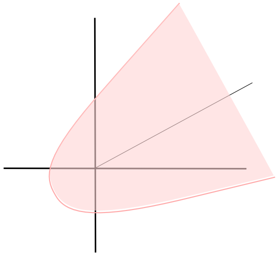
Hyperbolic geometry and continued fraction theory II

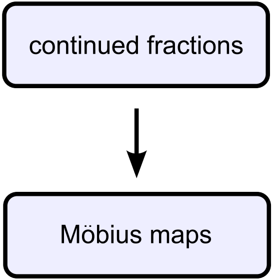
Ian Short 16 February 2010



http://maths.org/ims25/maths/presentations.php

Background

Schematic diagram



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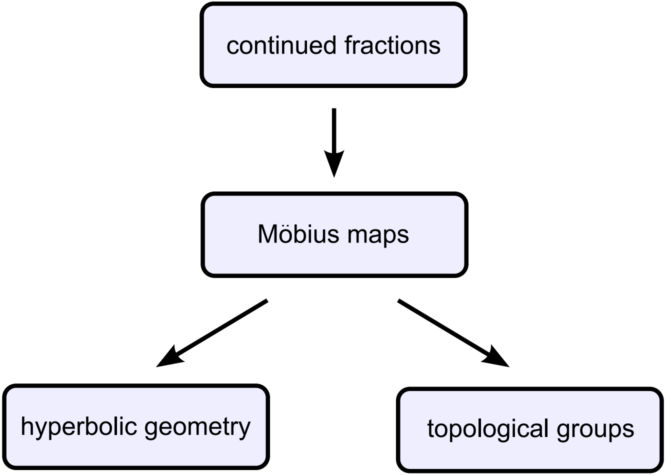

Möbius group

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- Group of hyperbolic isometries of $\mathbb{H}^3$.

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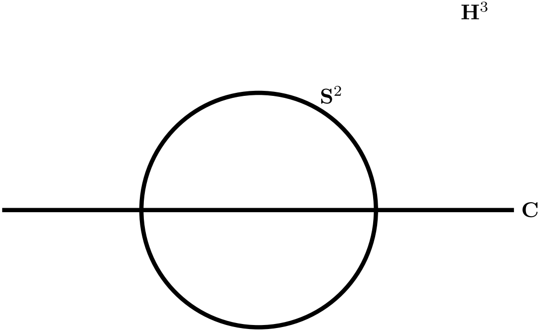
- Group of conformal automorphisms of $\mathbb{C}_\infty$.

Stereographic projection



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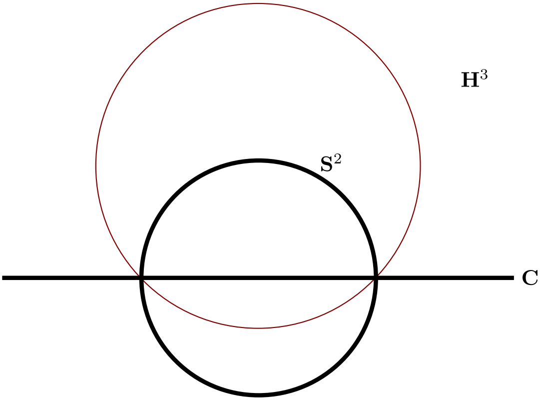

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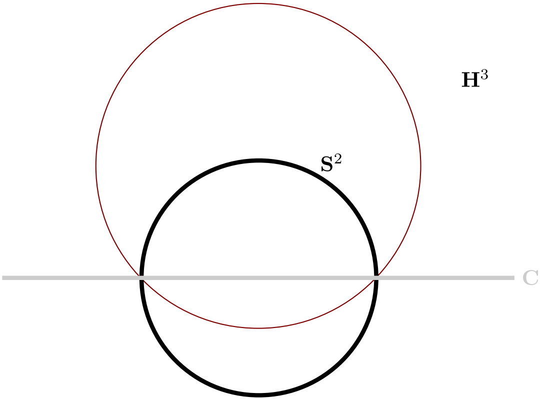

