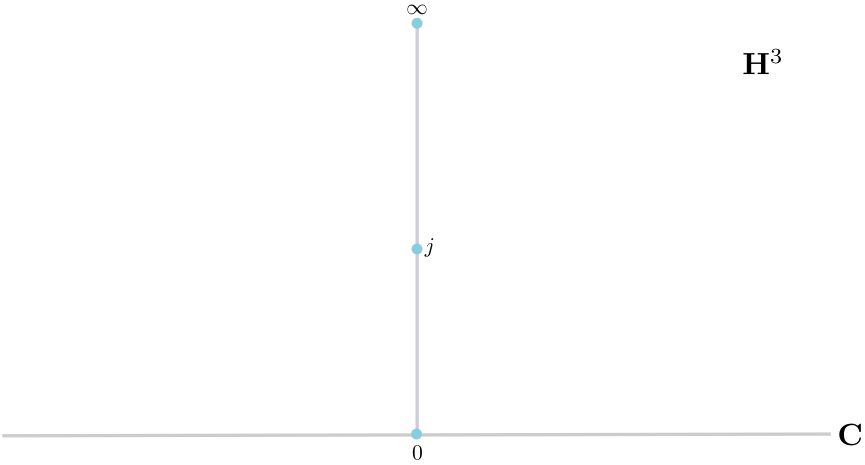

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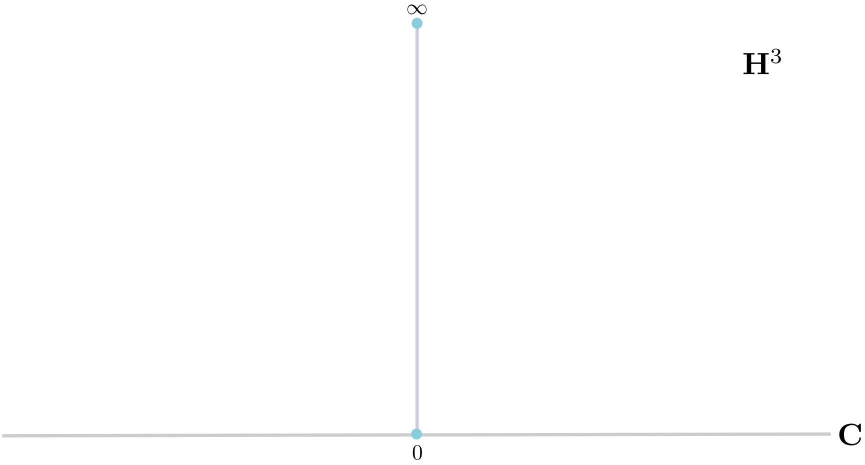

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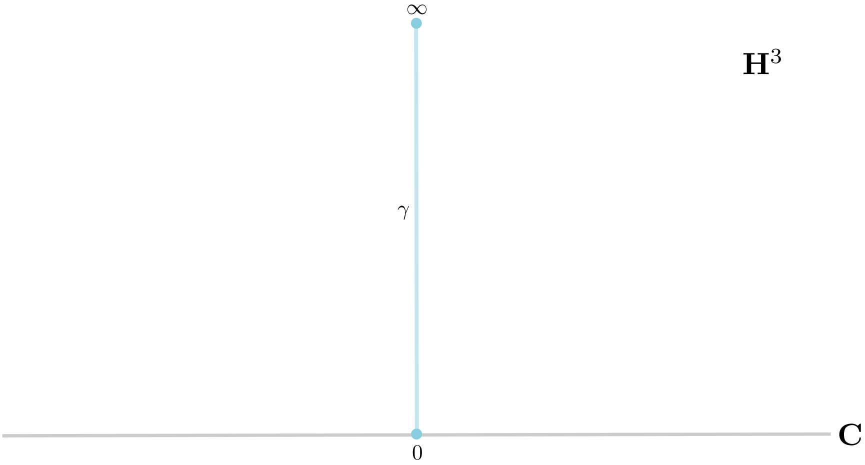

