On the small covering numbers $g_{1}^{(5)}(v)$

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Abstract
The minimum number of blocks having maximum size precisely five that is required to cover, exactly once, all pairs of elements from a set of cardinality $v$ is denoted by $g_{1}^{(5)}(v)$. As a prelude to further work, we give the values of $g_{1}^{(5)}(v)$ for $v \leq 25$, and construct all designs which attain this bound.

1 Introduction
The covering number $g_{\lambda}^{(k)}(v)$ is defined as the cardinality of the minimum pairwise balanced design (PBD) on a set of $v$ points that has largest block size $k$ and such that every pair occurs exactly $\lambda$ times in the design. The values of $g_{\lambda}^{(k)}(v)$ for $k = 4$ and all $\lambda$ have been determined, in a series of papers [4] to [17].

In the present paper we consider the case $k = 5$, $\lambda = 1$. A lower bound on $g_{1}^{(5)}(v)$ was given in [1] as

$$\frac{1}{20} \left\{ 6g_2 + 2g_3 + v \left( \left\lfloor \frac{v-1}{4} \right\rfloor - (v-1) \right) \right\},$$
where $g_2$ and $g_3$ are the number of blocks of lengths 2 and 3 respectively in the exact covering. Various applications were also given. However, this bound provides no useful information for small values of $v$ and so we here provide a table of the values of $g_1^{(5)}(v)$ for $v \leq 25$. The values for 18, 19, 20, 21, 22, 23, 24, 25, are already given in [1]. But in the present paper we also determine all designs which attain the lower bound $g_1^{(5)}(v)$ for $v \leq 25$. Of course $g_1^{(5)}(v + 1) \geq g_1^{(5)}(v)$, since deletion of a point not in all quintuples converts a design on $v + 1$ points into a design on $v$ points.

The values immediately following $v = 25$ will be the most difficult to determine, just as those following $v = 16$ (namely 17, 18, 19) were the most difficult in the case $k = 4$.

## 2 The cases $v = 5$ through 9

Trivially, $g_1^{(5)}(5) = 1$. For $v = 6, 7, 8$, there can be only one quintuple. Call it ABCDE and use numbers to denote the additional elements.

For $v = 6$, we need ABCDE with A1, B1, C1, D1, E1. So $g_1^{(5)}(6) = 6$. Similarly, for $v = 7$, we need A12, B1, B2, C1, C2, D1, D2, E1, E2, and thus $g_1^{(5)}(7) = 10$. Finally, for $v = 8$, we have a choice of A123, B1, B2, B3, ..., E1, E2, E3; or A12, A3, B13, B2, C23, C1, D1, D2, D3, E1, E2, E3. The first possibility requires 14 blocks, whereas the second requires only 13 blocks. Hence $g_1^{(5)}(8) = 13$.

For $v = 9$, there can be two quintuples ABCDE and A1234. However, this would require 18 blocks (these 2, plus 16 blocks containing pairs B1, B2, B3, B4, ..., E1, E2, E3, E4). We can do better than this by using a single quintuple ABCDE.

If we use ABCDE, we might take 1234 but this will require 22 blocks (these 2, plus 20 blocks containing pairs A1, A2, ..., E3, E4). Alternatively we might take ABCDE with A123, A4 along with B14, B2, B3, C24, C1, C3, D34, D1, D2, E1, E2, E3, E4, which requires 16 blocks.

But if we do not use any quadruple, we can obtain three non-isomorphic designs each having 15 blocks

(i) ABCDE, A12, A34, B13, B24, C14, C23, D1, D2, D3, D4, E1, E2, E3, E4;

(ii) ABCDE, A12, A34, B13, B24, C14, C23, D1, D2, D3, D4, E1, E2, E3, E4;
E2, E3, E4;
(iii) ABCDE, A12, A34, B13, B2, B4, C24, C1, C3, D14, D2, D3, E23, E1, E4.
The only other possibility using a triple is ABCDE with 123 which would require 19 blocks (these 2, plus 17 further blocks). A design using the quintuple ABCDE and only pairs would require 27 blocks. Hence \( g^{(5)}_1(9) = 15 \).

3 The cases \( v = 10 \) and 11

We first consider the case \( v = 11 \). If we take a block ABCDE and attach the five one-factors of a one-factorization of \( \{1, 2, 3, 4, 5, 6\} \) to the five letters A, B, C, D, E, we obtain a covering with 16 blocks.

Using two quintuples would require a minimum of 25 other blocks if the quintuples were disjoint. If the quintuples were ABCDE, A1234, there would have to be at least \( 3 + 4(4) = 19 \) blocks. So the minimum number of blocks certainly occurs when there is only one quintuple.

Suppose that there are \( a_j \) sub-blocks of length \( j \) (\( j = 1, 2, 3 \)) attached to the 5 letters A, B, C, D, E. Then by counting pairs

\[
3a_3 + 2a_2 + a_1 = 6(5) = 30,
3a_3 + a_2 = 15 - \epsilon,
\]

where \( \epsilon \) denotes the number of pairs not contained in the sub-blocks attached to A, B, C, D, E (a priori, \( \epsilon > 0 \) is unlikely).

Then \( a_1 + a_2 = 15 + \epsilon \), and so \( \epsilon = 0 \) (otherwise there would be more than 16 blocks). But then for the same reason we must have \( a_3 = 0 \). However the equations then give \( a_2 = 15 \) and \( a_1 = 0 \). In short, the design created using the one-factorization of \( K_6 \) is the unique minimal design and \( g^{(5)}_1(11) = 16 \).

For \( v = 10 \), we can produce a design in 16 blocks by deleting element 6 from the design on 11 elements. We show that no better design is possible.