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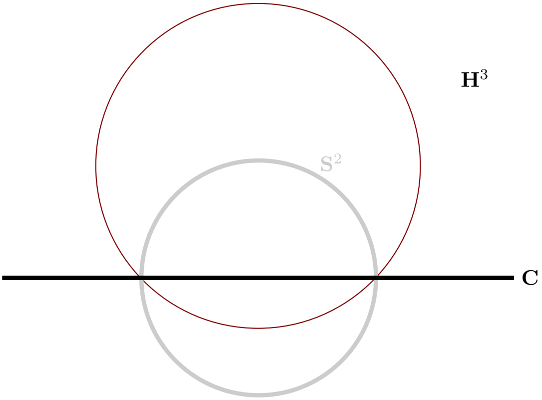
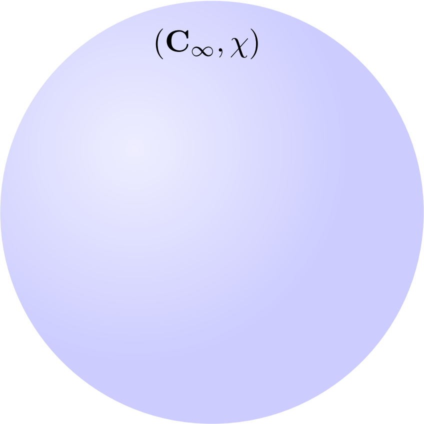


The chordal metric

$$\chi(w,z) = \frac{2|w-z|}{\sqrt{1+|w|^2}\sqrt{1+|z|^2}} \qquad \chi(w,\infty) = \frac{2}{\sqrt{1+|w|^2}}$$

The chordal metric



The supremum metric

Denote the Möbius group by $\mathcal{M}$.

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$$\chi_0(f,g) = \sup_{z\in\mathbb{C}_\infty} \chi(f(z),g(z))$$

$$f,g \in \mathcal{M}$$

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The metric of uniform convergence.

Möbius group

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- $(\mathcal{M}, \chi_0)$ is a complete metric space

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- $(\mathcal{M}, \chi_0)$ is a topological group
- right-invariant: $\chi_0(fk, gk) = \chi_0(f, g)$
- $h(z) = 1/z$ is a chordal isometry: $\chi_0(hf, hg) = \chi_0(f, g)$

The Stern–Stolz Theorem

Theorem. If $\sum_n |b_n|$ converges then $\mathbf{K}(1|\, b_n)$ diverges.

Notation

$$t_n(z) = \frac{1}{b_n + z}$$

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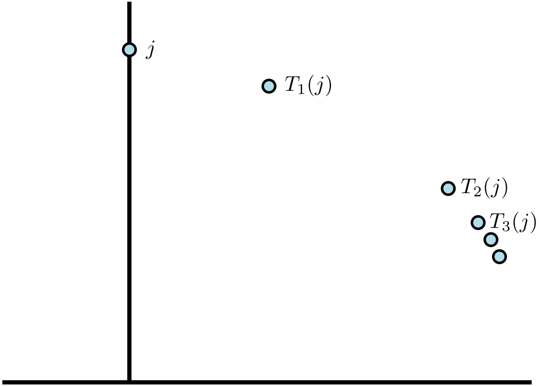
$$T_n = t_1 \circ t_2 \circ \cdots \circ t_n$$

$$h(z) = \frac{1}{z}$$

Hyperbolic geometry proof (Beardon)



Topological groups proof

$$\sum_n \chi_0(T_n, T_{n+2}) \leqslant \sum_n |b_n|$$

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$$T_{2n-1} \to g \qquad T_{2n} \to gh$$

$$T_{2n-1}(0) \to g(0) \qquad T_{2n}(0) \to g(\infty)$$

Today

Today

The Parabola Theorem

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The Parabola Theorem

'The queen of the convergence theorems' (Lorentzen)

