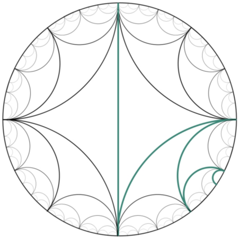
Hyperbolic geometry and continued fraction theory III

Ian Short 23 February 2010



http://maths.org/ims25/maths/presentations.php

Today

Today

Sets of divergence

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Hausdorff dimension of sets of divergence for continued fractions

Motivation

Iteration of holomorphic functions

$$f : \mathbb{C}_\infty \to \mathbb{C}_\infty$$

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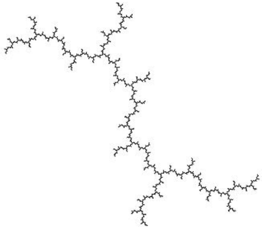
$$f : \mathbb{C}_\infty \to \mathbb{C}_\infty$$

$$f, \quad f^2, \quad f^3, \ldots$$

Julia sets



Julia set of $f(z) = z^2 + i$

Kleinian groups

Discrete subgroups of $\mathcal{M}$.

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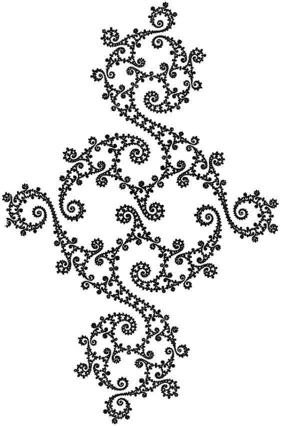
Kleinian groups are countable.

Limit sets



Limit set of a Schottky group

Continued fractions

$$\mathbf{K}(b_n) = \cfrac{1}{b_1 + \cfrac{1}{b_2 + \cfrac{1}{b_3 + \cfrac{1}{b_4 + \cdots}}}}$$

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$$T_n = t_1 \circ t_2 \circ \cdots \circ t_n$$

Julia sets for continued fractions

Common features

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- A sequence $F_1, F_2, \ldots$ of holomorphic maps of $\mathbb{C}_\infty$.

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- The derived set J of

$$\bigcup_{n=1}^{\infty} \{z \,:\, F_n(z) = w\},$$

  is, generally, independent of w.
- The complement of J is the largest open set on which $F_1, F_2, \ldots$ is a normal family.

Theorem to be proved today

Given *positive integers* $b_1, b_2, \ldots,$

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Then $T_n(z)$ converges to p for all points z outside a subset of $(-\infty, -1)$ of Hausdorff dimension 0.

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Given *positive integers* $b_1, b_2, \ldots$, let

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Then $T_n(z)$ converges to p for all points z outside a subset of $(-\infty, -1)$ of logarithmic Hausdorff dimension 1.

Open Problem IV

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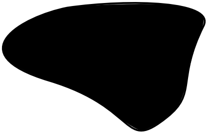
How are the coefficients of $\mathbf{K}(1|\, b_n)$ related to the (logarithmic) Hausdorff dimension of the associated set of divergence?

Hausdorff dimension

Dimensions



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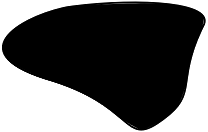

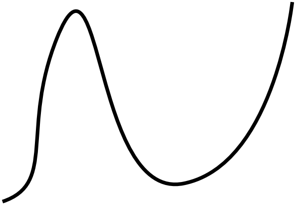

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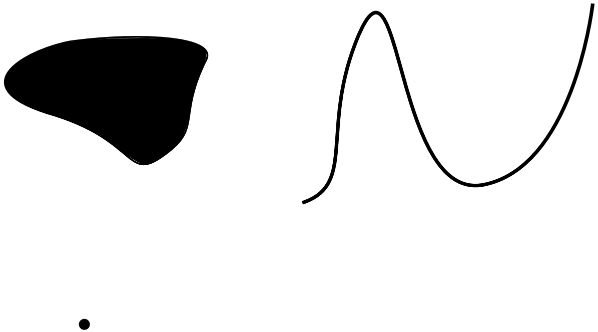

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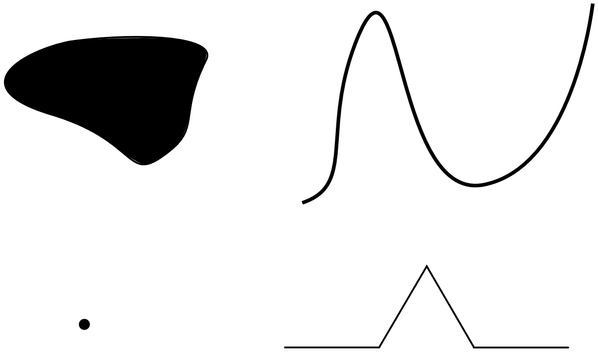

