Hyberbolic geometry and continued fraction theory III

Ian Short 23 February 2010

http://maths.org/ims25/maths/presentations.php
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**Today**
Today

Sets of divergence
Sets of divergence

Hausdorff dimension of sets of divergence for continued fractions
Motivation
Iteration of holomorphic functions

\[ f : \mathbb{C}_\infty \rightarrow \mathbb{C}_\infty \]
Iteration of holomorphic functions

\[ f : \mathbb{C}_\infty \rightarrow \mathbb{C}_\infty \]

\[ f, \quad f^2, \quad f^3, \ldots \]
Julia sets

Julia set of $f(z) = z^2 + i$
KLEINIAN GROUPS

Discrete subgroups of $\mathcal{M}$. 
KLEINIAN GROUPS

Discrete subgroups of $\mathcal{M}$.

Kleinian groups are countable.
Limit sets

Limit set of a Schottky group
CONTINUED FRACTIONS

\[ K(b_n) = \frac{1}{b_1 + \frac{1}{b_2 + \frac{1}{b_3 + \frac{1}{b_4 + \cdots}}}} \]
Continued fractions

\[ t_1(z) = \frac{1}{b_1 + z} \]

\[ K(b_n) = \frac{1}{b_1 + \frac{1}{b_2 + \frac{1}{b_3 + \frac{1}{b_4 + \cdots}}} \right] \]
**Continued fractions**

\[
K(b_n) = \cfrac{1}{b_1 + \cfrac{1}{b_2 + \cfrac{1}{b_3 + \cfrac{1}{b_4 + \cdots}}}}
\]

\[
t_1(z) = \frac{1}{b_1 + z}
\]

\[
t_2(z) = \frac{1}{b_2 + z}
\]
CONTINUED FRACTIONS

\[ K(b_n) = \frac{1}{b_1 + \frac{1}{b_2 + \frac{1}{b_3 + \frac{1}{b_4 + \cdots}}}} \]

\[ t_1(z) = \frac{1}{b_1 + z} \]
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\[ t_3(z) = \frac{1}{b_3 + z} \]
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\[ t_4(z) = \frac{1}{b_4 + z} \]
CONTINUED FRACTIONS

\[ K(b_n) = \frac{1}{b_1 + \frac{1}{b_2 + \frac{1}{b_3 + \frac{1}{b_4 + \cdots} \cdots}}} \]

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\[ t_2(z) = \frac{1}{b_2 + z} \]
\[ t_3(z) = \frac{1}{b_3 + z} \]
\[ t_4(z) = \frac{1}{b_4 + z} \]

\[ T_n = t_1 \circ t_2 \circ \cdots \circ t_n \]
Julia sets for continued fractions
COMMON FEATURES
COMMON FEATURES

- A sequence $F_1, F_2, \ldots$ of holomorphic maps of $\mathbb{C}_\infty$. 
COMMON FEATURES

○ A sequence $F_1, F_2, \ldots$ of holomorphic maps of $\mathbb{C}\infty$.
○ The derived set $J$ of

$$\bigcup_{n=1}^{\infty} \{ z : F_n(z) = w \},$$

is, generally, independent of $w$. 
COMMON FEATURES

- A sequence $F_1, F_2, \ldots$ of holomorphic maps of $\mathbb{C}_\infty$.
- The derived set $J$ of

$$\bigcup_{n=1}^{\infty} \{z : F_n(z) = w\},$$

is, generally, independent of $w$.
- The complement of $J$ is the largest open set on which $F_1, F_2, \ldots$ is a normal family.
Theorem to be proved today

Given positive integers $b_1, b_2, \ldots$, 

Motivation
Hausdorff dimension
Geodesic paths
The Farey graph
Rapid escape
Theorem proof
Given *positive integers* $b_1, b_2, \ldots$, let

$$p = \frac{1}{b_1 + \frac{1}{b_2 + \frac{1}{b_3 + \cdots}}}.$$
Theorem to be proved today

Given positive integers \( b_1, b_2, \ldots \), let

\[
p = \frac{1}{b_1 + \frac{1}{b_2 + \frac{1}{b_3 + \cdots}}}.
\]

Then \( T_n(z) \) converges to \( p \) for all points \( z \) outside a subset of \((−∞, −1)\) of Hausdorff dimension 0.
Given *positive integers* $b_1, b_2, \ldots$, let

$$p = \frac{1}{b_1 + \frac{1}{b_2 + \frac{1}{b_3 + \cdots}}}.$$ 

Then $T_n(z)$ converges to $p$ for all points $z$ outside a subset of $(-\infty, -1)$ of logarithmic Hausdorff dimension 1.
Open Problem IV
How are the coefficients of $K(1|b_n)$ related to the (logarithmic) Hausdorff dimension of the associated set of divergence?
HAUSDORFF DIMENSION
Dimensions
Dimensions
Dimensions
Dimensions
**DIMENSIONS**
**DIMENSIONS**