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The Parabola Theorem

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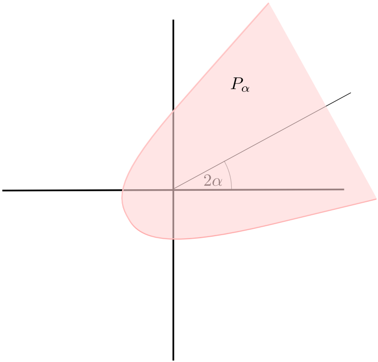
Topological groups techniques and hyperbolic geometry

The Parabola Theorem

Continued fractions

$$\mathbf{K}(a_n|\,1) = \cfrac{a_1}{1 + \cfrac{a_2}{1 + \cfrac{a_3}{1 + \cfrac{a_4}{1 + \cdots}}}}$$

Parabolic region



The Stern–Stolz series

$$\left|\frac{1}{a_1}\right| + \left|\frac{a_1}{a_2}\right| + \left|\frac{a_2}{a_1a_3}\right| + \left|\frac{a_1a_3}{a_2a_4}\right| + \left|\frac{a_2a_4}{a_1a_3a_5}\right| + \cdots$$

The Parabola Theorem

Suppose that $a_n \in P_\alpha$ for $n = 1, 2, \dots$.

The Parabola Theorem

Suppose that $a_n \in P_\alpha$ for $n = 1, 2, \ldots$. Then $\mathbf{K}(a_n | 1)$ converges if and only if the series

$$\left|\frac{1}{a_1}\right| + \left|\frac{a_1}{a_2}\right| + \left|\frac{a_2}{a_1 a_3}\right| + \left|\frac{a_1 a_3}{a_2 a_4}\right| + \left|\frac{a_2 a_4}{a_1 a_3 a_5}\right| + \cdots$$

diverges.

Long history of the Parabola Theorem

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Understanding the theorem

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What is the significance of the parabolic region?

Understanding the theorem

What is the significance of the parabolic region?

What is the significance of the series?

Years of confusion

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Split the theorem in two.

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Theorem involving the parabolic region.

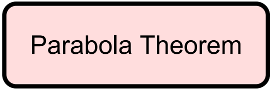
Years of confusion

Split the theorem in two.

Theorem involving the parabolic region.

Theorem involving the Stern–Stolz series.

Schematic diagram



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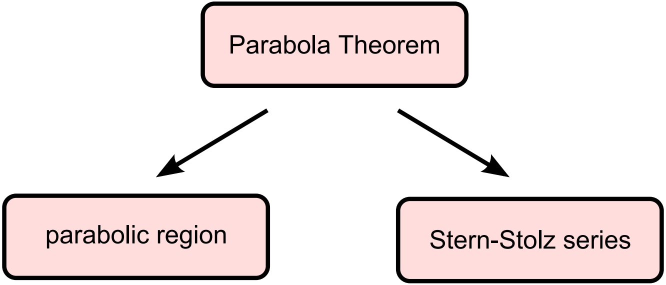
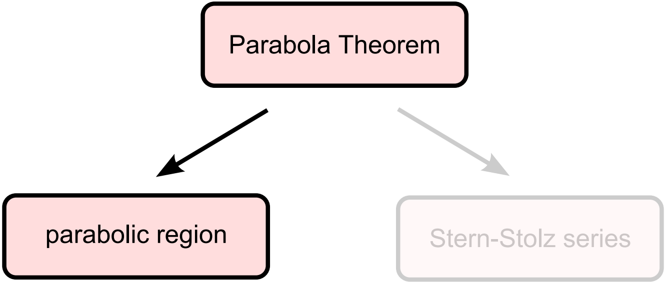


The parabolic region

Schematic diagram



Möbius transformations

$$t_n(z) = \frac{a_n}{1+z}$$

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$$T_n = t_1 \circ t_2 \circ \cdots \circ t_n$$

Convergence using Möbius transformations

The continued fraction $\mathbf{K}(a_n|\,1)$ converges if and only if $T_1(0), T_2(0), T_3(0), \ldots$ converges.