As for \( v = 11 \), it is clear that two disjoint quintuples require at least \( 2 + 5(5) = 27 \) blocks. Two intersecting quintuples require at least \( 3 + 4(4) = 19 \) blocks. So we must consider a single quintuple ABCDE. Using the same notation as before for the sub-blocks
attached to A, B, C, D, E, we have

\[ 3a_3 + 2a_2 + a_1 = 5(5) = 25, \]
\[ 3a_3 + a_2 = 10 - \epsilon. \]

Then \( a_1 + a_2 = 15 + \epsilon \). Hence \( \epsilon = 0 \) and \( a_3 = 0 \), since otherwise there would be more than 16 blocks. It follows that \( a_2 = 10 \) and \( a_1 = 5 \) and we have the usual one-factorization for \( K_5 \) (obtained as 5 sets, each comprising 2 pairs and a singleton) that results from deleting an element from \( K_6 \); hence \( g_1^{(5)}(10) = g_1^{(5)}(11) = 16 \).

4 The case \( v = 12 \)

We shall prove that \( g_1^{(5)}(12) = 18 \). Observe first that the presence of two disjoint quintuples requires at least \( 2 + 5(5) = 27 \) blocks. If there are two intersecting quintuples, the design requires at least \( 3 + 4(4) = 19 \) blocks. If there is a single quintuple, we use the previous notation and write

\[ 3a_3 + 2a_2 + a_1 = 35, \]
\[ 3a_3 + a_2 = 21 - \epsilon. \]

Then \( a_1 + a_2 = 14 + \epsilon, 3a_3 - a_1 = 7 - 2\epsilon \). So

\[ a_1 + a_2 + a_3 = \frac{49 + a_1 + \epsilon}{3}, \]

and we see that the design must contain at least \( 1 + 17 = 18 \) blocks. Furthermore, if \( \epsilon > 0 \) and \( a_1 + a_2 + a_3 = 17 \) then the design has at least 19 blocks. Hence any solution in 18 blocks requires \( \epsilon = 0 \), \( a_1 = 2 \), \( a_2 = 12 \) and \( a_3 = 3 \).

In such a case, each of A, B, C, D, E appears in blocks with the other seven points 1, 2, . . . , 7. Hence without loss of generality A, B, and C must each appear in one quadruple and two triples while D and E must each appear in one pair and three triples.

For each \( x \in \{1, 2, \ldots, 7\} \) let \( r_i(x) \) denote the number of occurrences of \( x \) in blocks of size \( i \). Since each \( x \) appears with A, B, C, D, E, \( r_2(x) + r_3(x) + r_4(x) = 5 \). Since each \( x \) appears with six other points from 1, 2, . . . , 7, \( r_3(x) + 2r_4(x) = 6 \). Thus \((r_2(x), r_3(x), r_4(x)) = \ldots \)
(0, 4, 1), (1, 2, 2) or (2, 0, 3). Suppose there are \(\alpha, \beta, \gamma\) points of each of the types respectively. Then \(\alpha + \beta + \gamma = 7\) and \(\beta + 2\gamma = 2\), the latter since \(a_1 = 2\). This gives \((\alpha, \beta, \gamma) = (5, 2, 0)\) or \((6, 0, 1)\). We now show that the former of these does not yield a design in 18 blocks but the latter does.

In the former case, take the two points of type \((1, 2, 2)\) to be 1 and 2. Without loss of generality this gives the design shown in Table 1,

<table>
<thead>
<tr>
<th>ABCDE</th>
<th>B145</th>
<th>C267</th>
<th>D2*</th>
<th>E1*</th>
</tr>
</thead>
<tbody>
<tr>
<td>A123</td>
<td>B2*</td>
<td>C1*</td>
<td>D**</td>
<td>E**</td>
</tr>
<tr>
<td>A**</td>
<td>B**</td>
<td>C**</td>
<td>D**</td>
<td>E**</td>
</tr>
<tr>
<td>A**</td>
<td></td>
<td></td>
<td>D1</td>
<td>E2</td>
</tr>
</tbody>
</table>

where the asterisks indicate undetermined entries. It is easy to see that the pairs B6 and B7 cannot both be included in this design.

Finally we show in Table 2 how to construct a design in 18 blocks having one point of type \((2, 0, 3)\), say 1, and six points of type \((0, 4, 1)\).

<table>
<thead>
<tr>
<th>ABCDE</th>
<th>B1bb</th>
<th>C1cc</th>
<th>Ddd</th>
<th>Eee</th>
</tr>
</thead>
<tbody>
<tr>
<td>A1aa</td>
<td>Bbb</td>
<td>Ccc</td>
<td>Ddd</td>
<td>Eee</td>
</tr>
<tr>
<td>Aaa</td>
<td>Bbb</td>
<td>Ccc</td>
<td>Ddd</td>
<td>Eee</td>
</tr>
<tr>
<td>Aaa</td>
<td></td>
<td></td>
<td>D1</td>
<td>E1</td>
</tr>
</tbody>
</table>

The lower case letters a, b, c, d, e in Table 2 are used as place holders for the five one-factors of the one-factorisation of \(K_6\) on the set \(\{2, 3, 4, 5, 6, 7\}\). Moreover, the quadruples, i.e. the blocks containing the pairs A1, B1 and C1, must also contain a one-factor on \(\{2, 3, 4, 5, 6, 7\}\). It is not difficult to complete Table 2 to obtain, to within isomorphism, the unique minimal design shown in Table 3.

<table>
<thead>
<tr>
<th>ABCDE</th>
<th>B145</th>
<th>C167</th>
<th>D25</th>
<th>E27</th>
</tr>
</thead>
<tbody>
<tr>
<td>A123</td>
<td>B26</td>
<td>C24</td>
<td>D36</td>
<td>E34</td>
</tr>
<tr>
<td>A46</td>
<td>B37</td>
<td>C35</td>
<td>D47</td>
<td>E56</td>
</tr>
<tr>
<td>A57</td>
<td></td>
<td></td>
<td>D1</td>
<td>E1</td>
</tr>
</tbody>
</table>
This establishes that $g_1^{(5)}(12) = 18$.

