Distance and fractional isomorphism in Steiner triple systems

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Abstract

In [8], Quattrochi and Rinaldi introduced the idea of n^{-1} - isomorphism between Steiner systems. In this paper we study this concept in the context of Steiner triple systems. The main result is that for any positive integer N, there exists $v_0(N)$ such that for all admissible $v \geq v_0(N)$ and for each STS(v) (say S), there exists an STS(v) (say S') such that for some $n > N$, S is strictly n^{-1}-isomorphic to S'. We also prove that for all admissible $v \geq 13$, there exist two STS(v)s which are strictly 2^{-1}-isomorphic.

Define the distance between two Steiner triple systems S and S' of the same order to be the minimum volume of a trade T which transforms S into a system isomorphic to S'. We determine the distance between any two Steiner triple systems of order 15 and, further, give a complete classification of strictly 2^{-1}-isomorphic and 3^{-1}-isomorphic pairs of STS(15)s.

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1 Introduction

A *Steiner triple system* of order v, briefly STS(v), is a pair $(V, \mathcal{B})$ where V is a set of cardinality v of *elements*, or *points*, and $\mathcal{B}$ is a collection of *triples*, also called *blocks* or *lines*, which has the property that every pair of distinct elements of V occurs in precisely one triple. It is well known that an STS(v) exists if and only if $v \equiv 1$ or 3 (mod 6). Such values of v are called *admissible*. An n-line *configuration* is a collection of n triples which has the property that every pair of distinct elements occurs in at most one triple. If $\mathcal{C}$ is a configuration, we denote the number of blocks by $b(\mathcal{C})$, the number of points by $p(\mathcal{C})$ and the set of points by $P(\mathcal{C})$. The *degree* of a point is the number of triples which contain it. Two configurations $\mathcal{C}$ and $\mathcal{D}$ are said to be *isomorphic*, $\mathcal{C} \cong \mathcal{D}$, if there exists a one-to-one mapping $\phi : P(\mathcal{C}) \to P(\mathcal{D})$ such that for each triple $T \in \mathcal{C}$, $\phi(T)$ is a triple in $\mathcal{D}$. Two STS(v)s, $(V, \mathcal{B})$ and $(V', \mathcal{B}')$ are *isomorphic* if $\mathcal{B} \cong \mathcal{B}'$. Up to isomorphism, the STS(3), STS(7) and STS(9) are unique. There are two STS(13)s, 80 STS(15)s and, as shown recently by Kaski and Östergård [4], 11,084,874,829 STS(19)s. In this paper we confine our attention mainly to the 80 pairwise non-isomorphic STS(15)s, and we refer to them by the standard numbering as given in [2, Chapter 5].

A *trade* $T = \{\mathcal{T}_1, \mathcal{T}_2\}$ is a pair of disjoint m-line configurations $\mathcal{T}_1$ and $\mathcal{T}_2$ which has the property that every pair of distinct elements occurs in precisely the same number (zero or one) of triples of $\mathcal{T}_1$ as of $\mathcal{T}_2$. Traditionally, the number of lines, m, is called the *volume* of the trade, denoted by vol(T), and the *foundation* of the trade, found(T), is the set of elements covered by $\mathcal{T}_1$ and $\mathcal{T}_2$. As it will be important to distinguish between a trade $T = \{\mathcal{T}_1, \mathcal{T}_2\}$ and either of the configurations $\mathcal{T}_1$ and $\mathcal{T}_2$, the latter will be referred to as *tradeable configurations*. If $S = (V, \mathcal{B})$ and $S' = (V, \mathcal{B}')$ are two Steiner triple systems and $T = \{\mathcal{C}, \mathcal{D}\}$ is a pair of configurations with $\mathcal{C} \subseteq \mathcal{B}$ such that S' is isomorphic to $(V, (\mathcal{B} \setminus \mathcal{C}) \cup \mathcal{D})$, then we say that T *transforms S into S'*. The trades $T = \{\mathcal{T}_1, \mathcal{T}_2\}$ and $T' = \{\mathcal{T}_1', \mathcal{T}_2'\}$ are said to be *isomorphic* if there exists a one-to-one mapping ϕ : found(T) $\to$ found(T') such that $\phi(\{\mathcal{T}_1, \mathcal{T}_2\}) = \{\mathcal{T}_1', \mathcal{T}_2'\}$. It is well-known that there exist trades of volume n only for $n = 4$ and $n \geq 6$ [6]; for example, there is a unique trade of volume 4, called a *Pasch switch*. A complete list of trades of up to 9 blocks is given in [3], from which it can be seen that every trade, $\{\mathcal{T}_1, \mathcal{T}_2\}$, of volume not exceeding 8 has $\mathcal{T}_1 \cong \mathcal{T}_2$.

We will be interested in three basic questions. The first of these is as follows. Given two STS(v)s, S and S', what is the minimum volume of a trade T which transforms S into (a system isomorphic to) S'? Formally, we define this to be the *distance*, $d(S, S')$, between S and S'. Observe that $d(S, S')$ is a metric in the usual sense. We investigate the distance problem for STS(15)s and we note in passing that the distance between the cyclic STS(13) and the non-cyclic STS(13) is 4.

The other questions are motivated by a paper of Quattrocchi and Rinaldi [8], who introduce the concept of n^{-1}-isomorphism. Two configurations $\mathcal{C}$ and $\mathcal{D}$ are said to be n^{-1}-*isomorphic* if there are partitions $\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_n$ of $\mathcal{C}$ and $\mathcal{D}_1, \mathcal{D}_2, \ldots, \mathcal{D}_n$ of $\mathcal{D}$ such that $\mathcal{C}_i \cong \mathcal{D}_i$ for $i = 1, 2, \ldots, n$. Two Steiner triple systems, $(V, \mathcal{B})$ and $(V', \mathcal{B}')$, are n^{-1}-isomorphic if $\mathcal{B}$ and $\mathcal{B}'$ are n^{-1}-isomorphic. For $n \geq 2$, two configurations are said to be *strictly* n^{-1}-*isomorphic* if they are n^{-1}-isomorphic but not $(n-1)^{-1}$-isomorphic; similarly for Steiner triple systems. It is natural to call this concept, *fractional isomorphism*. (Note that this has a different meaning to that given in [9], in relation to graphs.) Clearly, 1^{-1}-isomorphism is the same as isomorphism. However, unlike isomorphism, n^{-1}-isomorphism is not necessarily an equivalence relation if $n \geq 2$; reflexivity and symmetry are always satisfied but in general transitivity fails.