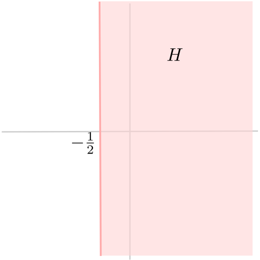
Question

What does the condition $a \in P_\alpha$ signify for the map $t(z) = a/(1+z)$?

Answer

The coefficient a belongs to P_α if and only if t maps a half-plane H_α within itself.

Proof ($\alpha = 0$)



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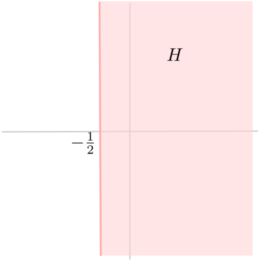

$t(z) = \frac{a}{1+z}$

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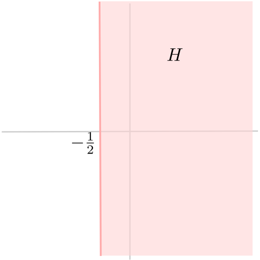
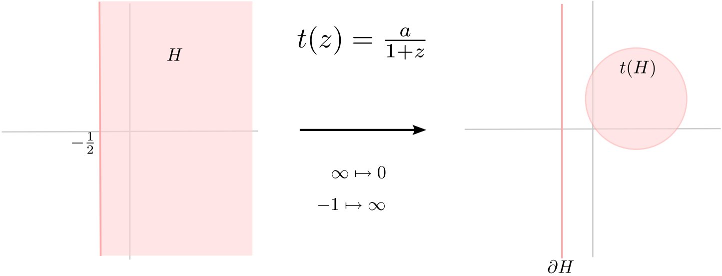


$t(z) = \frac{a}{1+z}$

$\infty \mapsto 0$

$-1 \mapsto \infty$

Proof $(\alpha = 0)$



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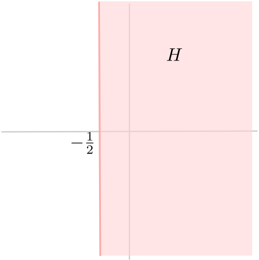

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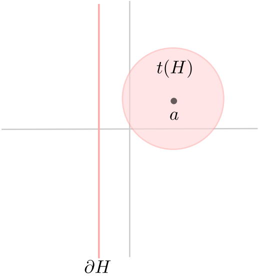

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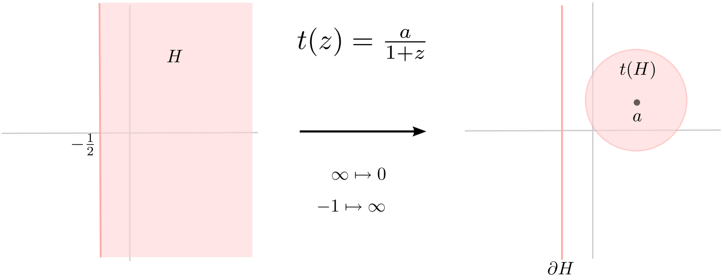

$$t(H) \subset H \iff |a - 0| \leqslant |a - \partial H|$$

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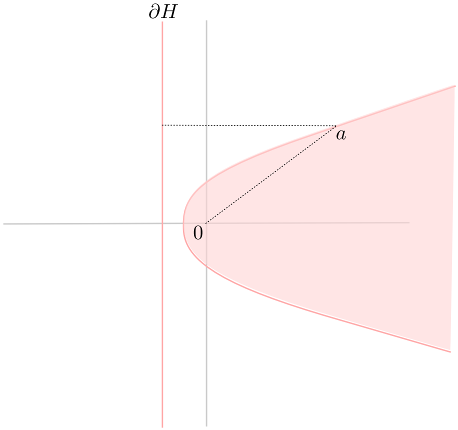
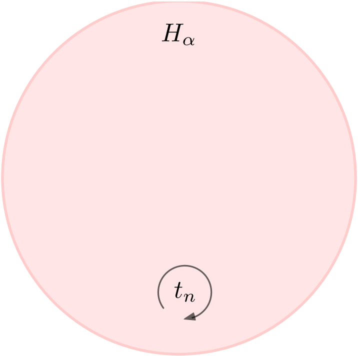