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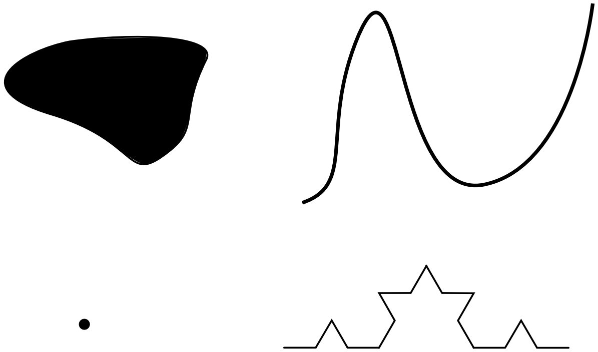

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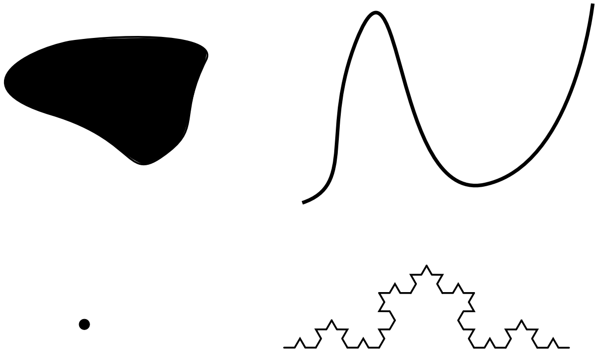

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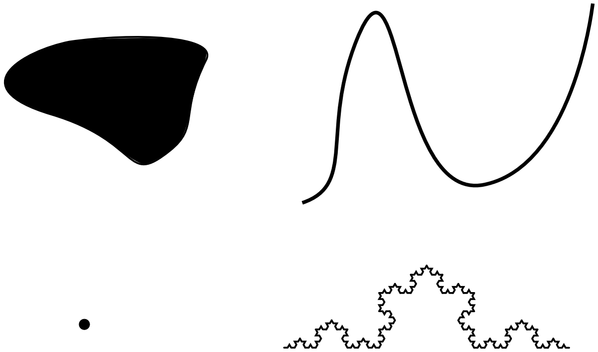
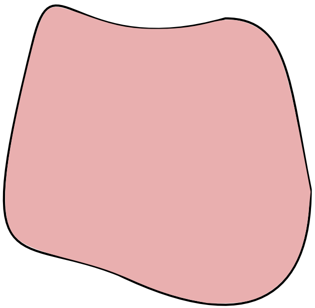


Hausdorff t-dimensional measure



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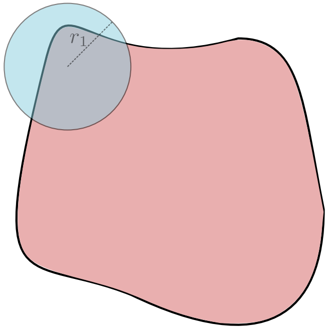

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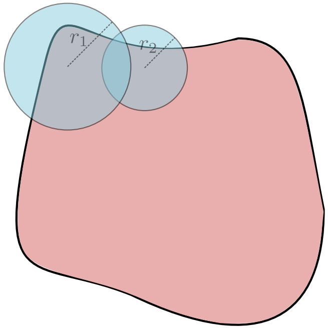

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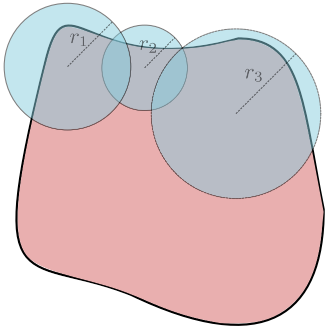

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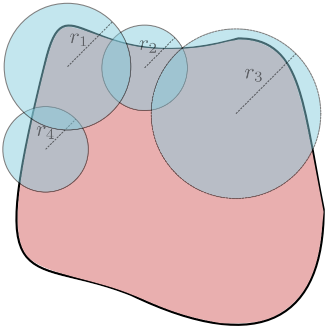

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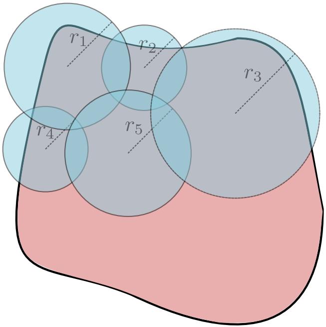

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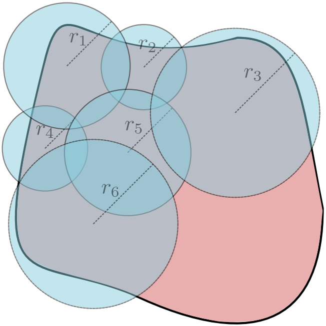