5 The case $v = 13$

If $v = 13$, we provide a design in 19 blocks and then show that it is minimal.

We can take three quintuples ABCDE, A1234, A5678. We adjoin 16 triples B15, B26, B37, B48, C16, C27, C38, C45, D17, D28, D35, D46, E18, E25, E36, E47. Since all 78 pairs are covered by these 3 quintuples and 16 triples, we have $g_1^{(5)}(13) \leq 19$.

As before, it is clear that any quadruple or quintuple must intersect ABCDE. Using the same notation as in the last section, we have

\[
\begin{align*}
4a_4 + 3a_3 + 2a_2 + a_1 &= 40, \\
6a_4 + 3a_3 + a_2 &= 28 - \epsilon.
\end{align*}
\]

If $a_4 = 0$, then $a_1 + a_2 = 12 + \epsilon$ and $3a_3 - a_1 = 16 - 2\epsilon$. Thus $a_1 + a_2 + a_3 + a_4 = (52 + a_1 + \epsilon)/3$. But if $\epsilon > 0$ then the design has at least $1 + 18 + 1 = 20$ blocks. Hence $\epsilon = 0$, $a_1 = 2$, $a_2 = 10$ and $a_3 = 6$. There are three possibilities for the design. The first of these is shown in Table 4.

| Table 4 |
|-------------------|-------------------|-------------------|-------------------|
| ABCDE | B147 | (5) | D** | E** |
| A123 | B258 | C*** | D** | E** |
| A456 | B36 | C** | D** | E** |
| A78 | D* | E** |

where the numeric entries are without loss of generality and the asterisks indicate undetermined entries. There are now two alternatives for the blocks containing C; namely C168, C357, C24 and C267, C348, C15 but the permutation (12)(45)(78) is an isomorphism between the two partial designs thus constructed. But now the blocks containing E cannot be completed.

7
The second possibility is shown in Table 5, using the same notation as above.

<table>
<thead>
<tr>
<th>ABCDE</th>
<th>B147</th>
<th>C168</th>
<th>D**</th>
<th>E**</th>
</tr>
</thead>
<tbody>
<tr>
<td>A123</td>
<td>B258</td>
<td>C357</td>
<td>D**</td>
<td>E**</td>
</tr>
<tr>
<td>A456</td>
<td>B36</td>
<td>C2</td>
<td>D**</td>
<td>E**</td>
</tr>
<tr>
<td>A78</td>
<td>C4</td>
<td>D**</td>
<td>E**</td>
<td></td>
</tr>
</tbody>
</table>

It is easy to see that the pairs 24, 26 and 27 cannot all be included in this design.

The third possibility is shown in Table 6, again using the previous notation.

<table>
<thead>
<tr>
<th>ABCDE</th>
<th>B147</th>
<th>C***</th>
<th>D***</th>
<th>E**</th>
</tr>
</thead>
<tbody>
<tr>
<td>A123</td>
<td>B258</td>
<td>C*</td>
<td>D*</td>
<td>E**</td>
</tr>
<tr>
<td>A456</td>
<td>B36</td>
<td>C**</td>
<td>D**</td>
<td>E**</td>
</tr>
<tr>
<td>A78</td>
<td>C**</td>
<td>D**</td>
<td>E**</td>
<td></td>
</tr>
</tbody>
</table>

Without loss of generality, there are now two alternatives for the remaining two quadruples; namely C168, D357 and C267, D348 but again the permutation (12)(45)(78) is an isomorphism between the two partial designs thus constructed. The design is now uniquely completable as shown in Table 7.

<table>
<thead>
<tr>
<th>ABCDE</th>
<th>B147</th>
<th>C168</th>
<th>D357</th>
<th>E15</th>
</tr>
</thead>
<tbody>
<tr>
<td>A123</td>
<td>B258</td>
<td>C5</td>
<td>D1</td>
<td>E38</td>
</tr>
<tr>
<td>A456</td>
<td>B36</td>
<td>C27</td>
<td>D26</td>
<td>E67</td>
</tr>
<tr>
<td>A78</td>
<td>C34</td>
<td>D48</td>
<td>E24</td>
<td></td>
</tr>
</tbody>
</table>

If \( a_4 = 1 \), then \( a_1 + a_2 = 14 + \epsilon \) and \( 3a_3 - a_1 = 8 - 2\epsilon \). Thus \( a_1 + a_2 + a_3 + a_4 = (53 + a_1 + \epsilon)/3 \). But if \( \epsilon > 0 \) then the design has at least \( 1 + 18 + 1 = 20 \) blocks. Hence \( \epsilon = 0 \), \( a_1 = 1 \), \( a_2 = 13 \) and \( a_3 = 3 \). Without loss of generality, there are two possibilities for the blocks containing A. The first of these is ABCDE, A1234, A567 and A8. The two further quadruples can then be taken to be
B158 and C268. But it is not now possible to complete the design. The second possibility is ABCDE, A1234, A56 and A78. Two of the three quadruples must contain the same letter, say B157 and B268. But we must then have B34; a contradiction.

Finally, noting that we cannot have $a_4 \geq 3$, if $a_4 = 2$ then $a_1 + a_2 = 16 + \epsilon$ and $3a_3 - a_4 = -2\epsilon$. Thus $a_1 + a_2 + a_3 + a_4 = (54 + a_1 + \epsilon)/3$. But if $\epsilon > 0$ then the design has at least $1 + 19 + 1 = 21$ blocks. Hence $\epsilon = 0$, $a_1 = 0$, $a_2 = 16$ and $a_3 = 0$. One solution of this form is the design given at the beginning of this section. However there is a second non-isomorphic solution ABCDE, A1234, A5678, B15, B26, B37, B48, C16, C25, C38, C47, D17, D28, D35, D46, E18, E27, E36, E45 obtained by using the Latin square based on the Cayley table of the Klein group $K_4$ rather than the cyclic group $C_4$. Both of these solutions are on the assumption that the three quintuples intersect in a common point. There are two other possibilities where the quintuples are either ABCDE, A1234, B5678 or ABCDE, A1234, B1567 but it is easy to see that neither of these can be completed to a design using only triples.