Parabola $|a - 0| = |a - \partial H|$

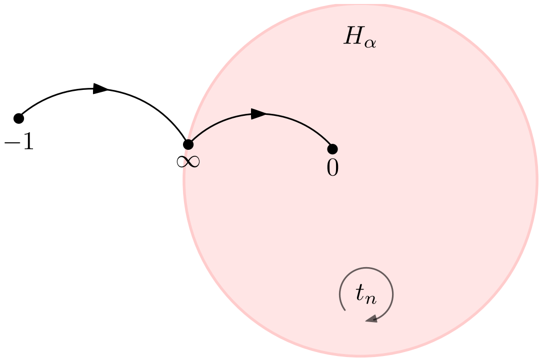
Original condition

$$a_n \in P_\alpha$$

New condition



New condition



New condition

$$t_n(-1) = \infty$$

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Does $T_n = t_1 \circ t_2 \circ \cdots \circ t_n$ converge at 0?

Extensive literature

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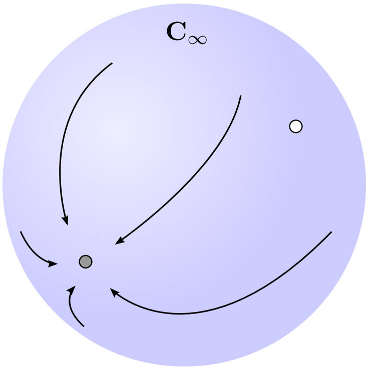
- Lorentzen (*Ramanujan J.*, 2007)

Conclusion

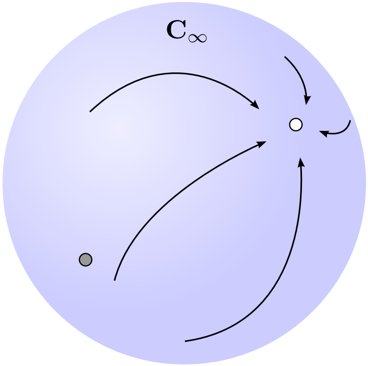
If $a_n \in P_\alpha$ then there are points p and q in H_α such that T_{2n-1} converges on H_α to p, and T_{2n} converges on H_α to q.

Divergence



Action of T_{2n-1}

Divergence



Action of T_{2n}

Summary

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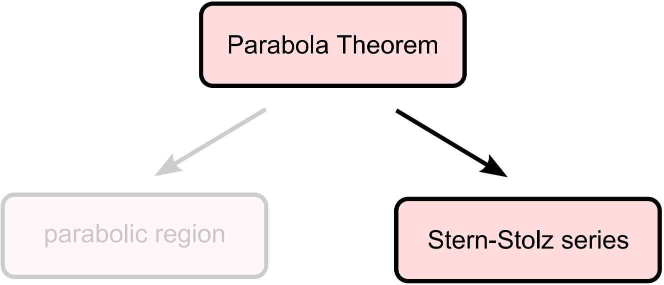
- refer to the literature

Summary

- $a_n \in P_\alpha$
- $t_n(H_\alpha) \subseteq H_\alpha$
- refer to the literature
- $T_{2n-1} \to p$ and $T_{2n} \to q$

The Stern–Stolz series

Schematic diagram



Recall the Parabola Theorem

Suppose $a_n \in P_\alpha$.

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Suppose $a_n \in P_\alpha$. Then $\mathbf{K}(a_n|\,1)$ converges if and only if the Stern–Stolz series diverges.

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The Stern–Stolz series

$$\left|\frac{1}{a_1}\right| + \left|\frac{a_1}{a_2}\right| + \left|\frac{a_2}{a_1 a_3}\right| + \left|\frac{a_1 a_3}{a_2 a_4}\right| + \left|\frac{a_2 a_4}{a_1 a_3 a_5}\right| + \cdots$$

Convergence of the Stern–Stolz series

$$t_n(z) = \frac{a_n}{1+z} \qquad \sim \qquad s_n(z) = \frac{a_n}{z}$$

Möbius transformations

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$$S_n = s_1 \circ s_2 \circ \cdots \circ s_n$$

Convergence of the Stern–Stolz series

Is $S_n \sim T_n$?