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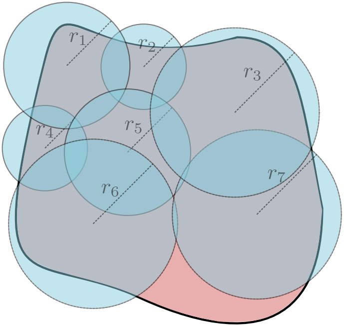

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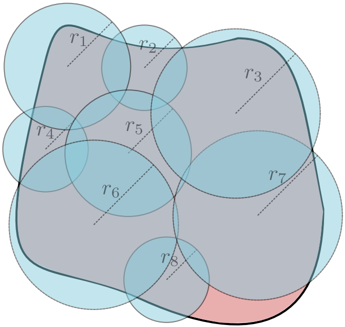

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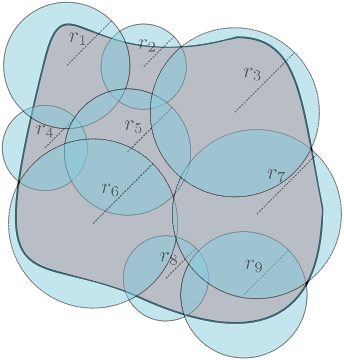

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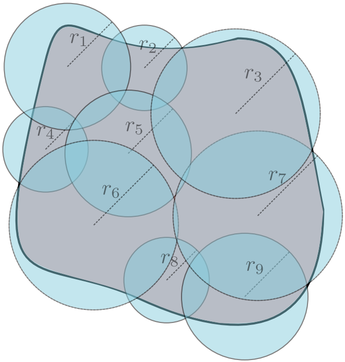
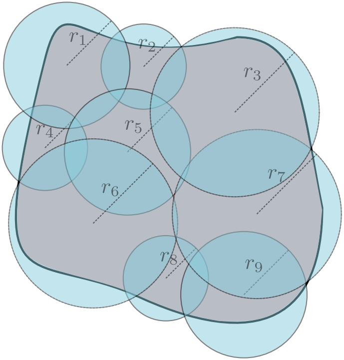


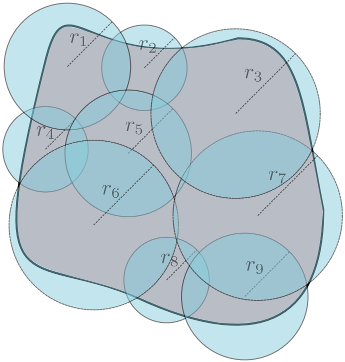
$$\sum r_i^t$$

Hausdorff t-dimensional measure



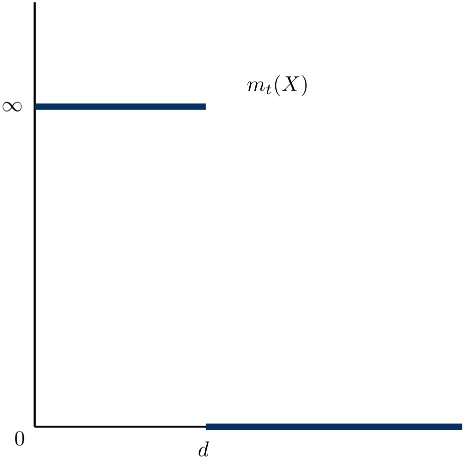
$$\inf_{r_i<\delta} \sum r_i^t$$

Hausdorff t-dimensional measure



$$m_t(X) = \lim_{\delta \to 0} \inf_{r_i < \delta} \sum r_i^t$$

Hausdorff dimension of a set X



The Cantor set

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Logarithmic Hausdorff dimension

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Instead of $f(x) = x^t$....

Logarithmic Hausdorff dimension

Instead of $f(x) = x^t$....

.... use $f(x) = \frac{1}{\left(\log \frac{1}{x}\right)^t}$.

Geodesic paths

Möbius transformations

Möbius transformations

- Conformal automorphisms of $\mathbb{C}_\infty$.

Möbius transformations

- Conformal automorphisms of $\mathbb{C}_\infty$.
- Conformal hyperbolic isometries of $\mathbb{H}^3$.