The second question is related to the first. If the trade $T = \{\mathcal{T}_1, \mathcal{T}_2\}$ transforms S to S' and $\mathcal{T}_1 \cong \mathcal{T}_2$, then S and S' are 2^{-1}-isomorphic. However, as noted in [3], there are trades consisting of non-isomorphic tradeable configurations. We ask the following question. For two non-isomorphic STS(v)s, S and S', what is the minimum volume of a trade T, consisting of isomorphic tradeable configurations, which transforms S into (a system isomorphic to) S'? Formally, we define this to be $h(S, S')$. If no such trade exists, we write $h(S, S') = \infty$. If $h(S, S') < \infty$, then S and S' are 2^{-1}-isomorphic. Although exceptions are relatively scarce, the converse is not necessarily true, as our investigations of the h function for STS(15)s will reveal.

The third question is for what values of n do there exist two STS(v)s, S and S', which are strictly n^{-1}-isomorphic.

2 Fractional isomorphism

It was Kirkman [5] who gave the first proof that for all admissible v there exists an STS(v). Later, Moore [7] proved that for all admissible $v \geq 13$, there exist two non-isomorphic STS(v)s; see [2, page 70]. Our next goal is to state and prove two existence theorems concerning n^{-1}-isomorphic STS(v)s.

Theorem 2.1 *For all admissible $v \geq 13$, there exist two STS(v)s which are strictly 2^{-1}-isomorphic.*

We conjecture that for each positive integer n, there exists $v_0(n)$ such that for all admissible $v \geq v_0(n)$ there exist two STS(v)s which are strictly n^{-1}-isomorphic. Whilst we are unable to prove this conjecture, we can establish a weaker result in the same direction.

Theorem 2.2 *For any positive integer N, there exists a positive integer $v_0(N)$ such that for all admissible $v \geq v_0(N)$ and for each STS(v) (say S), there exists an STS(v) (say S') such that for some $n > N$, S is strictly n^{-1}-isomorphic to S'.*

Before dealing with these theorems we prove some lemmas.

Lemma 2.1 *If $\mathcal{X}$ is a configuration, let $\rho(\mathcal{X})$ denote the (possibly empty) set of blocks obtained by removing from $\mathcal{X}$ all blocks containing points of degree 1. Suppose $\mathcal{C}$ and $\mathcal{D}$ are configurations which cover the same pairs, and suppose also that $\mathcal{C} \cong \mathcal{D}$. Then $\rho(\mathcal{C}) \cong \rho(\mathcal{D})$ and $\rho(\mathcal{C})$ covers the same pairs as $\rho(\mathcal{D})$.*

Proof. If $\mathcal{C}$ contains no points of degree 1, there is nothing to prove.

Otherwise let $\tau : P(\mathcal{C}) \rightarrow P(\mathcal{D})$ be an isomorphism from $\mathcal{C}$ to $\mathcal{D}$. Let $\mathcal{A}$ be the set of blocks of $\mathcal{C}$ which contain points of degree 1. Since $\mathcal{C}$ and $\mathcal{D}$ cover the same pairs, a block containing a point of degree 1 in one of the configurations $\mathcal{C}$ and $\mathcal{D}$ must also occur in the other configuration. Therefore $\mathcal{A} \subseteq \mathcal{C} \cap \mathcal{D}$. Then $\rho(\mathcal{C}) = \mathcal{C} \setminus \mathcal{A}$ and $\rho(\mathcal{D}) = \mathcal{D} \setminus \mathcal{A}$. Since $\tau(\mathcal{A}) = \mathcal{A}$, we have $\rho(\mathcal{C}) \cong \rho(\mathcal{D})$. Furthermore, since we have removed the same pairs from $\mathcal{C}$ and $\mathcal{D}$, the configurations $\rho(\mathcal{C})$ and $\rho(\mathcal{D})$ cover the same pairs. □

Lemma 2.2 *Suppose S and S' are Steiner triple systems and that $n \geq 1$. If there exists a trade $\{\mathcal{C}, \mathcal{D}\}$ with $\mathcal{C}$ n^{-1}-isomorphic to $\mathcal{D}$ that transforms S to S', then S is $(n+1)^{-1}$-isomorphic to S'.*

Proof. This follows directly from the definitions. □

The converse of Lemma 2.2 is not true. In an attempt to identify the reason for this, we define a *pseudo-trade* as a pair of configurations $\{\mathcal{C}, \mathcal{D}\}$ such that $\mathcal{C}$ and $\mathcal{D}$ cover the same pairs, $\mathcal{C} \cong \mathcal{D}$, $\mathcal{C} \cap \mathcal{D} \neq \emptyset$, and for any non-empty subset $\mathcal{A}$ of $\mathcal{C} \cap \mathcal{D}$ we have $\mathcal{C} \setminus \mathcal{A} \ncong \mathcal{D} \setminus \mathcal{A}$. By Lemma 2.1, $\mathcal{C}$ and $\mathcal{D}$ have no points of degree 1.