- Hausdorff dimension
- Geodesic paths
- The Farey graph
- Rapid escape
- Theorem proof

![Diagram of Hausdorff dimension, geodesic paths, the Farey graph, and rapid escape.](image-url)
DIMENSIONS
HAUSDORFF $t$-DIMENSIONAL MEASURE
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Hausdorff $t$-dimensional measure
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HAUSDORFF $t$-DIMENSIONAL MEASURE

\[ \sum r_i^t \]
HAUSDORFF $t$-DIMENSIONAL MEASURE

$$\inf_{r_i < \delta} \sum r_i^t$$
**HAUSDORFF $t$-DIMENSIONAL MEASURE**

\[ m_t(X) = \lim_{\delta \to 0} \inf_{r_i < \delta} \sum r_i^t \]
HAUSDORFF DIMENSION OF A SET $X$

$m_t(X)$

$0 \rightarrow d$
THE CANTOR SET
The Cantor set
The Cantor set
The Cantor set
Logarithmic Hausdorff dimension

Instead of
\[ f(x) = \sqrt[3]{x} \]
use
\[ f(x) = 1^{\frac{1}{\log_1 x}} \]
Logarithmic Hausdorff dimension

Instead of $f(x) = x^t$....
Logarithmic Hausdorff dimension

Instead of $f(x) = x^t$ ....

.... use $f(x) = \frac{1}{(\log \frac{1}{x})^t}$. 
GEODESIC PATHS
Möbius transformations
Möbius transformations

- Conformal automorphisms of $\mathbb{C}_\infty$. 
MöBIUS Transformations

- Conformal automorphisms of \( \mathbb{C}_\infty \).
- Conformal hyperbolic isometries of \( \mathbb{H}^3 \).
Notation

\[ t_n(z) = \frac{1}{b_n + z} \]
Notation

\[ t_n(z) = \frac{1}{b_n + z} \quad T_n = t_1 \circ t_2 \circ \cdots \circ t_n \]
**Notation**

\[ t_n(z) = \frac{1}{b_n + z} \quad T_n = t_1 \circ t_2 \circ \cdots \circ t_n \]

\[ T_n(\infty) = T_{n-1}(t_n(\infty)) = T_{n-1}(0) \]
THREE-DIMENSIONAL HYPERBOLIC SPACE
If $b_n$ are real (if they are integers, for example) then $T_n(\gamma)$ remains in the *vertical half plane*.
THE UPPER HALF-PLANE
The upper half-plane
Motivation
Hausdorff dimension
Geodesic paths
The Farey graph
Rapid escape
Theorem proof

\[ \gamma \]

\[ \infty \]

\[ \mathbb{H}^3 \]

\[ C \]
### Motivation
- Hausdorff dimension
- Geodesic paths
- The Farey graph
- Rapid escape
- Theorem proof

### Geodesic Paths in \( \mathbb{H}^3 \)

The diagram illustrates geodesic paths in hyperbolic 3-space \( \mathbb{H}^3 \). Geodesic paths are curves that locally minimize distance. The points 0, \( T_1(0) \), \( T_2(0) \), and \( T_3(0) \) represent endpoints or intermediate points along these paths. The infinite symbol \( \infty \) denotes a boundary point in \( \mathbb{H}^3 \) that cannot be reached by any finite geodesic path from 0.
The Farey graph
Positive integer coefficients henceforth

\[ b_n \in \mathbb{N} \]
THE VERTICAL HALF-PLANE
THE VERTICAL HALF-PLANE
The extended modular group

\[ \tilde{\Gamma} = \left\{ z \mapsto \frac{az + b}{cz + d} : a, b, c, d \in \mathbb{Z}, \ |ad - bc| = 1 \right\} \]
The Farey graph

\[ \tilde{\Gamma}(\gamma) \]
The Farey graph

Vertices = \mathbb{Q}

Join \frac{a}{b} to \frac{c}{d} if and only if |ad - bc| = 1.
The Farey graph

Vertices = \mathbb{Q}
The Farey graph

Vertices = \mathbb{Q}

Join \( \frac{a}{b} \) to \( \frac{c}{d} \) if and only if \( |ad - bc| = 1 \).
The Farey graph
Mediants

\[ \frac{a}{b} \quad \text{and} \quad \frac{c}{d} \]
**Mediants**

\[
\frac{a}{b} \quad \frac{a+c}{b+d} \quad \frac{c}{d}
\]
**Paths in the Farey Graph**

- **Motivation**
- **Hausdorff dimension**
- **Geodesic paths**
- **The Farey graph**
- **Rapid escape**
- **Theorem proof**
A correspondence
A correspondence

Integer continued fractions
A correspondence

Integer continued fractions

Paths from $\infty$ in the Farey graph
More about the Farey graph
More about the Farey graph