Notation

$$t_n(z) = \frac{1}{b_n + z}$$

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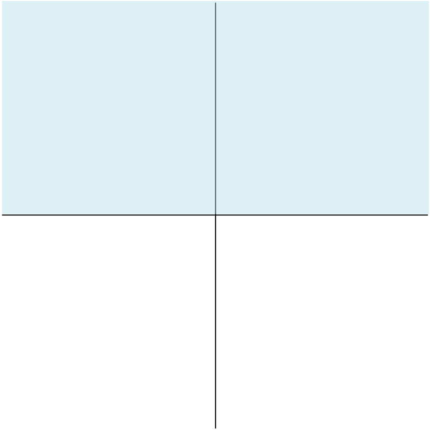
$$T_n(\infty) = T_{n-1}(t_n(\infty)) = T_{n-1}(0)$$

Three-dimensional hyperbolic space

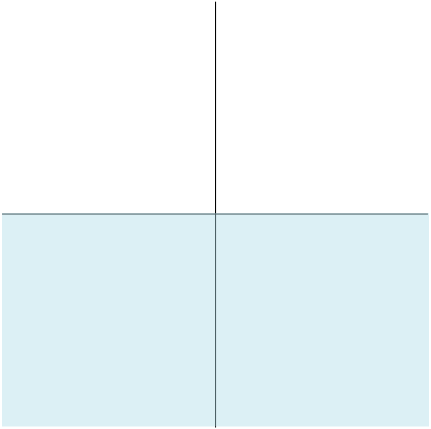
Important remark

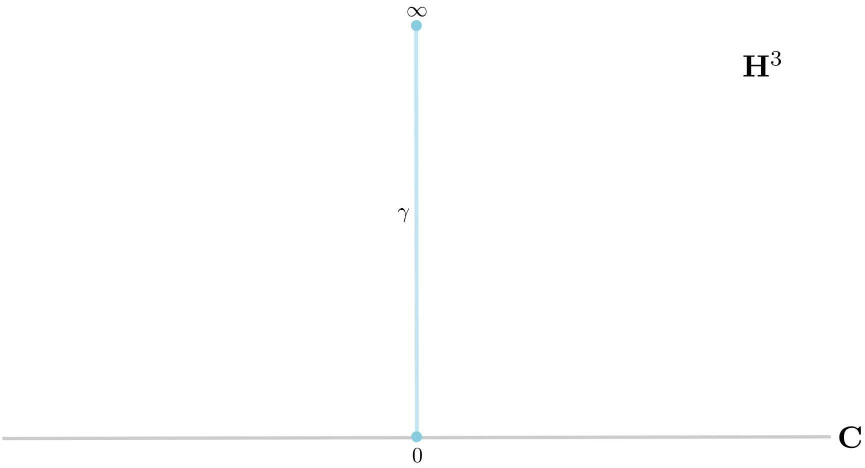
If b_n are real (if they are integers, for example) then $T_n(\gamma)$ remains in the *vertical half plane*.