Pseudo-trades of small volume may be enumerated by methods similar to those described in [3]. An example of a pseudo-trade is given by $\{\mathcal{C}, \mathcal{D}\}$ = {{012, 034, 056, 135, 146, 179, 1*bc*, 245, 37*b*, 47*c*, 49*b*}, {016, 024, 035, 125, 13*b*, 14*c*, 179, 347, 456, 49*b*, 7*bc*}}, where $\mathcal{C} \cap \mathcal{D} = \{179, 49b\}$. The table below gives, for small volumes, the number of labelled pseudo-trades $\{\mathcal{C}, \mathcal{D}\}$ where configuration $\mathcal{C}$ is canonically labelled [2, page 52].

$\lvert\mathcal{C}\rvert$	≤ 10	11	12	13
pseudo-trades $\{\mathcal{C}, \mathcal{D}\}$	0	8	24	168

With the definition of pseudo-trades in place we have the following result.

Lemma 2.3 *Let $S = (V, \mathcal{B})$ and $S' = (V, \mathcal{B}')$ be strictly 2^{-1}-isomorphic Steiner triple systems. Let $\beta = (|\mathcal{B}| - 1)/2$. Then there exist $T = \{\mathcal{C}, \mathcal{D}\}$ with $\mathcal{C} \cong \mathcal{D}$ and $|\mathcal{C}| \leq \beta$ where T is either a trade or a pseudo-trade and T transforms S to S'.*

Proof. Suppose there exists a 2^{-1}-isomorphism consisting of a partition of $\mathcal{B}$ into $\mathcal{B}_0$ and $\mathcal{B}_1$ with $|\mathcal{B}_1| \leq \beta$, a partition of $\mathcal{B}'$ into $\mathcal{B}'_0$ and $\mathcal{B}'_1$ with $|\mathcal{B}'_1| \leq \beta$, and one-to-one mappings $\phi_0 : V \to V$ and $\phi_1 : V \to V$ such that $\phi_0(\mathcal{B}_0) = \mathcal{B}'_0$ and $\phi_1(\mathcal{B}_1) = \mathcal{B}'_1$. Apply ϕ_0^{-1} to S', let $\mathcal{B}''_1 = \phi_0^{-1}(\mathcal{B}'_1)$ and consider the pair $\{\mathcal{B}_1, \mathcal{B}''_1\}$. Note that $|\mathcal{B}_1| \leq \beta$ and that $(\mathcal{B} \setminus \mathcal{B}_1) \cup \mathcal{B}''_1 \cong \mathcal{B}'$.

Let $\mathcal{F} = \mathcal{B}_1 \cap \mathcal{B}''_1$. If $\mathcal{F} = \emptyset$ then $\{\mathcal{B}_1, \mathcal{B}''_1\}$ is a trade which satisfies the conditions of the lemma. So we may assume that $\mathcal{F}$ is non-empty. If $\{\mathcal{B}_1, \mathcal{B}''_1\}$ is a pseudo-trade, we are done. Otherwise there exists a non-empty set $\mathcal{G}$ of maximum cardinality such that $\mathcal{G} \subseteq \mathcal{F}$ and $\mathcal{B}_1 \setminus \mathcal{G} \cong \mathcal{B}''_1 \setminus \mathcal{G}$. It is clear from the definition that $\{\mathcal{B}_1 \setminus \mathcal{G}, \mathcal{B}''_1 \setminus \mathcal{G}\}$ is a trade or a pseudo-trade with the required properties. □

The final lemma provides the main ingredient for the proof of Theorem 2.1.

Lemma 2.4 *For all admissible $v \geq 27$, there exists an STS(v) which contains precisely one sub-STS(13).*

Proof. For admissible v such that $27 \leq v \leq 63$, it is straightforward to generate STS(v)s with the desired property by Stinson's hill-climbing method [10].

For admissible $v > 63$ we employ a recursive construction. Let G be a $\{3\}$-GDD of type $g^t h^u$ and suppose we have an STS($g+13$) and an STS($h+13$), each having a unique sub-STS(13). Construct a new Steiner triple system, S, of order $tg + uh + 13$ as follows. Let T be an STS(13). On each group of size g, together with the points of T, put an STS($g+13$) such that the sub-STS(13) coincides with T. Similarly, on each group of size h, together with the points of T, put an STS($h+13$) such that the sub-STS(13) coincides with T.

Suppose, further, that G has at most four groups; i.e. $t+u \leq 4$. We show that the system S has a unique STS(13), namely T. To prove this, suppose U is a sub-STS(13) of S and $U \neq T$. Label the groups $G_1, G_2, \ldots, G_n$, where $n = 3$ or 4. Let A_i be the set of points of U which lie on G_i and let A be the set of points which are common to both U and T. Consider three cases according to the size of A.

(i) $|A| = 0$. For each i, we must have $|A_i| = 0$, 1 or 3. (We can rule out $|A_i| = 7$ and $|A_i| = 9$ because we know that neither STS(13) has a sub-STS(7) or a sub-STS(9).) As there are at most four groups, U cannot exist.

(ii) $|A| = 1$. Similarly we must have $|A_i| = 0$ or 2. Again, there are insufficient groups for U to exist.

(iii) $|A| = 3$. Now we are forced to have $|A_i| = 0$ for all i.

Thus the construction described above preserves the property of containing a unique sub-STS(13). By a theorem of Colbourn, Hoffman and Rees [1], there exist $\{3\}$-GDDs of the following types:

$$g^3$$
$$g^3h^1, \quad g \equiv h \equiv 0 \pmod 2),\ h \leq 2g$$

Using the construction with $\{3\}$-GDDs of these types and starter systems of orders 27, 31, ..., 63, we can generate the desired STS(v)s for all admissible $v > 63$ as follows.