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- $(\mathcal{M},\chi_0)$ a topological group

Recall supremum metric

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- χ_0 right-invariant
- $(\mathcal{M}, \chi_0)$ a topological group
- $(\mathcal{M}, \chi_0)$ a complete metric space

The Stern–Stolz series

$$\mu_1 = \frac{1}{a_1} \qquad \mu_2 = \frac{a_1}{a_2} \qquad \mu_3 = \frac{a_2}{a_1 a_3} \dots$$

The Stern–Stolz series

$$\mu_1 = \frac{1}{a_1} \qquad \mu_2 = \frac{a_1}{a_2} \qquad \mu_3 = \frac{a_2}{a_1 a_3} \dots$$

$$|\mu_1| + |\mu_2| + |\mu_3| + \cdots$$

The Stern–Stolz series

$$\mu_1 = \frac{1}{a_1} \qquad \mu_2 = \frac{a_1}{a_2} \qquad \mu_3 = \frac{a_2}{a_1 a_3} \dots$$

$$|\mu_1| + |\mu_2| + |\mu_3| + \cdots$$

$$S_{2n-1}(z) = \frac{1}{\mu_{2n-1} z} \qquad S_{2n}(z) = \mu_{2n} z$$

Conjugation

$$\mu z \circ (1+z) \circ \mu^{-1} z = \mu + z$$

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$$\mu z \circ (1+z) \circ \mu^{-1} z = \mu + z$$

$$S_{2n} \circ (1+z) \circ S_{2n}^{-1}(z)$$

Conjugation

$$\mu z \circ (1+z) \circ \mu^{-1} z = \mu + z$$

$$S_{2n} \circ (1+z) \circ S_{2n}^{-1}(z) = \mu_{2n} + z$$

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$$\mu z \circ (1+z) \circ \mu^{-1} z = \mu + z$$

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$$h(z) = \frac{1}{z}$$

$$S_{2n-1} \circ (1+z) \circ S_{2n-1}^{-1}(z)$$

Conjugation

$$\mu z \circ (1+z) \circ \mu^{-1} z = \mu + z$$

$$S_{2n} \circ (1+z) \circ S_{2n}^{-1}(z) = \mu_{2n} + z$$

$$h(z) = \frac{1}{z}$$

$$S_{2n-1} \circ (1+z) \circ S_{2n-1}^{-1}(z) = h \circ (\mu_{2n-1} + z) \circ h$$

Calculation

$$\chi_0(S_n T_n^{-1}, S_{n-1} T_{n-1}^{-1})$$

Calculation

$$\chi_0(S_nT_n^{-1}, S_{n-1}T_{n-1}^{-1}) = \chi_0(S_nt_n^{-1}, S_{n-1})$$

Calculation

$$\begin{aligned}\chi_0(S_nT_n^{-1}, S_{n-1}T_{n-1}^{-1}) &= \chi_0(S_nt_n^{-1}, S_{n-1}) \\ &= \chi_0(I, S_{n-1}t_nS_n^{-1})\end{aligned}$$

Calculation

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Calculation

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Calculation

$$\begin{aligned}
\chi_0(S_nT_n^{-1}, S_{n-1}T_{n-1}^{-1}) &= \chi_0(S_nt_n^{-1}, S_{n-1}) \\
&= \chi_0(I, S_{n-1}t_nS_n^{-1}) \\
&= \chi_0(I, S_n \circ (1+z) \circ S_n^{-1}) \\
&= \chi_0(I, \mu_n + z) \\
&\sim |\mu_n|
\end{aligned}$$