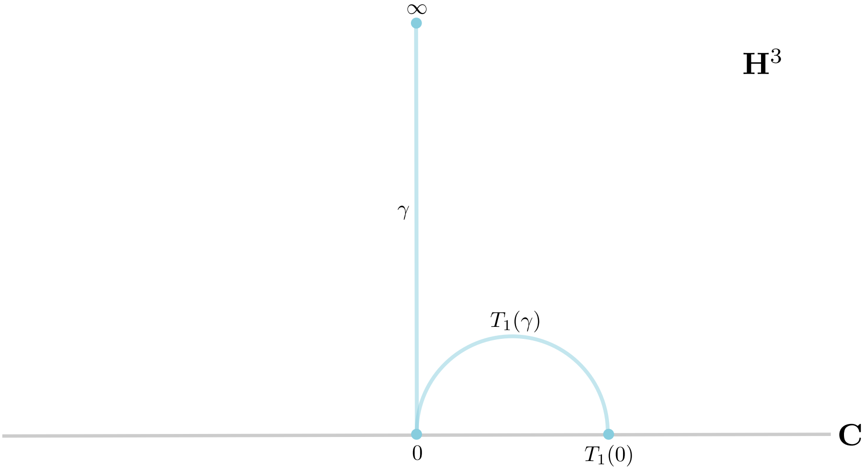
The upper half-plane

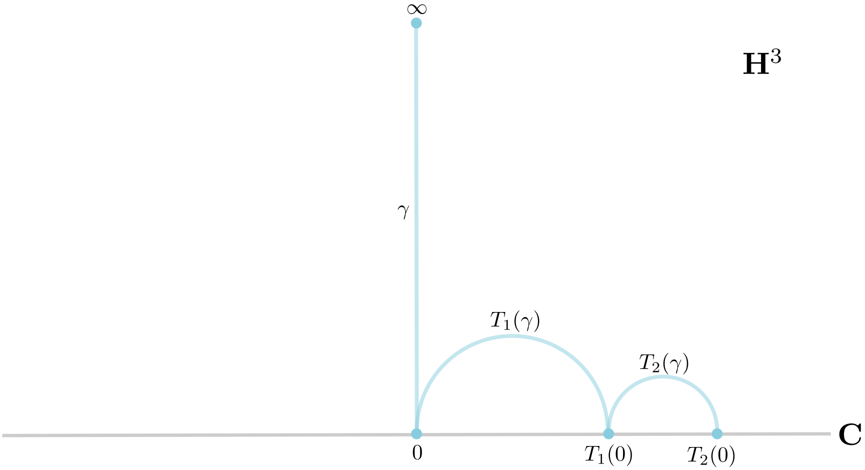


The upper half-plane




∞
$\mathbf{H}^3$
γ
$\mathbf{C}$
0


∞
$\mathbf{H}^3$
γ
$T_1(\gamma)$
$\mathbf{C}$
0
$T_1(0)$


∞
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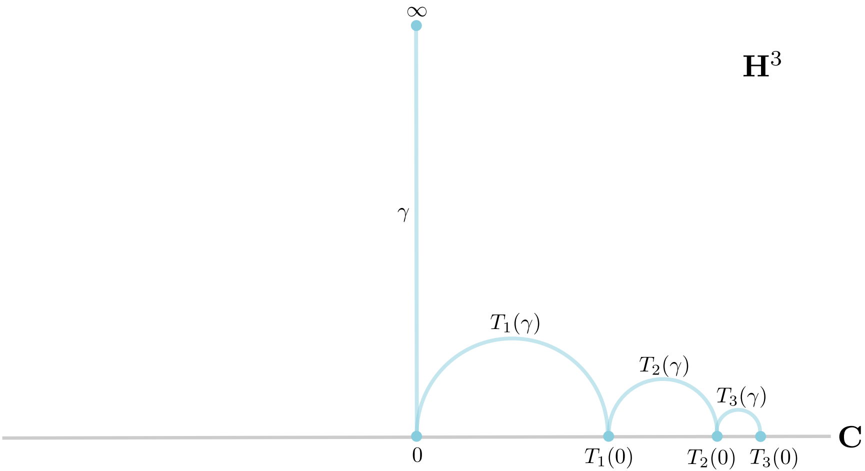
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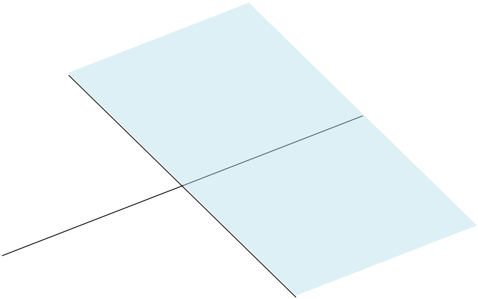
$T_1(0)$
$T_2(0)$
$T_3(0)$

The Farey graph

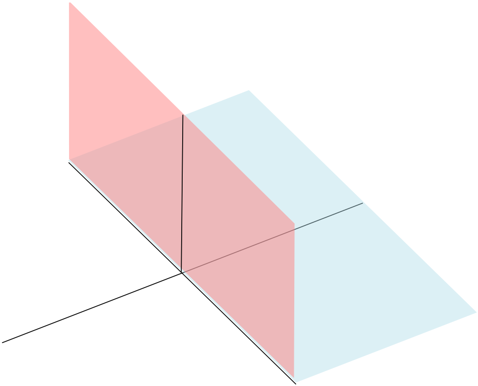
Positive integer coefficients henceforth

$$b_n \in \mathbb{N}$$

The vertical half-plane



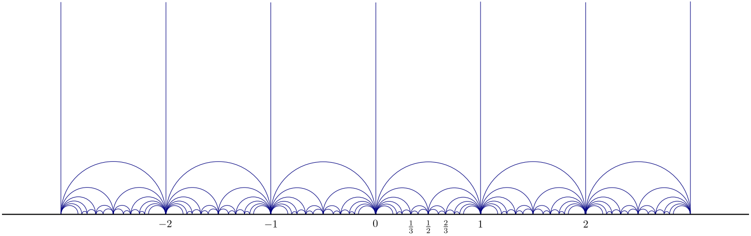
The vertical half-plane



The extended modular group

$$\widetilde{\Gamma} = \left\{ z \mapsto \frac{az+b}{cz+d} \;:\; a,b,c,d \in \mathbb{Z},\ |ad-bc| = 1 \right\}$$