First we construct a suitable STS(67) using a $\{3\}$-GDD of type 18^3. Then, using $\{3\}$-GDDs of type g^3h^1 with $g = 14$ and $h = 14$, 18, 20, 24 and 26, we construct suitable STS(v)s for $v = 69$, 73, 75, 79 and 81, respectively. Now let $k \geq 4$ and suppose that we already have suitable STS(u)s for admissible u in the range $27 \leq u \leq 3^k$. Let an admissible v be given such that $3^k < v \leq 3^{k+1}$, and write $v = 6r + e$, where $e = 1$ or 3. If $r \equiv 0$ or 1 (mod 3), put $s = 2r + 1$

and $t = 36 + e$; otherwise put $s = 2r + 3$ and $t = 30 + e$. Let $g = s - 13$ and $h = t - 13$. Then in either case $2g - h \geq 4r - e - 47 \geq 0$, since $r \geq 14$ and $e \leq 3$. It is easily verified that $27 \leq s, t \leq 3^k$ for admissible s and t; hence we can use a $\{3\}$-GDD of type g^3h^1 to construct a suitable Steiner triple system of order $3g + h + 13 = v$. □

Proof of Theorem 2.1. The two STS(13)s are 2^{-1}-isomorphic because one can be transformed into the other by a Pasch trade. For the same reason, STS(15) #1 is 2^{-1}-isomorphic to STS(15) #2. Pairs of 2^{-1}-isomorphic STS(v)s for $v = 19$, 21 and 25 are easily produced by choosing an appropriate system and transforming it by a Pasch trade.

So let $v \geq 27$ and let S be an STS(v) which contains a unique sub-STS(13), T, say. Lemma 2.4 guarantees that S exists. Choose a Pasch configuration in T which when traded transforms T into an STS(13) of the other isomorphism type. Perform this trade thus transforming S into S', say. By Lemma 2.2, S is 2^{-1}-isomorphic to S', but clearly S is not isomorphic to S'. □

Proof of Theorem 2.2. Given a positive integer N, take $v_0(N)$ so large that for all $v \geq v_0(N)$, the number of distinct STS(v)s, $D(v)$, satisfies

$$D(v) > (v!)^N \; N^{v(v-1)/6}.$$

This is possible because $D(v) = v^{v^2(1/6+o(1))}$ as $v \to \infty$ (see [11]). In fact, for large N, $v_0(N) \leq N + o(N)$.

Now take any STS(v), say S, with $v \geq v_0(N)$. Partition the $v(v-1)/6$ blocks of S into N sets, some of which may be empty. Such a partition can be represented by a vector of length $v(v-1)/6$ with entries from 1 to N, so that the number of possible partitions is at most $N^{v(v-1)/6}$.

For each set of the partition, apply a permutation to the base set. The number of combinations of permutations is $(v!)^N$. Most resulting sets of triples will not be STS(v)s but it is clear that this process can give rise to at most $(v!)^N N^{v(v-1)/6}$ STS(v)s which are N^{-1}-isomorphic to S, and any such systems (on the same base set) will arise at least once in this manner. Hence there must exist an STS(v), say S', which is not N^{-1}-isomorphic to S. But S' is certainly $(v(v-1)/6)^{-1}$-isomorphic to S. Hence there exists $n > N$ such that S and S' are strictly n^{-1}-isomorphic. □

The computational results of this paper suggest that for $v \geq 15$ there exists a pair of STS(v)s which are strictly 3^{-1}-isomorphic.

3 Algorithms

Other main results of this paper are two matrices, $D = [d_{i,j}]$ and $H = [h_{i,j}]$, showing relations between Steiner triple systems of order 15. The first is the 'distance table' for STS(15)s, where $d_{i,j}$ is the volume of the smallest trade that transforms STS(15) #i into STS(15) #j, the numbers i and j referring to the standard numbering of the 80 STS(15)s. In the second matrix, H, the entry $h_{i,j}$ is the volume of the smallest trade between isomorphic configurations which transforms STS(15) #i into STS(15) #j. We describe two algorithms for computing $[d_{i,j}]$ and $[h_{i,j}]$.

Algorithm 3.1

> For $b = 4, 6, 7, 8, \ldots$, make a list, L_b, of all possible trades and pseudo-trades $\{\mathcal{C}, \mathcal{D}\}$, where $\mathcal{C}$ is a b-block configuration which can occur in an STS(15).
>
> For each STS(15), S, for each $\{\mathcal{C}, \mathcal{D}\} \in L_b$:
>
> > For each occurrence $\phi(\mathcal{C})$ of an isomorphic copy of $\mathcal{C}$ in S: transform S to S', say, by the trade or pseudo-trade $\{\phi(\mathcal{C}), \phi(\mathcal{D})\}$. Record the designation $(01 - 80)$ of S and S' as well as information about the trade.

Algorithm 3.2

> For $b = 4, 6, 7, 8, \ldots$, for each STS(15), S, for each set $\mathcal{C}$ of b blocks of S:
>
> > For each trade or pseudo-trade $\{\mathcal{C}, \mathcal{D}\}$: record the designation of S and S', the STS(15) that results from transforming S by $\{\mathcal{C}, \mathcal{D}\}$, as well as information about $\{\mathcal{C}, \mathcal{D}\}$.

In spite of its apparent naivety, Algorithm 3.2 is the preferred option. It turns out that Algorithm 3.1 is not practicable for dealing with $b \geq 10$ because of the difficulty of constructing the list L_b. On the other hand, Algorithm 3.2 does not require a predetermined list and, furthermore, there is an efficient method, described in [3], for constructing all possible trades $\{\mathcal{C}, \mathcal{D}\}$, if any, from a given configuration $\mathcal{C}$. Also it is clear from [3] how to adapt the procedure to construct pseudo-trades. In fact, we used both methods for $b \leq 9$ and thereby gave ourselves extra confidence that our computer programming was sound.