Thus there are three non-isomorphic designs which attain the bound $g_1^{(5)}(13) = 19$.

6 The cases $v = 14$ through 17

We first consider the case $v = 17$. If we generate PG(2, 4) by developing the initial block {0, 1, 4, 14, 16} modulo 21, and then delete the five points of this block, we can write down the affine geometry on 16 points in the form

<table>
<thead>
<tr>
<th>A:</th>
<th>2, 5, 15, 17</th>
<th>2, 3, 6, 18</th>
<th>5, 8, 18, 20</th>
</tr>
</thead>
<tbody>
<tr>
<td>A:</td>
<td>6, 7, 10, 20</td>
<td>2, 7, 8, 11</td>
<td>5, 3, 10, 11</td>
</tr>
<tr>
<td>A:</td>
<td>3, 8, 9, 12</td>
<td>2, 9, 10, 13</td>
<td>5, 7, 12, 13</td>
</tr>
<tr>
<td>A:</td>
<td>11, 13, 18, 19</td>
<td>2, 12, 19, 20</td>
<td>5, 6, 9, 19</td>
</tr>
</tbody>
</table>

15, 6, 11, 12  17, 3, 7, 19
15, 7, 9, 18  17, 6, 8, 13
15, 3, 13, 20  17, 9, 11, 20
15, 8, 10, 19  17, 10, 12, 18
A design on 17 points in 20 blocks is obtained by appending a point A to the first four blocks in this array. Hence \( g_1^{(5)}(17) \leq 20 \). We show that equality holds.

As in earlier cases, there cannot be two disjoint quintuples since this would require at least \( 2 + 5(5) = 27 \) blocks. Two intersecting quintuples would require at least \( 4 + 4(4) = 20 \) blocks and this could only occur if the intersection point, call it A, was the intersection of four quintuples; these four quintuples would cover 40 pairs and the remaining 96 pairs would then have to be covered in 16 quadruples. Since deletion of A would produce the unique affine geometry on 16 points, we see that this configuration is uniquely minimal among all configurations containing more than one quintuple.

If there is a single quintuple ABCDE, then with the usual notation

\[
3a_3 + 2a_2 + a_1 = 60, \\
3a_3 + a_2 = 66 - \epsilon.
\]

Then \( a_1 + a_2 = \epsilon - 6 \) and \( 3a_3 - a_1 = 72 - 2\epsilon \). It follows that

\[
a_1 + a_2 + a_3 = \frac{54 + a_1 + \epsilon}{3}.
\]

Now \( \epsilon \geq 6 \) and so \( a_1 + a_2 + a_3 \geq 20 \). Since ABCDE still needs to be counted, this case leads to a design with more that 20 blocks. Thus, we have proved that \( g_1^{(5)}(17) = 20 \), and the 20 block design is unique.

As an aside, we note that if we take the design on 17 points specified above and delete elements 2, 5, 15, 17 we obtain a design on 13 points comprising the quintuples \{A, 6, 7, 10, 20\}, \{A, 3, 8, 9, 12\}, \{A, 11, 13, 18, 19\} and 16 triples which we previously obtained as one of the solutions for \( g_1^{(5)}(13) = 19 \); in fact the one based on the Cayley table of the Klein group \( K_4 \).

Now take the \( g_1^{(5)}(17) \) design and reduce it to a design on 14 points by deleting elements 3, 6, 18. The resulting design contains one quintuple, 9 quadruples, and 9 triples, and so contains 19 blocks; hence \( g_1^{(5)}(14) \leq 19 \). But \( g_1^{(5)}(14) \geq g_1^{(5)}(13) = 19 \). So we have established that \( g_1^{(5)}(14) = 19 \).

If \( v = 15 \), and noting that \( g_1^{(5)}(15) \leq g_1^{(5)}(17) = 20 \), we cannot have two disjoint quintuples, and two intersecting quintuples require
at least $4 + 4(4) = 20$ blocks. If there is only one quintuple, we find, using our previous notation, that

\[3a_3 + 2a_2 + a_1 = 50,\]
\[3a_3 + a_2 = 45 - \epsilon.\]

Then $a_1 + a_2 = 5 + \epsilon$, and $3a_3 - a_1 = 40 - 2\epsilon$. Hence

\[a_1 + a_2 + a_3 = \frac{55 + a_1 + \epsilon}{3}.\]

This number is at least 19 and so the design must contain at least 20 blocks. Thus, whether we allow one or more quintuples, at least 20 blocks are required. So $g_1^{(5)}(15) = 20$, and a solution is provided by deleting two points from the solution for $g_1^{(5)}(17)$.

Since $g_1^{(5)}(17) = g_1^{(5)}(15) = 20$, it follows that $g_1^{(5)}(16) = 20$, and a solution can be found by deleting one point from a solution for $g_1^{(5)}(17)$. We now show that the designs on 14 and 16 points are unique but there exist two non-isomorphic solutions for $v = 15$.