Continued fractions from the Farey graph
More about the Farey graph

Continued fractions from the Farey graph

Alan Beardon, Meira Hockman, and Ian Short
Open Problem V

Give necessary and sufficient conditions for an integer continued fraction $K(1|b_n)$ to converge.
Open Problem V

Give necessary and sufficient conditions for an integer continued fraction $K(1|b_n)$ to converge.
Rapid Escape
RAPID ESCAPE
RAPID ESCAPE

\[ \rho(\mathbb{z}_n, j) \to \infty \quad \text{fast} \]
Rapid escape

$$\rho(z_n, j) \to \infty \quad \text{fast}$$

$$\sum_{n=1}^{\infty} \exp[-s\rho(z_n, j)] < +\infty \quad \text{for each } s > 0.$$
Rapid escape

\[ \rho(z_n, j) \to \infty \quad \text{fast} \]

\[ \sum_{n=1}^{\infty} \exp[-s \rho(z_n, j)] < +\infty \quad \text{for each } s > 0 \]

That is, the sequence has critical exponent 0.
Path with positive integers
Rapidly shrinking geodesics
Rapidly shrinking geodesics

\[ |T_n(0) - T_n(\infty)| \leq \frac{1}{2n-1} \]
**Angle of Parallelism Lemma**

\[
\cosh \rho(j, \gamma) = \frac{2}{|a-b|}
\]
SOME EQUATIONS

\[ \exp[-\rho(j, T_n(j))] \]
SOME EQUATIONS

\[ \exp[-\rho(j, T_n(j))] \leq \frac{1}{\cosh[\rho(j, T_n(j))]}. \]
Some equations

\[
\exp[-\rho(j, T_n(j))] \leq \frac{1}{\cosh[\rho(j, T_n(j))]} \leq \frac{1}{\cosh[\rho(j, T_n(\gamma))]}\]
Some equations

\[ \exp[-\rho(j, T_n(j))] \leq \frac{1}{\cosh[\rho(j, T_n(j))]} \leq \frac{1}{\cosh[\rho(j, T_n(\gamma))]} = \frac{|T_n(0) - T_n(\infty)|}{2} \]
Some equations

\[ \exp[-\rho(j, T_n(j))] \leq \frac{1}{\cosh[\rho(j, T_n(j))]} \leq \frac{1}{\cosh[\rho(j, T_n(\gamma))]} = \frac{|T_n(0) - T_n(\infty)|}{2} \leq \frac{1}{2^n} \]
Conclusion

\[ \sum_{n=1}^{\infty} \exp[-s\rho(j, T_n(j))] \leq \sum_{n=1}^{\infty} \frac{1}{2^{ns}} < +\infty \]
CONCLUSION

\[ \sum_{n=1}^{\infty} \exp[-s \rho(j, T_n(j))] \leq \sum_{n=1}^{\infty} \frac{1}{2^n s} < +\infty \]

Hence \( T_n(j) \) is a rapid escape sequence.
Theorem proof
Recall our objective

For positive integers $b_1, b_2, \ldots$, 
Recall our objective

For positive integers $b_1, b_2, \ldots$, show that $T_n(z)$ converges to a constant $p$ for all points $z$ outside a subset of $(-\infty, -1)$ of Hausdorff dimension 0.
SWITCH TO BACKWARDS ORBITS

\[ T_1^{-1}(j), T_2^{-1}(j), T_3^{-1}(j), \ldots \]
Switch to backwards orbits

\[ T_1^{-1}(j), T_2^{-1}(j), T_3^{-1}(j), \ldots \]

\[ \rho \left( T_n^{-1}(j), j \right) = \rho \left( j, T_n(j) \right) \]
Backwards orbit accumulates in $(-\infty, -1)$
The sequence $T_n(\infty)$ converges to a constant $p$. 
The sequence $T_n(\infty)$ converges to a constant $p$.

For which other $z$ does $T_n(z)$ converge to $p$?
Choose a point $z$ outside $(-\infty, -1)$. 
$z \notin (-\infty, -1)$
$z \notin (-\infty, -1)$
$z \notin (-\infty, -1)$
Choose a point $z$ inside $(-\infty, -1)$. 
$z \in (-\infty, -1)$
$z \in (-\infty, -1)$
$z \in (-\infty, -1)$
In summary, $T_n(z)$ does not converge to $p$ if and only if there are infinitely many blue points.
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More formally, $T_n(z)$ does not converge to $p$ if and only if $z$ lies in the conical limit set of $T_n$. 
In summary, $T_n(z)$ does not converge to $p$ if and only if there are infinitely many blue points.

More formally, $T_n(z)$ does not converge to $p$ if and only if $z$ lies in the conical limit set of $T_n$.

How big is this conical limit set?
Recall that $\rho(T_n^{-1}(j), j) \to \infty$ rapidly.
Since the conical limit set consists of those points that lie in infinitely many shadows, it has Hausdorff dimension 0.