There are a number of ways to shorten the computational effort and reduce the amount of work to a reasonable level. We mention three observations. (i) A configuration that is part of a trade or a pseudo-trade has no points of degree one. (ii) To prove that two STS(15)s are 2^{-1}-isomorphic, we do not need to consider trades or pseudo-trades of volume greater than 17. This follows from the proof of Lemma 2.3. (iii) In computing the matrix H, after examining all trades of volume less than or equal to 17, a complete list of pairs (i, j) where $h_{i,j} > 17$ is known. If in addition we know that the smallest pseudo-trade which transforms STS(15) $\#i$ to STS(15) $\#j$ has volume $p \leq 17$, we can deduce that either $h_{i,j} \leq 35 - p$ or $h_{i,j} = \infty$, thus limiting the search space.

4 Results

The two matrices D and H are presented in tabular form. For clarity, only the upper half of the matrix is given; the other half follows by symmetry.

In Table 4.1, the entry (i, j), $i \leq j$, indicates $d_{i,j}$, the volume of the smallest trade that transforms STS(15) $\#i$ to STS(15) $\#j$. We do not distinguish between trades with isomorphic configurations and trades with non-isomorphic configurations. Numbers 10, 11, ..., 19 are represented by lower-case letters `a`, `b`, ..., `j`, respectively. We find that any STS(15) can be transformed into any other STS(15) by a trade of at most 19 blocks. Also 19 blocks are necessary only for the pairs $\{\text{STS(15) }\#01, \text{STS(15) }\#62\}$ and $\{\text{STS(15) }\#01, \text{STS(15) }\#71\}$. Eighteen blocks suffice for the rest. If STS(15) #01 is excluded, then 17 blocks are sufficient, and sometimes necessary.

Table 4.2 has the same format as Table 4.1 except that each trade consists of a pair of isomorphic configurations. The entry (i, j), $i \leq j$, indicates $h_{i,j}$, the volume of the smallest such trade that transforms STS(15) $\#i$ to STS(15) $\#j$. A dot indicates that no such trade exists: $h_{i,j} = \infty$. The same scheme as above is used for representing two-digit numbers, and entries that differ from the corresponding values in Table 4.1 are underlined. (Observe that values 4, 6, 7 and 8 occur at precisely the same locations in both tables.) There are only two values greater than 17: $h_{06,31} = 20$, represented by the letter `k` in the table, and $h_{07,25} = 24$, represented by the letter `o`.

Let ν be the smallest n such that every pair of STS(15)s is n^{-1}-isomorphic. Lemma 2.3 and the existence of pairs (i, j) where $h_{i,j} = \infty$ and $d_{i,j} > 17$, at $(01, 71)$ for example, implies that $\nu \geq 3$. However, from the information in Table 4.2 it is easy to deduce that $\nu \leq 4$. The table shows that STS(15) #11 is 2^{-1}-isomorphic to every other STS(15) except possibly STS(15) #01. Therefore it follows from Proposition 6 of [8] that for $2 \leq i < j \leq 80$, STS(15) #i is 4^{-1}-isomorphic to STS(15) #j. In a similar manner we can show that STS(15) #01 is 4^{-1}-isomorphic to STS(15) #j for $2 \leq j \leq 80$ by identifying a system STS(15) #k which is 2^{-1}-isomorphic to both STS(15) #01 and STS(15) #j.

Table 4.2 shows that all except 537 pairs of STS(15)s are 2^{-1}-isomorphic. However, to ascertain the full extent of 2^{-1}-isomorphism we must also determine 2^{-1}-isomorphic pairs of STS(15)s which are not indicated by Table 4.2. Their existence is possible because the converse of Lemma 2.2 is false. It suffices, by Lemma 2.3, to look for pseudo-trades of volume not greater than 17. In fact, for given i, j, all we need to do is allow common blocks in the search for $\mathcal{C}, \mathcal{D}$ with the smallest $|\mathcal{C}|$ such that $\mathcal{C}$ and $\mathcal{D}$ cover the same pairs, $\mathcal{C} \cong \mathcal{D}$ and $\{\mathcal{C}, \mathcal{D}\}$ transforms STS(15) #i to STS(15) #j. If $\mathcal{C} \cap \mathcal{D}$ is empty, $\{\mathcal{C}, \mathcal{D}\}$ is a trade; otherwise it is clear that $\{\mathcal{C}, \mathcal{D}\}$ is a pseudo-trade. Moreover, we can assume that $|\mathcal{C} \setminus \mathcal{D}| \geq 9$; for otherwise $\{\mathcal{C} \setminus \mathcal{D}, \mathcal{D} \setminus \mathcal{C}\}$ is a trade of volume at most 8 and therefore $\{\mathcal{C} \setminus \mathcal{D} \cong \mathcal{D} \setminus \mathcal{C}\}$. A complete search produces a further six 2^{-1}-isomorphic pairs. Specifically, let $e_{i,j}$ denote the smallest volume of a pseudo-trade, if any, that transforms STS(15) #i to STS(15) #j. Then we have the following values for pairs (i, j) where $i < j$ and $h_{i,j} = \infty$.

i, j	$05, 30$	$05, 34$	$12, 71$	$16, 29$	$19, 67$	$19, 72$
$e_{i,j}$	15	15	15	15	17	16

It is also worth mentioning that in the only two cases where $17 < h_{i,j} < \infty$ we have $e_{06,31} = e_{07,25} = 11$.

Thus 2^{-1}-isomorphism accounts for all except $537 - 6 = 531$ pairs of STS(15)s. To establish a 3^{-1}-isomorphism for the remaining pairs, three approaches may be used. Let S and S' be STS(15)s which are not 2^{-1}-isomorphic. (i) As in the proof of Proposition 6 of [8], it is sufficient to find an STS(15), S'', and trades, $\{\mathcal{C}, \mathcal{D}\}$ and $\{\mathcal{E}, \mathcal{F}\}$, where $\mathcal{C} \cong \mathcal{D}$ and $\mathcal{E} \cong \mathcal{F}$, such that $\{\mathcal{C}, \mathcal{D}\}$ transforms S'' to S, $\{\mathcal{E}, \mathcal{F}\}$ transforms S'' to S' and either $\mathcal{C} \bigcap \mathcal{E} = \emptyset$, or $\mathcal{C} \subseteq \mathcal{E}$, or $\mathcal{E} \subseteq \mathcal{C}$. (ii) We find a trade that consists of 2^{-1}-isomorphic configurations, possibly the one which was used to establish the value of the corresponding entry in Table 4.1, and then apply Lemma 2.2. (iii) We find a trade, $\{\mathcal{C}, \mathcal{D}\}$ that transforms S to S' and a set of blocks $\mathcal{X}$ of S disjoint from $\mathcal{C}$ such that $\mathcal{C} \cup \mathcal{X}$ is 2^{-1}-isomorphic to $\mathcal{D} \cup \mathcal{X}$.