The Farey graph



$\widetilde{\Gamma}(\gamma)$

The Farey graph

The Farey graph

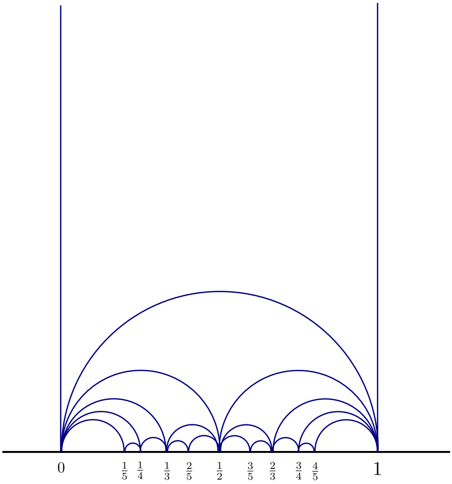
$$\text{Vertices} = \mathbb{Q}$$

The Farey graph

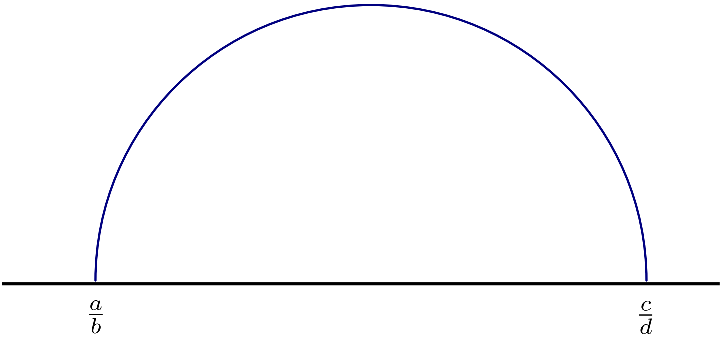
Vertices $= \mathbb{Q}$

Join $\frac{a}{b}$ to $\frac{c}{d}$ if and only if $|ad - bc| = 1$.

The Farey graph



Mediants



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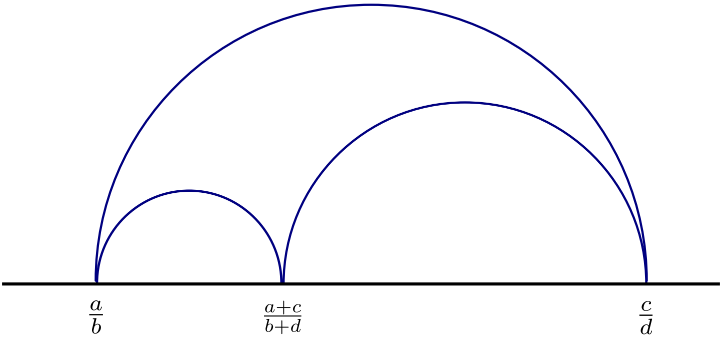
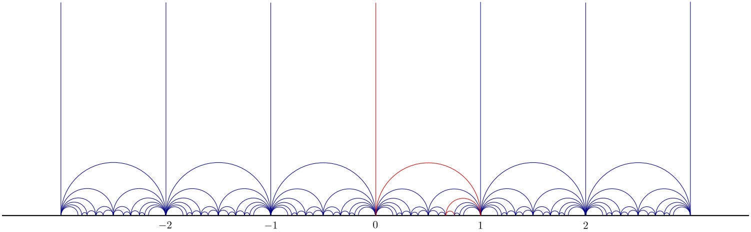


Paths in the Farey graph



A correspondence

A correspondence

Integer continued fractions

A correspondence

Integer continued fractions

Paths from ∞ in the Farey graph

More about the Farey graph

More about the Farey graph

Continued fractions from the Farey graph

More about the Farey graph

Continued fractions from the Farey graph

Alan Beardon, Meira Hockman, and Ian Short

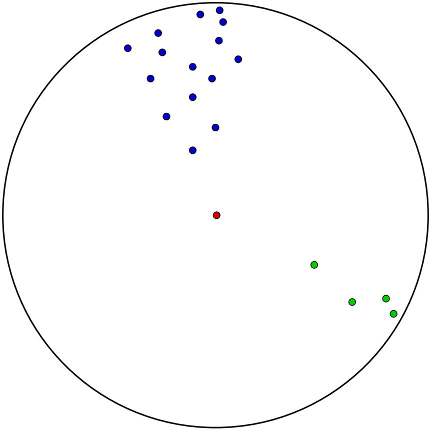
Open Problem V

Open Problem V

Give necessary and sufficient conditions for an integer continued fraction $\mathbf{K}(1|\, b_n)$ to converge.

Rapid escape

Rapid escape



Rapid escape

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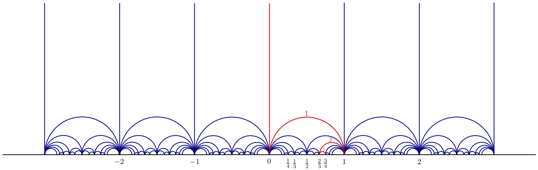
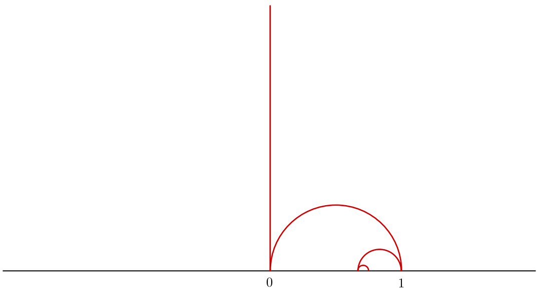
$$\sum_{n=1}^{\infty} \exp[-s\rho(z_n, j)] < +\infty \quad \text{for each } s > 0$$

That is, the sequence has *critical exponent* 0.