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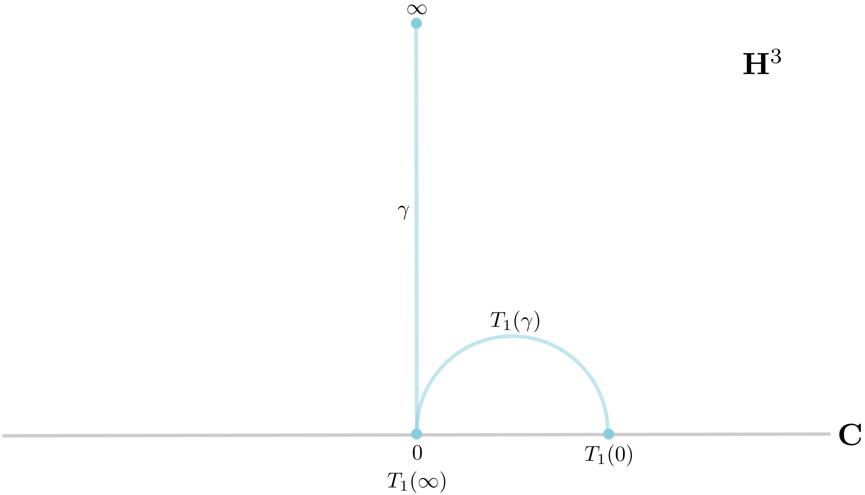

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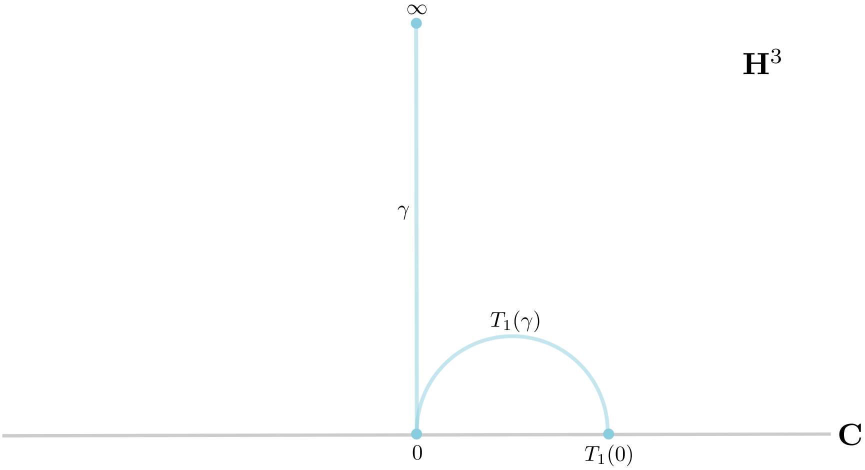

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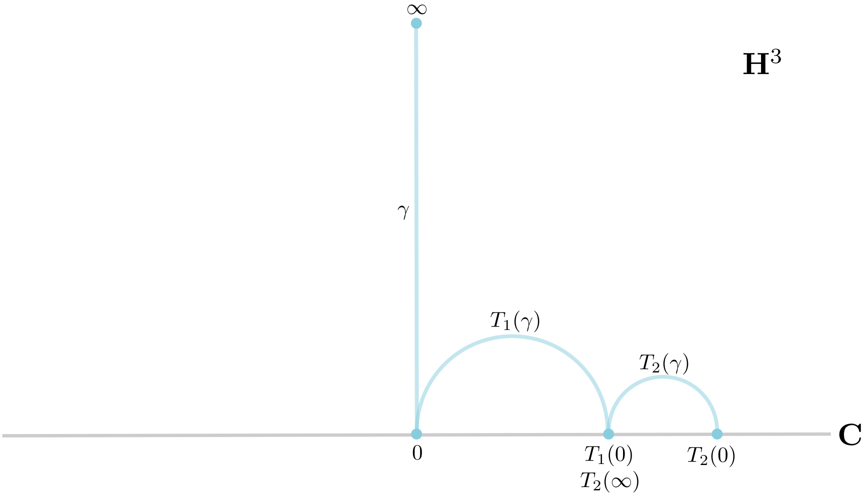