The second approach is particularly effective. Elementary computation shows that every trade of volume at most 12 consists of a pair of 2^{-1}-isomorphic configurations. This accounts for every pair where there is a value of `c` or less in Table 4.1. Further, with a little more computation we can use the same method to establish the required 2^{-1}-isomorphism for the trades corresponding to entries in Table 4.1 with values `d`, `e`, `f` and `g`. Hence for pairs $\{i, j\}$ where there is one of these letters in Table 4.1 and a dot in Table 4.2, we have that STS(15) $\#i$ is 3^{-1}-isomorphic to STS(15) $\#j$.

Of the remaining 39 cases, where the value in Table 4.1 is `h`, `i` or `j`, 3^{-1}-isomorphic pairs have been found; four by method (i): $(01, j)$, $j = 33, 64, 76, 79$; a further 33 pairs by method (ii): $(01, j)$, $j =$ 36, 37, 38, 41, 44, 45, 46, 48, 49, 50, 52, 53, 55, 56, 57, 58, 60, 61, 63, 65, 66, 67, 68, 69, 70, 71, 72, 74, 75, 77, as well as $(02, 77)$, $(03, 80)$ and $(16, 80)$; and two pairs by method (iii): $(01, 43)$ and $(01, 62)$.

Thus we have proved the following.

Theorem 4.1 *Any two Steiner triple systems of order 15 are* 3^{-1}*-isomorphic.*

Two particular cases of the final 39, namely (01, 43) and (01, 62), required considerable amounts of computer time, mainly because methods (i) and (ii) failed to produce the desired results. So it is appropriate to give details of these 3^{-1}-isomorphisms.

In the first case we have:

STS(15) #43: 012 034 057 06a 08c 09d 0be 135 146 17c 189 1ae
1bd 236 247 258 29a 2bc 2de 37d 38b 39e 3ac 459
48e 4ab 4cd 56b 5ad 5ce 67e 68d 69c 78a 79b;

extended trade: (06a 08c 09d 135 17c 189 1ae 247 29a 39e
3ac 459 4cd 5ad 5ce 68d 69c 78a 146,
069 08a 0cd 139 15a 178 1ce 249 27a 35c
3ae 45d 47c 59e 68c 6ad 89d 9ac 146) = $(\mathcal{C}, \mathcal{D})$;

$\mathcal{C}_1$: 08c 09d 29a 39e 4cd 5ad 68d 69c 78a,
$\mathcal{D}_1$: 069 08a 139 15a 178 249 68c 6ad 89d,
$\mathcal{C}_2$: 06a 135 17c 189 1ae 247 3ac 459 5ce 146,
$\mathcal{D}_2$: 0cd 1ce 27a 35c 3ae 45d 47c 59e 9ac 146.

In the second case we have:

STS(15) #62: 012 034 057 068 09b 0ad 0ce 135 146 17a 18b 19e
1cd 236 245 27b 28c 29d 2ae 37c 38d 39a 3be 47d
48e 49c 4ab 569 58a 5bc 5de 67e 6ac 6bd 789;

extended trade: (012 034 09b 0ad 0ce 17a 19e 1cd 29d 37c
39a 3be 49c 4ab 5bc 5de 67e 6ac 6bd 245,
01e 02d 03c 04b 09a 129 17c 1ad 349 37e
3ab 4ac 5bd 5ce 67a 6be 6cd 9bc 9de 245) = $(\mathcal{C}, \mathcal{D})$;

$\mathcal{C}_1$: 09b 19e 1cd 29d 3be 49c 5bc 67e 6ac 6bd,
$\mathcal{D}_1$: 01e 02d 03c 09a 1ad 4ac 5ce 67a 6be 6cd,
$\mathcal{C}_2$: 012 034 0ad 0ce 17a 37c 39a 4ab 5de 245,
$\mathcal{D}_2$: 04b 129 17c 349 37e 3ab 5bd 9bc 9de 245.

In each case one can verify that the extended trade transforms the given STS(15) into STS(15) #01.

References

[1] C. J. Colbourn, D. G. Hoffman and R. S. Rees, A new class of group divisible designs with block size three, *J. Combin. Theory Ser. A* **59** (1992), 73-89.

[2] C. J. Colbourn and A. Rosa, *Triple Systems*, Oxford University Press, New York 1999.

[3] A. D. Forbes, M. J. Grannell and T. S. Griggs, Configurations and trades in Steiner triple systems, *Australas. J. Combin.* **29** (2004), 75-84.

[4] P. Kaski and P. R. J. Östergård, The Steiner triple systems of order 19, *Math. Comp.* **73, no.248** (2004), 2075-2092.

[5] T. P. Kirkman, On a problem in combinatorics, *Cambridge and Dublin Math. J.* **2** (1847), 191-204.

[6] C. C. Lindner and A. Rosa, Steiner triple systems having a prescribed number of triples in common, *Canad. J. Math.* **27** (1975), 1166-1175. Corrigendum: **30** (1978), 896.

[7] E. H. Moore, Concerning triple systems, *Math. Ann.* **43** (1893), 271-285.

[8] G. Quattrocchi and G. Rinaldi, Steiner systems and n^{-1}-isomorphisms, *J. Geom.* **58** (1997), 146-157.

