Bounds on $g_1^{(5)}(v)$ for $v \equiv 9, 13, 17 \pmod{20}$

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Abstract

The minimum number of blocks having maximum size precisely five that is required to cover, exactly once, all pairs of elements from a set of cardinality v is denoted by $g_1^{(5)}(v)$. The exact values of $g_1^{(5)}(v)$ for $v = 20t + r$ are known for $r \in \{0, 1, 2, 3, 4, 5, 18, 19\}$. In this paper, we study the cases $r = 9, 13, 17$.

1 Introduction

The covering number $g_\lambda^{(k)}(v)$ is defined as the cardinality of the minimum pairwise balanced design (PBD) on a set of v points that has largest block size k and such that every pair occurs exactly λ times in the design. The values of $g_\lambda^{(k)}(v)$ for $k = 4$ and all λ have been determined, in a series of papers [5] to [18].

In the present paper we consider the case $k = 5$, $\lambda = 1$. A lower bound on $g_1^{(5)}(v)$ was given in [1] as

$$\frac{1}{20}\left\{6g_2 + 2g_3 + v(8\left\lceil\frac{v-1}{4}\right\rceil - (v-1))\right\},$$

where g_2 and g_3 are the number of blocks of lengths 2 and 3 respectively in the exact covering. The results for $v \leq 25$ are given in [4], and those for $20t + r$, where $r \in \{0, 1, 2, 3, 4, 5, 18, 19\}$ are given in [1]. In this paper, we study the cases $r = 9, 13, 17$ which share a feature with $r = 1$ and $r = 5$, namely, that an element can occur with all other elements in an exact number of quintuples. For any such packing by quintuples, define the *defect graph* to be the graph whose edges are the uncovered pairs. Then the valence of each vertex in the defect graph is 0 modulo 4. We will write $D(2, k, v)$ to denote the maximum number of edge-disjoint K_ks in a packing of K_v.

2 The case $v = 20t + 13$

For $t = 0$, we have $g_1^{(5)}(13) = 19$ (cf. [4]). The value of $g_1^{(5)}(33)$ is unknown, and we restrict ourselves to the case $t > 1$.

Since $\binom{20t+13}{2} = 200t^2 + 250t + 78$, and since any quintuple contains exactly ten pairs, we see that the packing number $D(2, 5, 20t + 13)$, the maximal number of possible quintuples, satisfies

$$D(2, 5, 20t + 13) \leq 20t^2 + 25t + 7.$$

However, if equality held, there would be exactly eight pairs not covered. The defect graph would then have a total valence of 16, and the vertices in it would have valences which are 0 modulo 4. The only possibility is four vertices of valence four each, and this must be rejected since it would lead to repeated edges in the defect graph. Hence we have the following well-known result.

Lemma 2.1. $D(2, 5, 20t + 13) \leq 20t^2 + 25t + 6.$

Also, since $200t^2 + 250t + 78$ pairs must be covered and each quintuple covers 10 pairs, we see that

$$g_1^{(5)}(v) \geq 20t^2 + 25t + 8 = g_0.$$

Furthermore, at least 18 pairs cannot be covered by quintuples and so these require at least three shorter blocks. Hence we may set $g_1^{(5)}(v) = g_0 + f$, where $f \geq 1$. Also, we have $g_1^{(5)}(v) = g_2 + g_3 + g_4 + g_5$, where g_i is the number of blocks of length i. We will require, for a minimal value of $g_1^{(5)}(v)$, a large number of quintuples. So we set

$$g_5 = 20t^2 + 25t + 6 - e.$$

Then $g_2 + g_3 + g_4 = g_1^{(5)}(v) - g_5 = 8 + f - 6 + e = 2 + f + e$.

We then calculate as follows.

$$\begin{aligned} 10(g_2 + g_3 + g_4 + g_5) &= 200t^2 + 250t + 80 + 10f, && (1)\\ g_2 + 3g_3 + 6g_4 + 10g_5 &= 200t^2 + 250t + 78, && (2)\\ 4(g_2 + g_3 + g_4) &= 8 + 4f + 4e && (3) \end{aligned}$$

Subtract equations (2) and (3) from (1) to obtain the following result.

Lemma 2.2. $5g_2 + 3g_3 = 6f - 4e - 6$.

Corollary 2.2.1. $f \neq 1$.
Proof. $f = 1$ implies $e = 0$. Thus $g_2 = 0$, $g_3 = 0$, $g_4 = 3$, $g_5 = 20t^2 + 25t + 6$. The defect graph is $3K_4$, it has 18 edges and total valence 36. A vertex of valence 8 or more would require a total valence at least $8 + 8 \times 4 > 36$. Thus the graph consists of nine vertices, each of valence 4. But it is not possible to use $3K_4$ to create such a graph, since every vertex in a K_4 has valence 3. So $f = 1$ is impossible.