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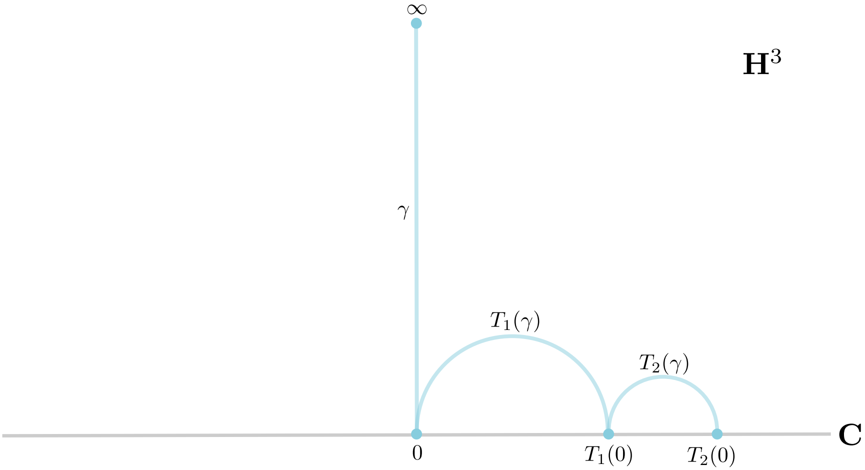

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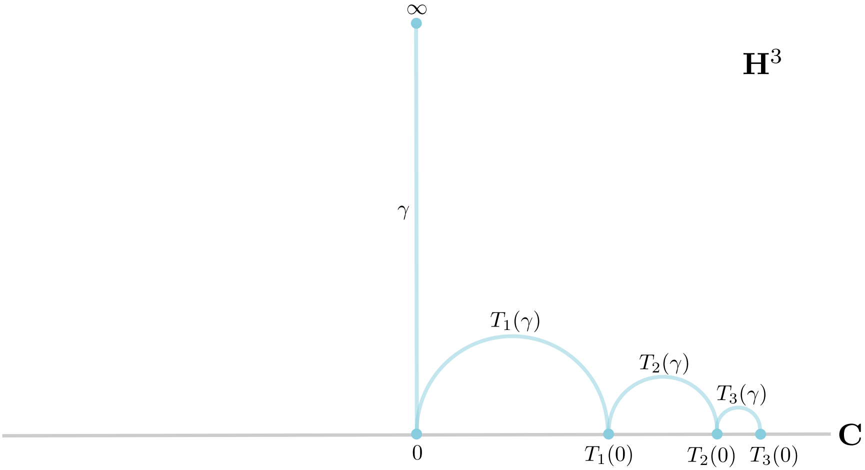

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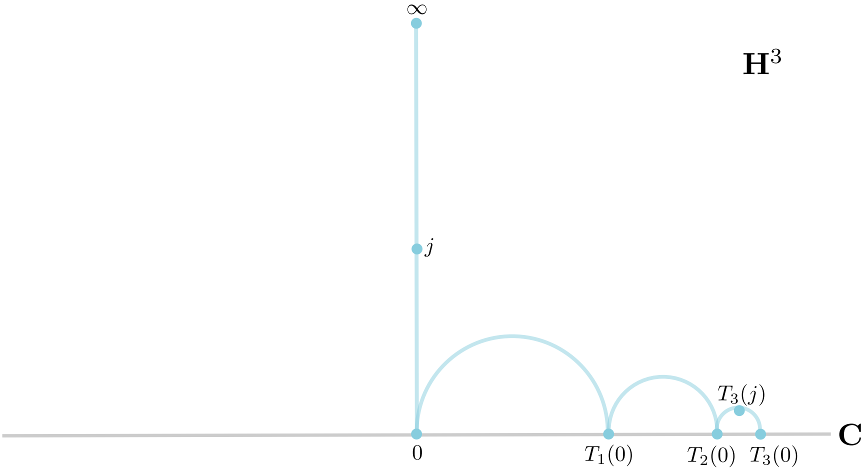

Convergence

If

$$T_n(0) \to p$$

Convergence

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then

$$T_n(j) \to p.$$

Convergence

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$$T_n(j) \to p.$$

(Lorentzen, Aebischer, Beardon)

Recall the Stern–Stolz Theorem

Theorem. If $\sum_n |b_n|$ converges then $\mathbf{K}(1|\, b_n)$ diverges.