If $v = 14$, we cannot have two disjoint quintuples, and two intersecting quintuples require at least $4 + 4(4) = 20$ blocks. So there is a single quintuple ABCDE, and with the usual notation,

\[a_3 + a_2 + a_1 = 18 - \delta,\]
\[3a_3 + 2a_2 + a_1 = 45,\]
\[3a_3 + a_2 = 36 - \epsilon,\]

where $\delta$ is the number of blocks disjoint from ABCDE and $\epsilon$ is the number of pairs in these $\delta$ blocks. Solving yields

\[a_1 = -\epsilon - 3\delta, \quad a_2 = 9 + 2\epsilon + 3\delta, \quad a_3 = 9 - \epsilon - \delta.\]

Therefore $a_1 = \epsilon = \delta = 0$ and $a_2 = a_3 = 9$.

The design must therefore be as shown in Table 8,

<table>
<thead>
<tr>
<th>ABCDE</th>
<th>B147</th>
<th>C***</th>
<th>D***</th>
<th>E***</th>
</tr>
</thead>
<tbody>
<tr>
<td>A123</td>
<td>B258</td>
<td>C**</td>
<td>D**</td>
<td>E**</td>
</tr>
<tr>
<td>A456</td>
<td>B369</td>
<td>C**</td>
<td>D**</td>
<td>E**</td>
</tr>
<tr>
<td>A789</td>
<td>C**</td>
<td>D**</td>
<td>E**</td>
<td></td>
</tr>
</tbody>
</table>
where the asterisks indicate undetermined entries. The triples to be
assigned to C, D and E must be three of 159, 267, 348, 168, 249 and
357 and further must form a parallel class, for otherwise the design
cannot be completed. Without loss of generality we choose C159,
D267 and E348. The design is now uniquely completable as shown
in Table 9.

<table>
<thead>
<tr>
<th>ABCDE</th>
<th>B147</th>
<th>C159</th>
<th>D267</th>
<th>E348</th>
</tr>
</thead>
<tbody>
<tr>
<td>A123</td>
<td>B258</td>
<td>C68</td>
<td>D18</td>
<td>E16</td>
</tr>
<tr>
<td>A456</td>
<td>B369</td>
<td>C24</td>
<td>D49</td>
<td>E29</td>
</tr>
<tr>
<td>A789</td>
<td>C37</td>
<td>D35</td>
<td>E57</td>
<td></td>
</tr>
</tbody>
</table>

If \( v = 15 \), we have already noted that we cannot have two disjoint
quintuples. Let ABCDE be a quintuple, then with the usual nota-
tion,

\[
\begin{align*}
4a_4 + 3a_3 + 2a_2 + a_1 &= 50, \\
6a_4 + 3a_3 + a_2 &= 45 - \epsilon,
\end{align*}
\]

where \( \delta \) is the number of blocks disjoint from ABCDE and \( \epsilon \) is the
number of pairs in these \( \delta \) blocks. Solving yields

\[
a_1 = 2 - a_4 - \epsilon - 3\delta, \quad a_2 = 3 + 3a_4 + 2\epsilon + 3\delta, \quad a_3 = 14 - 3a_4 - \epsilon - \delta.
\]

Since \( \delta > 0 \) implies \( a_1 < 0 \), and \( \delta = 0 \) implies \( \epsilon = 0 \), the solution
reduces to

\[
a_1 = 2 - a_4, \quad a_2 = 3 + 3a_4, \quad a_3 = 14 - 3a_4, \quad 0 \leq a_4 \leq 2.
\]

Suppose \( a_4 = 0 \). Then \( a_1 = 2, \quad a_2 = 3 \) and \( a_3 = 14 \). For each
\( x \in \{1, 2, \ldots, 10\} \), let \( r_i(x) \) denote the number of occurrences of \( x \) in
blocks of size \( i \). Then

\[
\begin{align*}
r_2(x) + r_3(x) + r_4(x) &= 5, \\
r_3(x) + 2r_4(x) &= 9, \\
r_2(x) &\leq 2, \quad r_3(x) \leq 3.
\end{align*}
\]

But the only solution is \( (r_2(x), r_3(x), r_4(x)) = (0, 1, 4) \); consequently
the design does not exist.
Suppose $a_4 = 1$. Then $a_1 = 1$, $a_2 = 6$ and $a_3 = 11$. With $r_i(x)$ as before, we have
\[
\begin{align*}
    r_2(x) + r_3(x) + r_4(x) + r_5(x) &= 5, \\
    r_3(x) + 2r_4(x) + 3r_5(x) &= 9, \\
    r_2(x) &\leq 1, \quad r_5(x) \leq 1,
\end{align*}
\]
which for $\alpha$, $\beta$, $\gamma$ has admissible solutions $(r_2(x), r_3(x), r_4(x), r_5(x)) = (0, 1, 4, 0)$ for $\alpha$ values of $x$, $(0, 2, 2, 1)$ for $\beta$ values of $x$ and $(1, 0, 3, 1)$ for $\gamma$ values of $x$. Since $a_1 = a_4 = 1$, we have $\gamma = 1$, $\beta = 3$, and hence $\alpha = 10 - \beta - \gamma = 6$. Assuming that the pattern $(1, 0, 3, 1)$ applies to 1, pattern $(0, 2, 2, 1)$ applies to 2, 3 and 4, and denoting 10 by X, we can construct the design as follows.