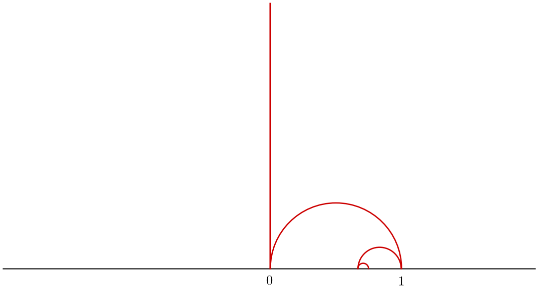
Path with positive integers



Rapidly shrinking geodesics

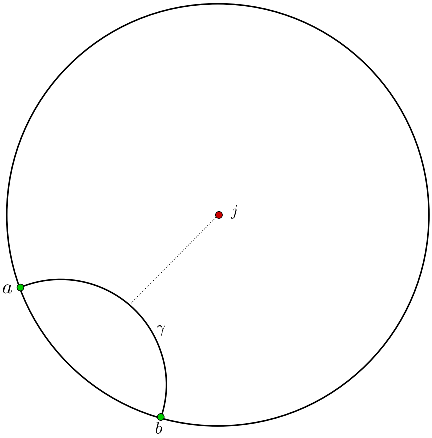


Rapidly shrinking geodesics



$$|T_n(0) - T_n(\infty)| \leqslant \frac{1}{2^{n-1}}$$

Angle of parallelism lemma



$$\cosh \rho(j,\gamma) = \frac{2}{|a-b|}$$

Some equations

$$\exp[-\rho(j, T_n(j))]$$

Some equations

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Some equations

$$
\begin{aligned}
\exp[-\rho(j, T_n(j))] &\leqslant \frac{1}{\cosh[\rho(j, T_n(j))]} \\
&\leqslant \frac{1}{\cosh[\rho(j, T_n(\gamma))]} \\
&= \frac{|T_n(0) - T_n(\infty)|}{2}
\end{aligned}
$$

Some equations

$$\begin{aligned}\exp[-\rho(j, T_n(j))] &\leqslant \frac{1}{\cosh[\rho(j, T_n(j))]} \\ &\leqslant \frac{1}{\cosh[\rho(j, T_n(\gamma))]} \\ &= \frac{|T_n(0) - T_n(\infty)|}{2} \\ &\leqslant \frac{1}{2^n}\end{aligned}$$

Conclusion

$$\sum_{n=1}^{\infty} \exp[-s\rho(j, T_n(j))] \leqslant \sum_{n=1}^{\infty} \frac{1}{2^{ns}} < +\infty$$

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$$\sum_{n=1}^{\infty} \exp[-s\rho(j, T_n(j))] \leqslant \sum_{n=1}^{\infty} \frac{1}{2^{ns}} < +\infty$$

Hence $T_n(j)$ is a rapid escape sequence.

Theorem proof

Recall our objective

For positive integers $b_1, b_2, \ldots,$

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For positive integers $b_1, b_2, \ldots$, show that $T_n(z)$ converges to a constant p for all points z outside a subset of $(-\infty, -1)$ of Hausdorff dimension 0.

Switch to backwards orbits

$$T_1^{-1}(j), T_2^{-1}(j), T_3^{-1}(j), \ldots$$

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$$T_1^{-1}(j), T_2^{-1}(j), T_3^{-1}(j), \dots$$

$$\rho\left(T_n^{-1}(j), j\right) = \rho\left(j, T_n(j)\right)$$

Backwards orbit accumulates in $(-\infty, -1)$

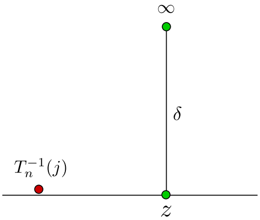
The sequence $T_n(\infty)$ converges to a constant p.

The sequence $T_n(\infty)$ converges to a constant p.

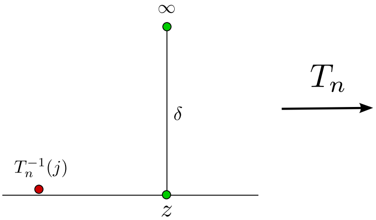
For which other z does $T_n(z)$ converge to p?