Recall the hyperbolic metric

$$\sinh \tfrac{1}{2}\rho(x,y) = \frac{|x-y|}{2\sqrt{x_3 y_3}}$$

Hyperbolic distance calculations

$$\rho(j, t_n(j))$$

Hyperbolic distance calculations

$$\rho(j, t_n(j)) = \rho(h(j), ht_n(j))$$

Hyperbolic distance calculations

$$\begin{aligned}\rho(j, t_n(j)) &= \rho(h(j), ht_n(j)) \\ &= \rho(j, b_n + j)\end{aligned}$$

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$$\sinh \tfrac{1}{2}\rho(j, t_n(j)) = \frac{|b_n|}{2}$$

$$\rho(j, t_n(j)) = \rho(T_{n-1}(j), T_n(j))$$

Hyperbolic distance calculations

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Hence if $\sum_n |b_n| < +\infty$ then

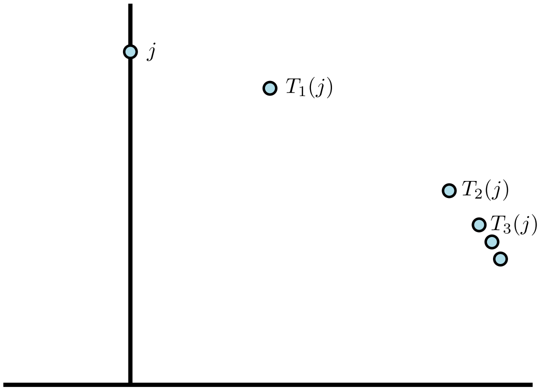
Hyperbolic distance calculations

$$\sinh \tfrac{1}{2}\rho(T_{n-1}(j), T_n(j)) = \frac{|b_n|}{2}$$

Hence if $\sum_n |b_n| < +\infty$ then

$$\sum_n \rho(T_{n-1}(j), T_n(j)) < +\infty$$

Cannot reach the boundary (Beardon)



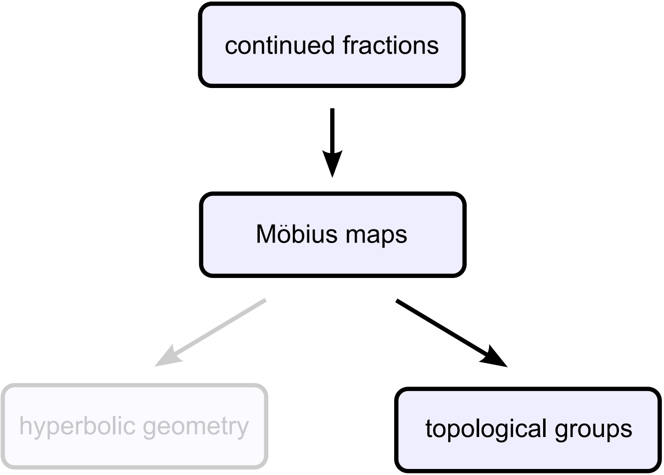
Open Problem II

Open Problem II

What is the geometric significance of the *argument* of b_n to the orbit $T_1(j), T_2(j), \dots$?

Toplogical groups

Schematic diagram



Möbius group

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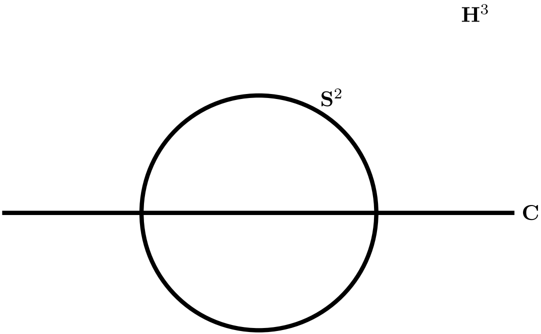
- Group of hyperbolic isometries of $\mathbb{H}^3$.