Without loss of generality, we have blocks ABCDE, A1234, B156, C178, D19X and E1. The remaining blocks containing B or C or D must be one of size 4 and two of size 3. So the remaining blocks containing A must be two of size 4 and containing E must be three of size 4. The design must therefore be as shown in Table 10,

\[
\begin{array}{cccccc}
\text{Table 10} \\
\hline
\text{ABCDE} & \text{B156} & \text{C178} & \text{D19X} & \text{E1} \\
\text{A1234} & \text{B***} & \text{C***} & \text{D***} & \text{E2**} \\
\text{A579} & \text{B**} & \text{C*} & \text{D**} & \text{E3**} \\
\text{A68X} & \text{B**} & \text{C*} & \text{D**} & \text{E4**} \\
\end{array}
\]

where the asterisks indicate undetermined entries. There are two alternatives for the missing pairs in the quadruples containing E; namely 58, 7X, 96 and 5X, 76, 98 but the permutation $(56)(78)(9X)$ is an isomorphism between the two partial designs thus constructed, The design is now uniquely completable as shown in Table 11.

\[
\begin{array}{cccccc}
\text{Table 11} \\
\hline
\text{ABCDE} & \text{B156} & \text{C178} & \text{D19X} & \text{E1} \\
\text{A1234} & \text{B389} & \text{C45X} & \text{D267} & \text{E258} \\
\text{A579} & \text{B2X} & \text{C29} & \text{D35} & \text{E37X} \\
\text{A68X} & \text{B47} & \text{C36} & \text{D48} & \text{E469} \\
\end{array}
\]
Suppose $a_4 = 2$. Then $a_1 = 0$, $a_2 = 9$ and $a_3 = 8$. With $r_i(x)$ as before, we have

\[
\begin{align*}
r_3(x) + r_4(x) + r_5(x) &= 5, \\
r_3(x) + 2r_4(x) + 3r_5(x) &= 9, \\
r_5(x) &\leq 2,
\end{align*}
\]

which has admissible solutions $(r_3(x), r_4(x), r_5(x)) = (1, 4, 0)$ for $\alpha$ values of $x$, $(2, 2, 1)$ for $\beta$ values of $x$ and $(3, 0, 2)$ for $\gamma$ values of $x$.

Suppose $\gamma > 0$. Then, since $a_4 = 2$, it follows that $\gamma = 1$, i.e. there exists exactly one number, say 1, which occurs in two blocks of length 5, say A1234 and B1567. Also the pairs A5, A6, A7, B2, B3, B4, and \{C, D, E\} × \{1, 2, 3, 4\} must occur in separate blocks. Together with ABCDE, there are now at least 21 blocks. Therefore we may assume that $\gamma = 0$, and hence $\beta = 8$ and $\alpha = 2$.

Since we cannot have two disjoint blocks of length 5, we may assume without loss of generality that the design contains blocks ABCDE, A1234, A5678 and A9X. The remainder of the design contains 8 blocks of size 4 and 8 blocks of size 3. Elementary counting further shows that each of B, C, D and E are contained in two blocks of each size. Therefore without loss of generality we can further assume that the design contains blocks B159, B26X, B37 and B48. This leaves only the triples 17X, 18X, 279, 289, 35X, 369, 389, 38X, 45X, 469, 479, 47X which can be assigned to C, D and E. Since the point 1 must occur in one quadruple and two triples with C, D and E there are, again without loss of generality, eight possibilities for the quadruples containing C,

\[
\begin{align*}
(i) & \quad C17X, \quad C289, & (v) & \quad C18X, \quad C279, \\
(ii) & \quad C17X, \quad C369, & (vi) & \quad C18X, \quad C469, \\
(iii) & \quad C17X, \quad C389, & (vii) & \quad C18X, \quad C479, \\
(iv) & \quad C17X, \quad C469, & (viii) & \quad C18X, \quad C369.
\end{align*}
\]

But the permutation (34)(78) is an isomorphism between the partial designs constructed by adjoining the respective possibilities (i) and (v), (ii) and (vi), (iii) and (vii), (iv) and (viii) to the chosen blocks containing A and B. So there are just four possibilities to consider.
Possibility (i). This has a unique completion as shown in Table 12.

Table 12

<table>
<thead>
<tr>
<th>ABCDE</th>
<th>B159</th>
<th>C17X</th>
<th>D369</th>
<th>E38X</th>
</tr>
</thead>
<tbody>
<tr>
<td>A1234</td>
<td>B26X</td>
<td>C289</td>
<td>D45X</td>
<td>E479</td>
</tr>
<tr>
<td>A5678</td>
<td>B37</td>
<td>C35</td>
<td>D18</td>
<td>E16</td>
</tr>
<tr>
<td>A9X</td>
<td>B48</td>
<td>C46</td>
<td>D27</td>
<td>E25</td>
</tr>
</tbody>
</table>

Possibility (ii). The partial design can be extended with blocks D45X, E479 and D289. But now the triple D3* cannot be completed.

Possibility (iii). The partial design can be extended with blocks D279, E35X and E469. But now the other quadruple containing D cannot be completed.

Possibility (iv). The partial design can be extended with blocks D389, E35X and E279. But again the other quadruple containing D cannot be completed.

If \( v = 16 \), we cannot have two disjoint quintuples. Let ABCDE be a quintuple, then with the usual notation,

\[
\begin{align*}
a_4 + a_3 + a_2 + a_1 & = 19 - \delta, \\
4a_4 + 3a_3 + 2a_2 + a_1 & = 55, \\
6a_4 + 3a_3 + a_2 & = 55 - \epsilon,
\end{align*}
\]

where \( \delta \) is the number of blocks disjoint from ABCDE and \( \epsilon \) is the number of pairs in these \( \delta \) blocks. Solving yields

\[
\begin{align*}
a_1 & = 2 - a_4 - \epsilon - 3\delta, \\
a_2 & = 3a_4 - 2 + 2\epsilon + 3\delta, \\
a_3 & = 19 - 3a_4 - \epsilon - \delta.
\end{align*}
\]

Since \( \delta > 0 \) implies \( a_1 < 0 \), and \( \delta = 0 \) implies \( \epsilon = 0 \), the solution reduces to

\[
\begin{align*}
a_1 & = 2 - a_4, \\
a_2 & = 3a_4 - 2, \\
a_3 & = 19 - 3a_4, \\
1 & \leq a_4 \leq 2.
\end{align*}
\]

Suppose \( a_4 = 1 \). Then \( a_1 = a_2 = 1 \) and \( a_3 = 16 \). With \( r_i(x) \) denoting the number of occurrences of \( x \) in blocks of size \( i \), as before, we have

\[
\begin{align*}
r_2(x) + r_3(x) + r_4(x) + r_5(x) & = 5, \\
r_3(x) + 2r_4(x) + 3r_5(x) & = 10, \\
r_2(x) & \leq 1, \\
r_3(x) & \leq 1, \\
r_5(x) & \leq 1,
\end{align*}
\]

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which for some $\alpha, \beta$ has admissible solutions $(r_2(x), r_3(x), r_4(x), r_5(x)) = (0, 0, 5, 0)$ for $\alpha$ values of $x$ and $(0,1,3,1)$ for $\beta$ values of $x$. Hence $r_2(x) = 0$ for all $x$ and the design does not exist.

