The Proof of the Anti-Pasch Conjecture

It was in the morning of 27th October 1999 that we finally completed the proof of the anti-Pasch conjecture. To explain the conjecture, we need to take a few steps back from it and start by explaining the idea of a *Steiner triple system*, STS(v). This comprises a set of v points and a collection of triples of these points. The triples are usually called *blocks* and they collectively have the property that every pair of distinct points occurs in precisely one of the blocks. Examples for $v = 7$ and $v = 9$ are given below.

STS(7)	STS(9)	
1 2 3	1 2 3	1 5 9
1 4 5	4 5 6	2 6 7
1 6 7	7 8 9	3 4 8
2 4 6	1 4 7	1 6 8
2 5 7	2 5 8	2 4 9
3 4 7	3 6 9	3 5 7
3 5 6		

The blocks are written horizontally without brackets or commas so, for example, 1 2 3 means the triple {1, 2, 3}. In the case of the STS(7), there are seven blocks and in the case of the STS(9) there are 12 blocks. It is easy to count the number of blocks in an STS(v). Since there are $v(v\text{-}1)/2$ pairs of points and each triple $a\ b\ c$ contains the three pairs $a\ b$, $a\ c$ and $b\ c$, the number of triples must be $v(v\text{-}1)/6$. Of course this number must be an integer, so either v or v-1 must be divisible by 3. A further necessary condition may be discovered by considering those triples containing a single specified point x. There are v-1 points other than x and each triple containing x accounts for two of these v-1 points. Thus v-1 has to be even and so v must be odd. Putting the two conditions together, it is easy to deduce that an STS(v) can only exist if v is of the form $6s$+1 or $6s$+3, where s is a positive integer (or zero in the case $v = 3$).

Steiner triple systems should, arguably, be named after the Rev. Thomas Pennington Kirkman who was the first person to prove their existence for all values of v having the form $6s$+1 or $6s$+3. Kirkman established this result in 1847 [1], but his work was overlooked. The question of their existence was raised again in 1853 by Jakob Steiner [2]. However, Steiner did not provide a solution and he was unaware of the work of Kirkman.

Steiner triple systems form only one instance of a more general set of structures known as *block deigns*. Variations may be played on the theme by having a different size (k) for the blocks, requiring that the pairs occur an equal fixed number of times (λ) rather than just once, or replacing the condition for pairs by an analogous condition for t-tuples ($t \geq 2$). The generalisation is called a t-(v, k, λ) block design. In this terminology, an STS(v) is a 2-(v, 3, 1) block design. Surprisingly little is known in the general case. Indeed, with

$\lambda = 1$ and $v > k > t \geq 4$ only a finite number of designs are known, and none of these have $t > 5$.

Specifying the value of v in an STS(v) does not generally determine the design uniquely. We have to be careful what we mean by "uniquely" here, since we may call the points anything we like. We say that two STS(v)s are isomorphic if it is possible to rename the points of one design so that the blocks become identical to those of the other. Up to isomorphism, there is only one STS(3), one STS(7) and one STS(9). However, there are two non-isomorphic STS(13)s, 80 non-isomorphic STS(15)s and, it is estimated, about $1{\cdot}1 \times 10^9$ non-isomorphic STS(19)s. It has been shown that, for large v, the number of non-isomorphic STS(v)s is approximately $v^{v^2/6}$ (see [3]).

It can be quite difficult to decide if two STS(v)s are isomorphic. One could examine all possible $v!$ permutations of the points, but even for modest values of v this is prohibitively lengthy. One therefore looks for structural features which may differ in non-isomorphic STS(v)s. One such feature is the number of quadrilaterals or Pasch configurations which a system contains.

A *quadrilateral* or *Pasch configuration* is a set of four blocks of an STS(v) which collectively contain only six points. Any such set must have the form $\{a\ b\ c,\ a\ y\ z,\ x\ b\ z,\ x\ y\ c\}$ for six distinct points a, b, c, x, y, z. Returning to the example of the STS(7) we see that the blocks 1 2 3, 1 4 5, 2 4 6 and 3 5 6 form a quadrilateral. In fact the STS(7) contains seven quadrilaterals. On the other hand, the STS(9) contains no quadrilaterals; it is *quadrilateral-free* or *anti-Pasch*. In the case of $v = 13$, the two non-isomorphic systems contain different (but non-zero) numbers of quadrilaterals. For $v = 15$ the 80 non-isomorphic systems contain quadrilaterals ranging in number between 0 and 105. At least 1010 of the non-isomorphic STS(19)s are anti-Pasch.

The number of quadrilaterals in an STS(v) can help to distinguish it from other STS(v)s, but the anti-Pasch conjecture is really concerned with the extreme cases: can we find an anti-Pasch STS(v) for all v of the forms $v = 6s+1$ and $6s+3$ (apart from the known exceptions for $v = 7$ and $v = 13$)? Perhaps surprisingly, the existence of anti-Pasch STS(v)s does have a practical application to the encoding of data on computer disks. In theoretical terms the conjecture is interesting because the Pasch configuration is the smallest collection of blocks of an STS(v) which is potentially avoidable.

In 1976 Paul Erdös made a very comprehensive conjecture that for given L and for all sufficiently large v there exists an STS(v) which avoids all configurations having l blocks and $l+2$ points for $4 \leq l \leq L$. The anti-Pasch conjecture relates to the case $L = 4$. Many researchers, including ourselves, have worked on this problem over the past 20 years. References [4] to [9] at the end of this article give some of the relevant papers.