Summary

$$\left|\frac{1}{a_1}\right| + \left|\frac{a_1}{a_2}\right| + \left|\frac{a_2}{a_1a_3}\right| + \left|\frac{a_1a_3}{a_2a_4}\right| + \left|\frac{a_2a_4}{a_1a_3a_5}\right| + \cdots < +\infty$$

if and only if

$$\sum_n \chi_0(S_nT_n^{-1}, S_{n-1}T_{n-1}^{-1}) < +\infty$$

Convergence of the Stern–Stolz series

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$$\chi_0(gS_n, T_n) \to 0$$

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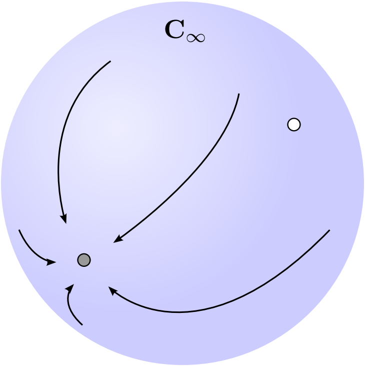
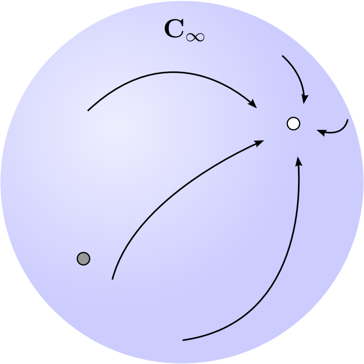
So if $T_n \sim gS_n$ then $T_{2n-1} \to g(\infty)$ and $T_{2n} \to g(0)$.

Oscillation



Action of T_{2n-1}

Oscillation



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Open Problem III

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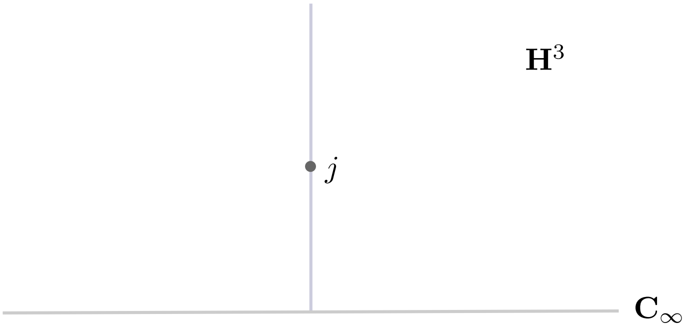
Hyperbolic geometry

Hyperbolic space



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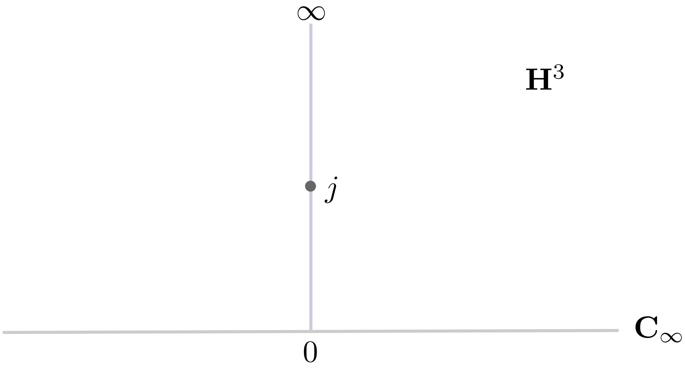
Dynamics of S_n in hyperbolic space



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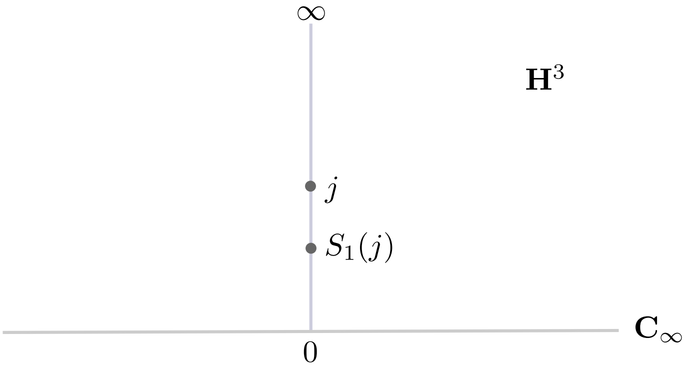