Corollary 2.2.2. $f \neq 2$.
Proof. $f = 2$ implies $5g_2 + 3g_3 = 6 - 4e$. Hence $e = 0$, $g_2 = 0$, $g_3 = 2$, $g_4 = 2$. The defect graph now comprises $2K_3 + 2K_4$ and again has 18 edges. But we cannot obtain nine vertices of valence 4 by combining K_3s and K_4s, since all vertices of K_3 have valence 2 and all vertices of K_4 have valence 3. So $f = 2$ is impossible.

Corollary 2.2.3. $f \neq 3$.
Proof. $f = 3$ implies $5g_2 + 3g_3 = 12 - 4e$.
Case 1. If $e = 3$, then $g_2 = g_3 = 0$, $g_4 = 8$. The defect graph has 48 edges and a total valence of 96. But eight K_4s can only produce vertices of valence 12 or 24, the latter being impossible as it would require a total valence in excess of 96. Hence there must be precisely

eight vertices each of valence 12. This would require repeated edges and is consequently impossible.

Case 2. If $e = 2$, then $5g_2 + 3g_3 = 4$, which is impossible.

Case 3. If $e = 1$, then $g_2 = g_3 = 1$, $g_4 = 4$. Here the defect graph has 28 edges and so it can contain vertices of valence 4 and valence 8, but not valence 12 or greater. But $K_2 + K_3 + 4K_4$ can give at most two vertices of valence 4 $(= 1 + 3)$. So there would have to be at least six vertices of valence 8, and this is impossible (we can get at most three such vertices as $3 + 3 + 2$).

Case 4. If $e = 0$, then $g_2 = 0$, $g_3 = 4$, $g_4 = 1$. This gives a defect graph, as before, with nine vertices of valence 4, and it cannot be made from $4K_3 + K_4$.

So we have established that $f = 3$ is impossible.

Corollary 2.2.4. $f \neq 4$ unless it is possible to have $D(2, 5, 20t + 13) = 20t^2 + 25t + 6$ with a defect graph on nine vertices consisting of six triangles.

Proof. $f = 4$ implies $5g_2 + 3g_3 = 18 - 4e$.

Case 1. Clearly, $e = 4$ is impossible.

Case 2. If $e = 3$ then $g_2 = 0$, $g_3 = 2$, $g_4 = 7$. The defect graph has 48 edges and total valence 96. With the defect graph being $2K_3 + 7K_4$, there is at most one vertex of valence 4. There cannot be a vertex of valence 12 or greater since this would require total valence at least $12 + (4 + 11 \times 8) > 96$. If there is one vertex of valence 4 $(= 2 + 2)$, there are at most four vertices of valence 8 $(= 3 + 3 + 2)$, and if there are no vertices of valence 4, there are at most six vertices of valence 8. In either case, a vertex of valence at least 12 is required, a contradiction.

Case 3. If $e = 2$, then $g_2 = 2$, $g_3 = 0$, $g_4 = 6$. The defect graph has 38 edges and total valence 76. There are at most four vertices of valence 4, and at most two vertices of valence 8. So there must be several vertices of valence at least 12. The total valence is then at least $12 + (4 \times 4 + 2 \times 8 + 6 \times 12) > 76$, (even disregarding the fact that we cannot have four vertices of valence 4 at the same time as two vertices of valence 8), a contradiction.

Case 4. If $e = 1$, then $g_2 = 1$, $g_3 = 3$, $g_4 = 3$. The defect graph has 28 edges and total valence 56. A vertex of valence 12 would require total valence at least $12 + (12 \times 4) > 56$, and so we can have only vertices of valence 4 and valence 8. The numbers of these respectively must be 0 and 7, or 2 and 6, or 4 and 5 (since $4 = 2+2 = 1+3$, there are at most five vertices of valence 4). But to form a vertex of valence 8 requires the combination of two K_4s and a K_3 ($8 = 3+3+2$) or of a K_4, two K_3s and a K_2 ($8 = 3+2+2+1$). However only three of these combinations can be formed. So there are at most three vertices of valence 8. This contradicts the requirement that the number of vertices of valence 8 is at least five.

Case 5. Finally, if $e = 0$, then $5g_2 + 3g_3 = 18$. There are two possibilities. First we deal with the possibility $g_2 = 3$, $g_3 = 1$, $g_4 = 2$. The defect graph has 18 edges and total valence 36. This implies that there can be no vertices of valence 8 or greater because $8 + 8 \times 4 > 36$. Hence there must be nine vertices of valence 4. But this cannot be achieved. The second possibility is $g_2 = 0$, $g_3 = 6$, $g_4 = 0$. This is the possibility noted in [1].

So we have the following theorem.