Suppose $a_4 = 2$. Then $a_1 = 0$, $a_2 = 4$ and $a_3 = 13$. Again, we have

\begin{align*}
    r_3(x) + r_4(x) + r_5(x) &= 5, \\
    r_3(x) + 2r_4(x) + 3r_5(x) &= 10, \\
    r_3(x) &\leq 4, \quad r_5(x) \leq 2,
\end{align*}

which for some $\alpha, \beta, \gamma$ has admissible solutions $(r_3(x), r_4(x), r_5(x)) = (0, 5, 0)$ for $\alpha$ values of $x$, $(1,3,1)$ for $\beta$ values of $x$ and $(2,1,2)$ for $\gamma$ values of $x$. If $r_5(x) = 2$ for some $x$, we can assume that there exist blocks A1234 and B1567; hence the pairs $\{A\} \times \{5, 6, 7\}$, $\{B\} \times \{2, 3, 4\}$ and $\{C, D, E\} \times \{1, 2, 3, 4\}$ must occur in distinct blocks, implying a total block count exceeding 20. Therefore $\gamma = 0$, $\beta = 8$ and $\alpha = 3$.

Denote 10 by $X$ and 11 by $Y$. Elementary counting shows that one of A, B, C, D or E has two quadruples and one triple attached whilst the other four points have three triples and one pair attached. Also the points contained in the two attached quadruples are the same as the points contained in the four attached pairs. To construct the design therefore, without loss of generality we begin with blocks ABCDE, A1234, A5678, A9XY, B15, C26, D37, E48.

The design now has a unique completion to within isomorphism. We can add further blocks as shown in Table 13,

<table>
<thead>
<tr>
<th></th>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
<th>E</th>
</tr>
</thead>
<tbody>
<tr>
<td>ABCDE</td>
<td>B15</td>
<td>C26</td>
<td>D37</td>
<td>E48</td>
<td></td>
</tr>
<tr>
<td>A1234</td>
<td>B27*</td>
<td>C35*</td>
<td>D45*</td>
<td>E25*</td>
<td></td>
</tr>
<tr>
<td>A5678</td>
<td>B38*</td>
<td>C47*</td>
<td>D28*</td>
<td>E36*</td>
<td></td>
</tr>
<tr>
<td>A9XY</td>
<td>B46*</td>
<td>C18*</td>
<td>D16*</td>
<td>E17*</td>
<td></td>
</tr>
</tbody>
</table>

where the asterisks indicate undetermined entries. The points 9, $X$ and $Y$ can now be assigned to the incomplete blocks B27*, B38* and B46* in any order, say B279, B38X and B46Y whence the completion
of the design is unique as shown in Table 14.

<table>
<thead>
<tr>
<th>ABCDE</th>
<th>B15</th>
<th>C26</th>
<th>D37</th>
<th>E48</th>
</tr>
</thead>
<tbody>
<tr>
<td>A1234</td>
<td>B279</td>
<td>C35Y</td>
<td>D459</td>
<td>E25X</td>
</tr>
<tr>
<td>A5678</td>
<td>B38X</td>
<td>C47X</td>
<td>D28Y</td>
<td>E369</td>
</tr>
<tr>
<td>A9XY</td>
<td>B46Y</td>
<td>C189</td>
<td>D16X</td>
<td>E17Y</td>
</tr>
</tbody>
</table>

7 The cases $v = 18$ through 25

Both the projective plane of order 4, PG(2, 4), and the affine plane of order 5, AG(2, 5), are 2-transitive and unique up to isomorphism. Hence for each one, the designs produced by removing two points are pairwise isomorphic. The same is obviously true if only one point is removed. So by Theorem 3 of [1] the designs for $v = 19, 20, 21, 23, 24$ and 25 are unique up to isomorphism. It remains to deal with $v = 18$ and $v = 22$. The results which follow are based on the full automorphism groups of PG(2, 4) and AG(2, 5) obtained from designtheory.org [3, 2].

Consider the case $v = 18$. By [1, Theorem 3], the design is obtained from PG(2, 4) by removing three points. We construct PG(2, 4) by developing $\{0, 1, 4, 14, 16\}$ modulo 21. We then delete points 19 and 20 to obtain the 19-point, 21-block design $D$;

$\{0, 1, 4, 14, 16\}, \{0, 2, 7, 8, 11\}, \{1, 2, 5, 15, 17\},$
$\{1, 3, 8, 9, 12\}, \{2, 3, 6, 16, 18\}, \{2, 4, 9, 10, 13\},$
$\{3, 5, 10, 11, 14\}, \{4, 6, 11, 12, 15\}, \{5, 7, 12, 13, 16\},$
$\{6, 8, 13, 14, 17\}, \{7, 9, 14, 15, 18\}, \{0, 10, 12, 17, 18\},$
$\{0, 3, 13, 15\}, \{3, 4, 7, 17\}, \{0, 5, 6, 9\},$
$\{4, 5, 8, 18\}, \{1, 6, 7, 10\}, \{8, 10, 15, 16\},$
$\{1, 11, 13, 18\}, \{9, 11, 16, 17\}, \{2, 12, 14\}.$