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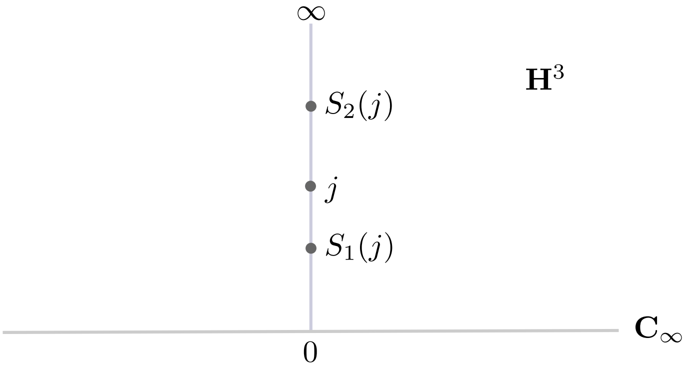

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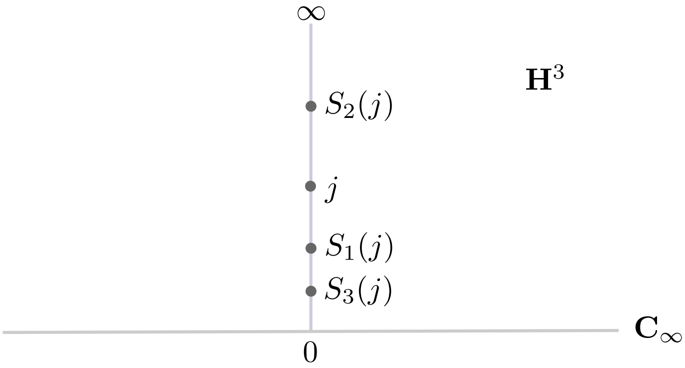

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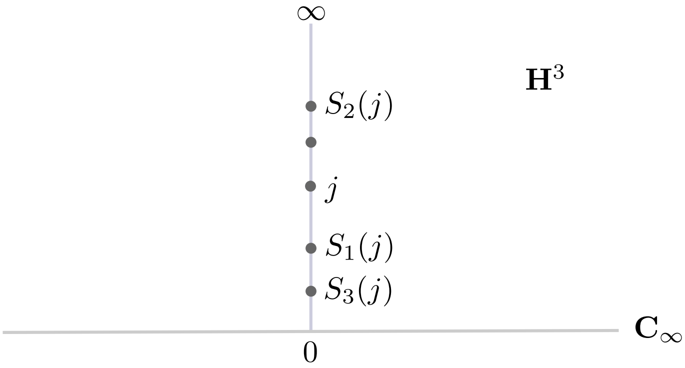

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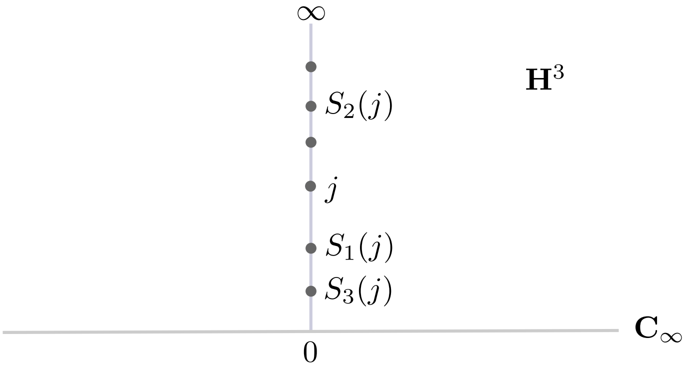

Equivalent conditions

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$$\cfrac{1}{\mathrm{T}+\cfrac{1}{\mathrm{H}+\cfrac{1}{\mathrm{A}+\cfrac{1}{\mathrm{N}+\cfrac{1}{\mathrm{K}+\cfrac{1}{\mathrm{Y}+\cfrac{1}{\mathrm{O}+\cfrac{1}{\mathrm{U}+\frac{1}{!}}}}}}}}}$$