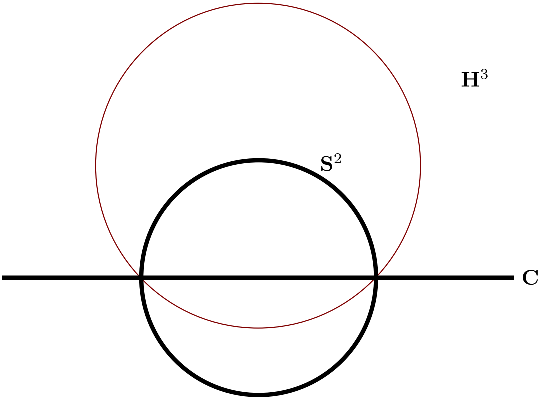
Möbius group

- Group of hyperbolic isometries of $\mathbb{H}^3$.
- Group of conformal automorphisms of $\mathbb{C}_\infty$.

Stereographic projection



Stereographic projection



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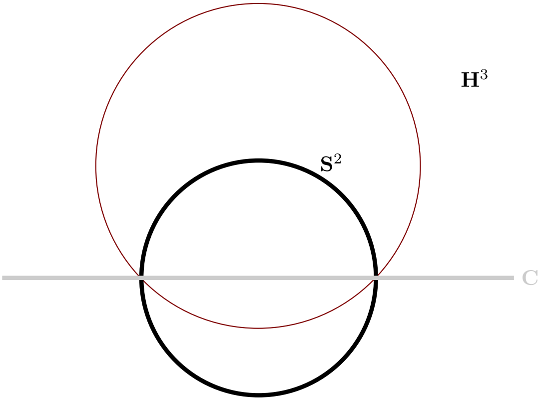

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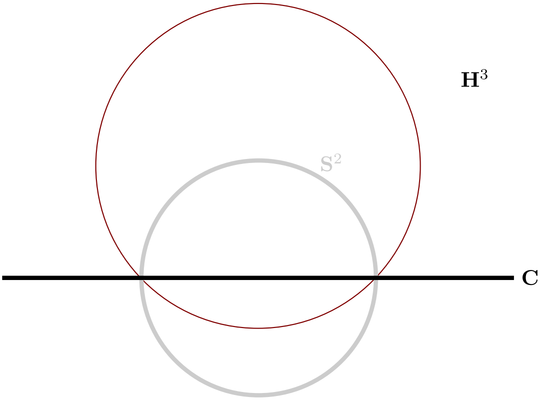
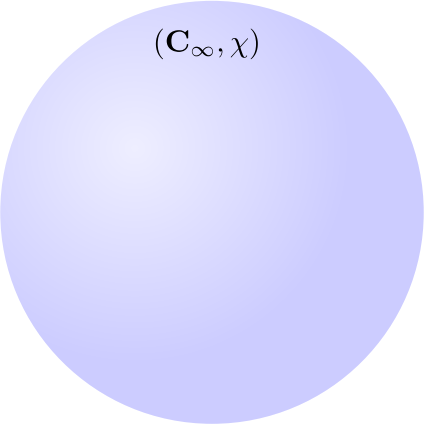


The chordal metric

$$\chi(w,z) = \frac{2|w-z|}{\sqrt{1+|w|^2}\sqrt{1+|z|^2}} \qquad \chi(w,\infty) = \frac{2}{\sqrt{1+|w|^2}}$$

The chordal metric



The supremum metric

$$\chi_0(f,g) = \sup_{z\in\mathbb{C}_\infty} \chi(f(z), g(z))$$

$$f, g \in \mathcal{M}$$

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$$f,g \in \mathcal{M}$$

The metric of uniform convergence.

Möbius group

Möbius group

- $(\mathcal{M}, \chi_0)$ is a complete metric space

Möbius group

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- $(\mathcal{M}, \chi_0)$ is a topological group

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- right-invariant: $\chi_0(fk, gk) = \chi_0(f, g)$

Möbius group

- $(\mathcal{M}, \chi_0)$ is a complete metric space
- $(\mathcal{M}, \chi_0)$ is a topological group
- right-invariant: $\chi_0(fk, gk) = \chi_0(f, g)$
- $h(z) = 1/z$ is a chordal isometry: $\chi_0(hf, hg) = \chi_0(f, g)$

Recall the Stern–Stolz Theorem

Theorem. If $\sum_n |b_n|$ converges then $\mathbf{K}(1|\, b_n)$ diverges.