[9] M. V. Ramana, E. R. Scheinermann and D. Ullman, Fractional isomorphism of graphs, *Discrete Math.* **132** (1994), 247-265.

[10] D. R. Stinson, Hill-climbing algorithms for the construction of combinatorial designs, *Ann. Discrete Math.* **26** (1995), 311-334.

[11] R. M. Wilson, Nonisomorphic Steiner triple systems, *Math. Z.* **135** (1974), 303-313.

Table 4.1: $d_{i,j}$, the distance between STS(15)s, part I

	0000000001111111111222222222233333333334 1234567890123456789012345678901234567890	4444444445555555555666666666677777777778 1234567890123456789012345678901234567890
01	04688ccaccdccae7deefffgeedffggghggiihgg	hgiiiigiiighhghhihghhjiiiihiiijhgiiiigif
02	044488688a886a6aaccccccacccdcdedefgfdd	eeggffefffeeedeefedfefefffefffgfeffehegf
03	04486487a7648478cbbbba98aabcbcdddefddc	ceeeeedeeecdccddedddefeeefeefefeeefegefh
04	044844474446886a8998886998a8acbccedba	bcedcdccddcccbcccbadcccdcdcddcddecdcecef
05	04446477468868a9bb8988aaab8bcccdedcc	ccddcdcdddcdcccddcbdcddddecddcedccddfcef
06	0486647774c8488999999889898aabcccbb	bcccccbcccccbbcccbccacacccdccccddccddddf
07	08977b678a48abbcac9abbbaaaccdecdcd	dccceccceddcfeddedddccfddddededdeecedfdccf
08	0448744444488bb776477797aa9abdba9	9cbbbcbbcbaba9abbbabcbacccccccbcccbcaecdf
09	04474448748499444666477788899cb99	9bba9b9ababa9a9aa97a99aaabbbb9bcdaa8ccde
10	04444487477994644777747989acaa9	9baa9b9bbbaba99ab99aaaababbabbb9b9dabe
11	04774c774466667744747477 98b988	8a889b89999a98aa988a897aa9aaa8aab989aaac
12	0777ab864664444884 76788aadb88	8bba9b9abbaa999bb87a9a9aababbabbdaa8dcde
13	04484477bb7747886879989beab9	9aaabbabbbabaa9cbaaabbacbccbcccccbbbdddd
14	07667b7bb7764888989a9aaeba9	9cccbcbbcbaca9babaabbcccccccccadcecdbdcef
15	0844479977774778 77aab8aaab	9aaabcaab9baaaab99aaba7ababbaabbb9aadcce
16	068ccddaa97bbbcbcdbcdfddb	cedededeeecdcbddeeddfedeeefeeefefefdgffh
17	0477bb9a77779aaa99bacbbb	baaabacbbbaccbbcacaaccacccbdbacccbcddccd
18	074a97777747a7977bbb99a	bcba9ba9ab9aa9aaab8a9a9ababbb8abcabbdaef
19	0668999977a79799aacb99	9baaab8bb9bbaaccbaaa9a4acbbbbabbababcaac
20	086777777779744aac9aa	abba99a999989899a84869989bb998aabaaaabce
21	0498889888a8aa499844	4a998848889a77a9a7ab67899aa9a99aa8a9bbbd
22	0899989a9a8888bb977	79baaa7a99884488848944a79999aa9ababbbadd
23	04444477447497b988	88a76868899988a89687b87a8888978ab877aabd
24	04444764444a7a968	8887886888976898987 7b8889888979ab874a9bc
25	044444766468b886	6a779889888888698877b9799a9a98a9ba97baad
26	04447477787b986	6a998a88a88986888889b9899aaa97a9b997babe
27	0474747794b888	68779a88869888896679a948998887988677aaac
28	0777744878668	8aa9688688678688994 7a9778998a478a897b8cc
29	0874478aca88	6ba788999897899a8969a668a9a9a8abbb7aaacd
30	0444684b886	6844888899986888987 9aa77888a8798b879977d
31	0777a8a899	9b778b8baaba9999a9a9caaaa8bab99a8697c88c
32	06487a968	6884886868876688984 7a64886899478884877ac
33	0487a888	8878668866747687867789948888848888 97a99b
34	089a688	8897868688676686867499776888648888 88a8bc
35	08b474	4b86787798774898a899a6669a977899a7bb98bd
36	08896	8a44888989879a9999abcb7a98b8a99bc8aab88c
37	0bab	94ba886988ab98998ac9cbaa977abaa9899bbb9c
38	047	7888477774666864 8888b99877744688a98a968c
39	04	7998448444687776848898a967977687778698bb
40	0	4968477477887777488999887969989a488b98b

Table 4.1: $d_{i,j}$, the distance between STS(15)s, part II

	4444444444555555555566666666667777777777 8
	1234567890123456789012345678901234567890
41	0988474774787476778898887888886 8a79899ab
42	088668667889668888 6b9b9644888964a6b88b9
43	04898898979aa989abcba988a8ba9bb8abb74c
44	0688689747987998ac87988899879a979947c
45	0747646687477788b798448878678869888c
46	044488887686766a9887466866647889ab9
47	0447697478878898687886889774898a9b
48	0477887667788b98847686688879a99bb
49	077686774888aa984676866477688ab8
50	068877766a8aa6874677686888966ab
51	0444444876b797466767789899a4bc
52	047647747c7987888644688a7a8ab
53	0474748884847886877 7a99887bd
54	0464466778766867888878a97ac
55	074766a4b98766466688 6888bc
56	07477b7a767846644aa9887bc
57	0486978666687887688a87ba
58	098778798679977978998bc
59	04a966688687888a8babcc
60	0ba99666764466 94778ba
61	07a4b9baaabbccdddddf
62	09689997869ba898bcd
63	09a9888798b97a9aca
64	08688878a99aabaac
65	06464664898997aa
66	04468776a89889a
67	0466746a8969cb
68	04747898999cb
69	08649a8a68bc
70	0488884a6ba
71	068a7788ca
72	08a6b68c9
73	0a9ba886
74	0898668
75	0a4aac
76	0b9dc
77	0bbb
78	08b
79	0c
80	0