How does one set about proving that the anti-Pasch conjecture is true? The method used is to construct anti-Pasch STS(v)s for all possible values of v. For the case $v = 6s+3$, this was first done by Brouwer in 1977. This left $v = 6s+1$, and for this case a number of different constructions were employed, the majority of which are recursive. It isn't possible to show all of these constructions here as this would be far too lengthy. However, the following "product" construction contains some of the typical features. It produces an STS(uv) from an STS(u) and an STS(v). It is easy to check that if the original systems are anti-Pasch then so is the resulting system.

Suppose that the points of the STS(u) are 1, 2, . . . , u and that the points of the STS(v) are 1, 2, . . . , v. We take the points of the STS(uv) to be the ordered pairs (1, 1), (1, 2), . . . , (1, v), (2, 1), (2, 2), . . . , (2, v), . . . , (u, v). The blocks of the STS(uv) are of three types.

Type I	$(a, x)\ (a, y)\ (a, z)$	where $1 \le a \le u$ and $x\ y\ z$ is a block of the STS(v).
Type II	$(a, x)\ (b, x)\ (c, x)$	where $a\ b\ c$ is a block of the STS(u) and $1 \le x \le v$.
Type III	$(a, x)\ (b, y)\ (c, z)$	where $a\ b\ c$ is a block of the STS(u) and $x\ y\ z$ is a block of the STS(v).

Note that the Type III blocks come in sets of six because the blocks $a\ b\ c$ of the STS(u) and $x\ y\ z$ of the STS(v) give rise to six blocks of the STS(uv) obtained by varying the orders of the elements:

$$(a, x)\ (b, y)\ (c, z) \qquad (a, x)\ (c, y)\ (b, z) \qquad (a, y)\ (b, x)\ (c, z)$$
$$(a, y)\ (b, z)\ (c, x) \qquad (a, z)\ (b, x)\ (c, y) \qquad (a, z)\ (b, y)\ (c, x)$$

The total number of blocks is therefore

$$u\left(\frac{v(v-1)}{6}\right) + v\left(\frac{u(u-1)}{6}\right) + 6\left(\frac{u(u-1)}{6}\cdot\frac{v(v-1)}{6}\right) = \frac{uv(uv-1)}{6},$$

as expected. It is easy to verify that every pair of distinct points $\{(\alpha, \beta), (\gamma, \delta)\}$ appears in one of the blocks.

By the spring of 1999 we had established the result apart from values of v of the forms $v = 5040t+1579$ and $v = 5040t+3595$. By this stage we had developed a recursive construction which could also potentially fill both of these gaps in the argument. However, these values required us to construct anti-Pasch STS(v)s for $v = 139$ and $v = 235$ having additional structural properties.

In September 1999, after a three-month computer run, our colleague Carol Whitehead at Goldsmiths College finally obtained the smaller of these two designs. The computer produced designs of this type by adding triples at random to a base set of triples, checking that the structural properties required were consistent with the triples already added, and removing any "obvious" obstructing triples. Because of the danger of getting into an endless loop of additions and removals, each attempt to produce the required design was terminated after a certain point. The computer must have examined several billion possibilities before finding a solution.

The fact that the design on 139 points found by Carol had taken the computer so long made us very pessimistic about producing the corresponding design on 235 points which we knew would finish the problem. The likely running time for this design would be an order of magnitude greater than three months. After numerous re-examinations of our constructions to try to find an alternative strategy, it finally dawned on us on 26th October that we could use our recursive construction in a different way based on an STS(61) having even more stringent structural properties. The computer was set loose to find this exotic object. In an hour it found a solution which we checked and found to be incorrect –

we'd made a mistake in setting up the data. In an irritated but optimistic state we retired to bed. The next morning, with the data corrected, the computer found the final missing ingredient. By 10.00 a.m. the three of us had a solution to this 20 year-old problem.

References

The following papers are given in chronological order.

[1] T. P. Kirkman, *On a problem in combinations*, Cambridge and Dublin Mathematical Journal, 2, **1847**, 191-204.

[2] J. Steiner, *Combinatorische Aufgabe*, Journal für die Reine und Angewandte Mathematik, 45, **1853**, 181-182 (Gesammelte Werke II, 435-438).

[3] R. M. Wilson, *Nonisomorphic Steiner triple systems*, Mathematische Zeitschrift, 135, **1974**, 303-313.

[4] P. Erdös, *Problems and results in combinatorial analysis*, Creation in Mathematics, 9, **1976**, 25.

[5] A. E. Brouwer, *Steiner triple systems without forbidden subconfigurations*, Mathematisch Centrum Amsterdam, ZW 104/77, **1977**.

[6] T. S. Griggs, J. Murphy and J. S Phelan, *Anti-Pasch Steiner triple systems*, Journal of Combinatorics, Information and Systems Sciences, 15, **1990**, 79-84.

[7] D. R. Stinson and Y. J. Wei, *Some results on quadrilaterals in Steiner triple systems*, Discrete Mathematics, 105, **1992**, 207-219.

[8] A. C. H. Ling, C. J. Colbourn, M. J. Grannell and T. S. Griggs, *Construction techniques for anti-Pasch Steiner triple systems*, Journal of the London Mathematical Society, Series 2, 61, **2000**, 641-657.

[9] M. J. Grannell, T. S. Griggs and C. A. Whitehead, *The resolution of the anti-Pasch conjecture*, Journal of Combinatorial Designs, 8, **2000**, 300-309.

A good source of further information about Steiner triple systems is the following book.

C. J. Colbourn and A. Rosa, *Triple Systems*, Oxford University Press, **1999**, 560pp, ISBN 0198535767.

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