Key observation

If b_n small then

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$$t_n(z) = \frac{1}{b_n + z} \sim h(z) = \frac{1}{z}.$$

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If b_n small then

$$t_n(z) = \frac{1}{b_n + z} \sim h(z) = \frac{1}{z}.$$

We must calculate $\chi_0(t_n, h)$.

Calculate $\chi_0(t_n, h)$

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Calculate $\chi_0(t_n, h)$

$$\chi_0(t_n, h) = \chi_0(ht_n, I)$$

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$$\begin{aligned}\chi_0(t_n, h) &= \chi_0(ht_n, I) \\ &= \chi_0(z + b_n, z)\end{aligned}$$

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\begin{aligned}
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&= \sup_{z \in \mathbb{C}_\infty} \frac{2|b_n|}{\sqrt{1+|z|^2}\sqrt{1+|z+b_n|^2}} \\
&\leq 2|b_n|
\end{aligned}
$$

Calculate $\chi_0(T_n^{-1}, T_{n+2}^{-1})$

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Calculate $\chi_0(T_n^{-1}, T_{n+2}^{-1})$

$$\begin{aligned}\chi_0(T_n^{-1}, T_{n+2}^{-1}) &= \chi_0(t_{n+1}t_{n+2}, I)\\ &= \chi_0(ht_{n+1}t_{n+2}, h)\end{aligned}$$

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$$\begin{aligned}
\chi_0(T_n^{-1}, T_{n+2}^{-1}) &= \chi_0(t_{n+1}t_{n+2}, I) \\
&= \chi_0(ht_{n+1}t_{n+2}, h) \\
&\leq \chi_0(ht_{n+1}t_{n+2}, t_{n+2}) + \chi_0(t_{n+2}, h)
\end{aligned}$$

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\chi_0(T_n^{-1}, T_{n+2}^{-1}) &= \chi_0(t_{n+1}t_{n+2}, I) \\
&= \chi_0(ht_{n+1}t_{n+2}, h) \\
&\leq \chi_0(ht_{n+1}t_{n+2}, t_{n+2}) + \chi_0(t_{n+2}, h) \\
&= \chi_0(t_{n+1}, h) + \chi_0(t_{n+2}, h) \\
&\leq 2|b_{n+1}| + 2|b_{n+2}|
\end{aligned}$$

Conclusion

If $\sum_n |b_n| < +\infty$ then

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$$\sum_n \chi_0(T_n^{-1}, T_{n+2}^{-1}) < +\infty.$$

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Hence T_{2n-1}^{-1} converges uniformly to a Möbius map f.

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$$\sum_n \chi_0(T_n^{-1}, T_{n+2}^{-1}) < +\infty.$$

Hence T_{2n-1}^{-1} converges uniformly to a Möbius map f.

Hence T_{2n-1} converges uniformly to a Möbius map g.

Conclusion

If $\sum_n |b_n| < +\infty$ then

$$\sum_n \chi_0(T_n^{-1}, T_{n+2}^{-1}) < +\infty.$$

Hence T_{2n-1}^{-1} converges uniformly to a Möbius map f.

Hence T_{2n-1} converges uniformly to a Möbius map g.

Hence $T_{2n} = T_{2n-1}t_{2n}$ converges uniformly to gh, where $h(z) = 1/z$.

Conclusion

In summary, if $\sum_n |b_n| < +\infty$ then there is a Möbius map g such that

$$T_{2n-1} \to g \qquad T_{2n} \to gh.$$

Conclusion

In summary, if $\sum_n |b_n| < +\infty$ then there is a Möbius map g such that

$$T_{2n-1} \to g \qquad T_{2n} \to gh.$$

Hence

$$T_{2n-1}(0) \to g(0) \qquad T_{2n}(0) \to gh(0) = g(\infty).$$

Thank you!