Let $\sigma$ be the subgroup of the full automorphism group of PG(2, 4) which stabilizes the points 19 and 20. Then $\sigma$ is generated by the
permutations

(0 5)(7 1)(8 15)(11 17)(4 13)(18 3)(14 12),
(2 12)(7 17)(8 18)(11 10)(1 16)(15 13)(9 6),
(0 5 9)(7 1 10)(8 17 13)(11 15 4)(16 3 18),
(0 15)(7 1)(8 5)(11 17)(4 18)(10 16)(9 6)(13 3),
(7 11)(1 17)(4 18)(10 16)(9 6)(13 3)(14 12),
(0 9 6)(7 13 16)(8 4 18)(11 10 3)(1 15 17),

and is an automorphism group of $D$. Furthermore, under the action of $\sigma$ the points of $D$ fall into two orbits $T = \{2, 12, 14\}$ and $U = \{0, 1, 3, 4, \ldots, 11, 13, 15, 16, 17, 18\}$. Therefore, denoting by $D_i$ the design obtained from $D$ by removing point $i$, we see that the $D_i$ fall into two isomorphism classes, $i \in T$ and $i \in U$.

Finally, in the case $v = 22$, the design is obtained from $AG(2, 5)$ by removing three points $[1, Theorem 3]$. We construct $AG(2, 5)$ as

$\{(0, 0), (0, 1), (1, 2), (1, 4), (3, 3)\} + (i, j) : 0 \leq i \leq 4, 0 \leq j \leq 4\} \cup$

$\{(0, 0), (1, 0), (2, 0), (3, 0), (4, 0)\} + (0, j) : 0 \leq j \leq 4\}$

with arithmetic in $\mathbb{Z}_5 \times \mathbb{Z}_5$. Then points $(4, 3)$ and $(4, 4)$ are deleted to obtain the 23-point, 30-block design $E$, where a point $(a, b)$ of $AG(2, 5)$ is denoted by the integer $5a + b$;

\begin{align*}
\{0,1,7,9,18\}, & \quad \{1,2,5,8,19\}, \quad \{0,3,14,21,22\}, \\
\{2,3,6,9,15\}, & \quad \{0,4,6,8,17\}, \quad \{2,4,13,20,21\}, \\
\{3,4,5,7,16\}, & \quad \{0,5,10,15,20\}, \quad \{1,6,11,16,21\}, \\
\{2,7,12,17,22\}, & \quad \{7,8,11,14,20\}, \quad \{5,9,11,13,22\}, \\
\{8,9,10,12,21\}, & \quad \{2,10,14,16,18\}, \quad \{3,10,11,17,19\}, \\
\{4,11,12,15,18\}, & \quad \{0,12,13,16,19\}, \quad \{1,13,14,15,17\}, \\
\{6,18,19,20,22\}, & \quad \{1,3,12,20\}, \quad \{1,4,10,22\}, \\
\{5,6,12,14\}, & \quad \{6,7,10,13\}, \quad \{3,8,13,18\}, \\
\{4,9,14,19\}, & \quad \{7,15,19,21\}, \quad \{8,15,16,22\}, \\
\{9,16,17,20\}, & \quad \{5,17,18,21\}, \quad \{0,2,11\}.
\end{align*}

Let $\tau$ be the subgroup of the full automorphism group of $AG(2, 5)$ which stabilizes the points $(4, 3) = 23$ and $(4, 4) = 24$. Then $\tau$ is
generated by the permutations
\[(1 \ 8)(19 \ 5)(16 \ 20)(18 \ 4)(14 \ 21)(10 \ 13)(6 \ 7)(9 \ 17)(15 \ 12)(3 \ 22),\]
\[(1 \ 5 \ 8 \ 19)(16 \ 9 \ 20 \ 17)(18 \ 15 \ 4 \ 12)(14 \ 3 \ 21 \ 22)(10 \ 6 \ 13 \ 7),\]
\[(1 \ 18 \ 9 \ 7)(8 \ 16 \ 15 \ 22)(19 \ 10 \ 3 \ 17)(5 \ 14 \ 6 \ 12)(21 \ 4 \ 13 \ 20),\]
\[(1 \ 19 \ 8 \ 5)(16 \ 17 \ 20 \ 9)(18 \ 12 \ 4 \ 15)(14 \ 22 \ 21 \ 3)(10 \ 7 \ 13 \ 6).\]

and is an automorphism group of \(E\). Let \(E_i\) denote the design obtained from \(E\) by removing point \(i\). Then for \(i\) in the triple of \(E\), we have \(E_0 \simeq E_2 \simeq E_{11}\) via isomorphisms generated by the permutations
\[(2 \ 0)(4 \ 3)(7 \ 5)(9 \ 8)(12 \ 10)(14 \ 13)(17 \ 15)(19 \ 18)(22 \ 20) : \ E_0 \mapsto E_2,\]
\[(11 \ 2)(4 \ 16 \ 5 \ 3)(19 \ 20 \ 22 \ 6)(12 \ 10 \ 21 \ 8)(15 \ 14 \ 17 \ 13) : \ E_0 \mapsto E_{11}.\]

On the other hand, the points of \(E\) which are not in the triple form a single orbit under the action of \(\tau\) and therefore the corresponding \(E_i\) form a single isomorphism class.

8 Summary

We conclude with a table of the results. The values of \(n^{(5)}_1(v)\) are the numbers of non-isomorphic designs which achieve the bound \(g^{(5)}_1(v)\).

<table>
<thead>
<tr>
<th>(v)</th>
<th>(g^{(5)}_1(v))</th>
<th>(n^{(5)}_1(v))</th>
<th>(v)</th>
<th>(g^{(5)}_1(v))</th>
<th>(n^{(5)}_1(v))</th>
</tr>
</thead>
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19
References