Choose a point z outside $(-\infty, -1)$.

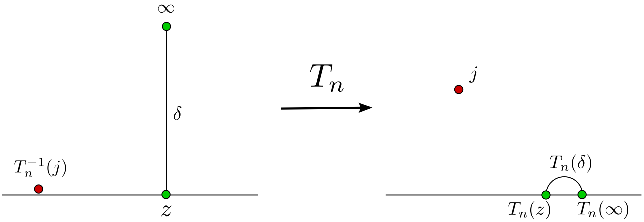
$z \notin (-\infty, -1)$



$z \notin (-\infty, -1)$

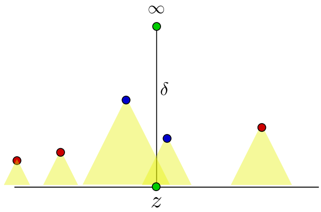


$z \notin (-\infty, -1)$

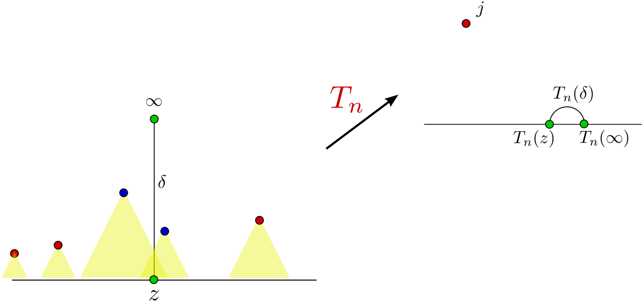


Choose a point z inside $(-\infty, -1)$.

$z \in (-\infty, -1)$



$z \in (-\infty, -1)$



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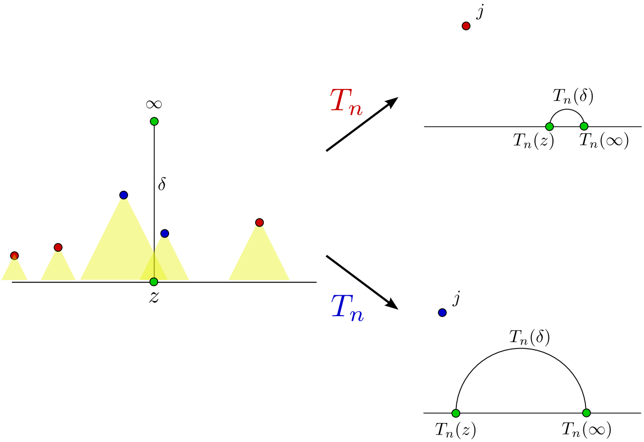

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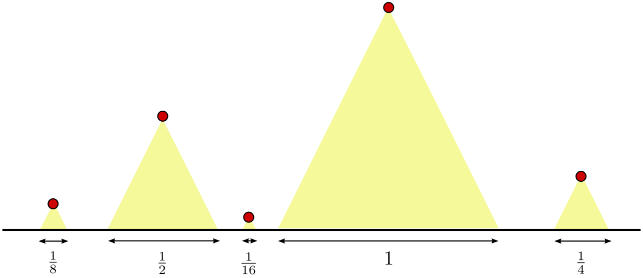
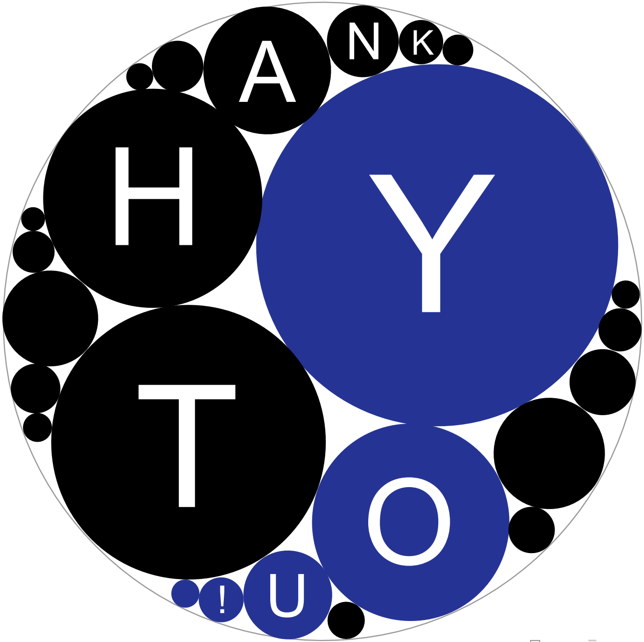
How big is this conical limit set?

Recall that $\rho(T_n^{-1}(j), j) \to \infty$ *rapidly*



Finale

Since the conical limit set consists of those points that lie in infinitely many shadows, it has Hausdorff dimension 0.


THANK YOU!