Theorem 2.1. $g_1^{(5)}(20t+13) \geq g_0 + 4$, *with equality if and only if it is possible to have* $D(2,5,20t+13) = 20t^2 + 25t + 6$, *with a defect graph on nine vertices consisting of six triangles.*

If $t \geq 2$, then the set of all $\binom{20t+13}{2}$ pairs can be covered by one block of length 13 and a set of $20t^2 + 25t$ quintuples [2, 3]. Replacing the block of length 13 by the blocks of a $(13, 13, 4, 4, 1)$-BIBD gives the total number of blocks as $20t^2 + 25t + 13 = g_0 + 5$. This gives the following theorem.

Theorem 2.2. *For* $t \geq 2$, $g_0 + 4 \leq g_1^{(5)}(20t+13) \leq g_0 + 5$, *and the value* $g_0 + 4$ *is only possible if the conditions of Theorem 2.1 are met.*

3 The case $v = 20t + 9$

For $t = 0$, we have $g_1^{(5)}(9) = 15$ (cf. [4]). The values of $g_1^{(5)}(29)$ and $g_1^{(5)}(49)$ are unknown, and we restrict ourselves to the case $t > 2$.

Since $\binom{20t+9}{2} = 200t^2 + 170t + 36$, we immediately have $D(2,5,20t+9) \leq 20t^2 + 17t + 3$. As in the last section, if this bound were met, the defect graph would have six edges and total valence 12. Also, all vertices would have to have valences which are 0 modulo 4, and so the defect graph could have only three vertices. This is impossible without repeated edges. So we have the following well-known result.

Lemma 3.1. $D(2,5,20t+9) \leq 20t^2+17t+2$.

We proceed as in Section 2. Since $200t^2+170t+36$ pairs must be covered,

$$g_1^{(5)}(v) \geq 20t^2+17t+4 = g_0.$$

Furthermore, at least 16 pairs cannot be covered by quintuples and so these require at least three shorter blocks. Hence we may set $g_1^{(5)}(v) = g_0+f$, where $f \geq 1$. As before we have $g_1^{(5)}(v) = g_2+g_3+g_4+g_5$, and we now define e by

$$g_5 = 20t^2+17t+2-e.$$

Then $g_2+g_3+g_4 = g_1^{(5)}(v)-g_5 = g_0+f-g_5 = 2+f+e$.

As in Section 2, $10(g_2+g_3+g_4+g_5)-(g_2+3g_3+6g_4+10g_5)-4(g_2+g_3+g_4) = 5g_2+3g_3$. But this expression also reduces to $10g_1^{(5)}(v)-\binom{20t+9}{2}-4(g_2+g_3+g_4) = 40+10f-36-4e-4f-8 = 6f-4e-4$. So we have the following result.

Lemma 3.2. $5g_2+3g_3 = 6f-4e-4$.

Corollary 3.2.1. $f \neq 1$.

Corollary 3.2.2. $f \neq 2$.
Proof. $f=2$ implies $5g_2+3g_3 = 8-4e$.
Case 1. If $e=2$, then $g_2=g_3=0$, $g_4=6$. The defect graph has 36 edges and total valence 72. Since it is made up of K_4s, each valence must be a multiple of 12, and clearly no vertex can have valence 24. Hence there are six vertices, each of valence 12. This cannot be achieved from six K_4s, and so $e=2$ is impossible.
Case 2. If $e=1$, then $5g_2+3g_3=4$, and there is no solution.
Case 3. If $e=0$, then $g_2=g_3=1$, $g_4=2$. The defect graph has 16 edges and total valence 32. A vertex of valence 8 or more

would require total valence at least $8+8\times 4>32$. So there are eight vertices, each of valence 4 in the defect graph. The two K_4s must be disjoint and the K_3 cannot be attached to either. So, $f=2$ is impossible.

Corollary 3.2.3. $f\neq 3$.
Proof. $f=3$ implies $5g_2+3g_3=14-4e$.
Case 1. Clearly, $e=3$ is impossible.
Case 2. If $e=2$, then $g_2=0$, $g_3=2$, $g_4=5$. The defect graph has 36 edges and total valence 72. There is at most one vertex of valence 4 ($=2+2$), and there cannot be a vertex of valence 12 or more because $12+(1\times 4+11\times 8)>72$. So there are at least nine vertices of valence 8. But this is impossible since there are only six vertices in the two K_3s.
Case 3. If $e=1$, then $g_2=2$, $g_3=0$, $g_4=4$. The defect graph has 26 edges and total valence 52. There are at most four vertices of valence 4 ($=3+1$), and there cannot be any vertices of valence 12. So there are at least five vertices of valence 8 ($=3+3+1+1$), which is clearly impossible with only two K_2s.
Case 4. If $e=0$, then $g_2=1$, $g_3=3$, $g_4=1$. The defect graph has 16 edges, and hence eight vertices, each of valence 4. But each of the four vertices of the K_4 requires a contribution of one further edge, and this is impossible with only one K_2.