Table 4.2: $h_{i,j}$ for pairs of STS(15)s, part I

	0000000001111111111222222222233333333334 1234567890123456789012345678901234567890	4444444445555555555666666666677777777778 1234567890123456789012345678901234567890
01	04688.ca.e.c...7.............g.........	....f
02	044488688c886a6aa.c..ccca..cee.........	................................eh......
03	04486487a7648478cbcccca8c.bcce........	..e................f...........e.......
04	044844474446886a8aa8886aa8c8bcbe..gbb	d...ded....dcdfh.fcdc..e..d......f.c....
05	04446477468868ab.c8a88bba.8bc..e.f..	.c.ef...d............e......d..ch....g.
06	0486647774c8488acbbbb88c8k8cc....bb	.cccfcchd.c..ce.ebcg..ecdf...c.d..c.e...
07	08a77.678.48a..f..o...b.......c...	....f...............c.....d.....c...eccf
08	0448744444488..7764777a7abb.c.caa	b.ccccbdd.cbcacb.cab..ccd....bd..fec....
09	044744487484aa444666477788bb.daa	ac.bbbabbbbaaadada7baaaacddbcbdd.bb8cf..
10	04444487477aa4644777747a8ab.bbb	acdbabcbbbcbaaadca9bacabccabcaccd.bbec..
11	04774c7744666677447474 77a8ea88	8a88ab8babbba8abb88a8a7cacaac8cabb8bcbgc
12	0777cb86466444488476788ae.c88	8h.cbbacbcbaaaccc87c9ccbce..fb.d.cc8....
13	04484477cc774788687998eb.b.a	ba.edbad.badabdddcac..acbfcgdc.h.c.b.d..
14	07667.7cf7764888989bab....e	a..d....g..dac..cdaccccc...e.b.....ff...
15	084447aa7777477877aab8abab	aaaaccbacabcccaccbbac.7.cabcgabb.daagd..
16	068cc....b7......dbc.....	.......................f................
17	0477fc9a7777abea99.ecd..	eaaafaffeba.....b.eaffacfd.db.c..de....d
18	074aa7777747a7977.b.bba	fcbabbaaacbaabbaab8a.a9cbabbc8ac.acc....
19	0668ba9977b797bcc..b.b	gbccac8eeacbcad.daca9c4ccc.fcab.a.abcaa.
20	08677777777a744aacbaa	acgbaaaba.b8a8aaa8486da8acbac8aabaaaac.f
21	04a888a888c8aa4a.844	4b.a884888aa77aca7ac678aacbababcc8a.cf..
22	08aaa8acae8888dda77	7bdbaa7baa884488848a44a7bbaabaaacde.cb..
23	04444477447497ba88	88a768688a9988a8a687.87b8888978ab877aa..
24	04444764444a7aa68	8887886888a768989877b888a888b79ab874a9c.
25	044444766468d886	6a779889888888698877.a799a9aa8a9ba97caae
26	04447477787.986	6a998a88a88986888889.a899baa97a9ca97cc..
27	04747477a4c888	68779a88869888896679e948998887988677aaaf
28	0777744878668	8aa9688688678688a947c97789a8a478a897b8..
29	0874478aca88	6cc788a998978aaa8a69c668aba9b8abd.7aaa.g
30	0444684e886	684488889a9868889879cb77888a8798b879977d
31	0777c8a8aa	ab778b8cbabaaa99baa9.caac8cbb99a8697e88e
32	06487a968	6884886868876688a847c6488689a478884877cd
33	0487b888	8878668866747687867 78a94888884888897ab9b
34	089b688	88a786868867668 68674e977688864888888a8c.
35	08e474	4b867877987748a8a899f6669a9778a9a7dc98..
36	088a6	8a4488898a879a9999accb7a98b8a99bc8aab88c
37	0bab	d4ea886a88bb98a98aca..aa977bbaa98bdeb.9.
38	047	78884777746668648888eba877744688a98aa68f
39	04	7aa84484446877768488.8a9679776877786b8..
40	0	4a684774778877777488aaa887a69989a488ba8.

Table 4.2: $h_{i,j}$ for pairs of STS(15)s, part II

	4444444444555555555566666666667777777778 1234567890123456789012345678901234567890
41	0b884747747874767788c88878888868a7989a.b
42	088668667889668886.cb9644888964d6c88cc
43	0489889897aaa98aab.fa988b8ba9be8ccc74f
44	0688689747987998bd8798889987aaa79947e
45	0747646687477788c7a8448878678869888.
46	044488887686766a9887466866647889acb
47	04476a74788788c868788688a7748a8abe
48	0477887667788d98847686688879a99..
49	077686774888aaa84676866477688ad8
50	068877766a8.a6874677686888b66af
51	0444444876b7974667677898a9a4d.
52	047647747d7a87888644688a7a8ab
53	0474748884847886877a9a887bd
54	046446677876686788878a97a.
55	074766b4b9876646668886888d.
56	07477c7a767846644ba9887..
57	0486e78666687887688a87ca
58	0a87787a8679a77a78a98d.
59	04c966688687888a8bbb..
60	0baa966676446694778ca
61	07.4dcbaad.bd....d..
62	09689997869da898b..
63	09ba888798ca7a9aca
64	08688878a99acbaae
65	06464664898ba7d.
66	04468776a89889a
67	0466746a8a69cb
68	047478a8999ce
69	08649a8a68cc
70	0488884a6ea
71	068a7788ca
72	08a6b68d9
73	0aada886
74	0898668
75	0a4aa.
76	0bc..
77	0b..
78	08b
79	0.
80	0