So we have established that $f=3$ is impossible.

Corollary 3.2.4. $f\neq 4$ unless it is possible to have $D(2,5,20t+9)=20t^2+17t+2$, with a defect graph on eight vertices consisting of two quadruples and four pairs.
Proof. $f=4$ implies $5g_2+3g_3=20-4e$.
Case 1. If $e=5$, then $g_2=g_3=0$, $g_4=11$. The defect graph has 66 edges and total valence 132. Since all vertices have valence at least 12 (only K_4s are available), there are at most 11 vertices. This is a contradiction since a vertex of valence 12 requires at least 13 vertices in the defect graph.
Case 2. If $e=4$, then $5g_2+3g_2=4$, which has no solutions.
Case 3. If $e=3$, then $g_2=g_3=1$, $g_4=7$. The defect graph has 46 edges and total valence 92. There are at most two vertices of valence 4, and at most three vertices of valence 8. The remaining vertices

must have valence at least 12. But $12+(2\times 4+3\times 8+7\times 12) > 92$.
Case 4. If $e = 2$, then $g_2 = 0$, $g_3 = 4$, $g_4 = 4$. The defect graph has 36 edges and total valence 72. All valences are at least 8, and so there must be nine vertices, each of valence 8. This would require each of the vertices of a particular K_4 to have another K_4 attached $(8 = 3 + 3 + 2)$, and this is impossible, since there are only three other K_4s.
Case 5. If $e = 1$, then $g_2 = g_3 = 2$, $g_4 = 3$. The defect graph has 26 edges and total valence 52. There are no vertices of valence 12 or more since $12 + 12 \times 4 > 52$. Two K_4s can share at most one vertex, so the three K_4s must combine in such a way that at least six vertices appear in only one of the K_4s. Each of these vertices would require a single additional edge from a K_2, but there are only two K_2s.
Case 6. Finally, if $e = 0$, then $5g_2 + 3g_3 = 20$. There are two possibilities. First we deal with the possibility $g_2 = 1$, $g_3 = 5$, $g_4 = 0$. The defect graph has 16 edges and total valence 32. There cannot be a vertex of valence 8 or more since $8 + 8 \times 4 > 32$. Hence there must be eight vertices, each of valence 4. But this cannot be achieved with only one K_2. The second possibility is $g_2 = 4$, $g_3 = 0$, $g_4 = 2$. As in the first possibility, there must be eight vertices, each of valence 4. This is the possibility obtained in a different manner in [1].
So we have the following theorem.

Theorem 3.1. $g_1^{(5)}(20t + 9) \geq g_0 + 4$, *with equality if and only if it is possible to have* $D(2, 5, 20t + 9) = 20t^2 + 17t + 2$, *with a defect graph on eight vertices consisting of two quadruples and four pairs.*

We next consider what happens if $f \neq 4$, that is if $g_1^{(5)}(20t+9) > g_0 + 4$.

Corollary 3.2.5. $f \neq 5$.
Proof. $f = 5$ implies $5g_2 + 3g_3 = 26 - 4e$.
Case 1. Clearly $e = 6$ is impossible.
Case 2. If $e = 5$, then $g_2 = 0$, $g_3 = 2$, $g_4 = 10$. The defect graph has 66 edges and total valence 132. By considering the K_3s, we see that there are either six vertices of valence 8, or one vertex of valence 4 and

four of valence 8. These vertices of valence at most 8 can contribute at most 48 to the total valence. A vertex of valence 16 would give a total valence at least $16 + (1 \times 4 + 4 \times 8 + 11 \times 12) > 132$, and so cannot occur. Thus all remaining vertices have valence 12, and there must be at least seven such vertices. These can only arise when four K_4s share a common vertex; we will call such a vertex a 4-vertex. Now define a new graph G whose vertices are the ten K_4s and join two of these vertices by an edge if and only if the corresponding K_4s are incident at a 4-vertex. The graph G has ten vertices and its edge set decomposes into at least seven edge-disjoint K_4s. This implies that $D(2, 4, 10) \geq 7$, contradicting the fact that $D(2, 4, 10) = 5$ [3].

Case 3. If $e = 4$, then $g_2 = 2$, $g_3 = 0$, $g_4 = 9$. The defect graph has 56 edges and total valence 112. There are either four vertices of valence 4, or one vertex of valence 8 and two vertices of valence 4. All remaining vertices have valence at least 12. So the total valence is at least $12 + (4 \times 4 + 8 \times 12) > 112$, a contradiction.

Case 4. If $e = 3$, then $g_2 = 1$, $g_3 = 3$, $g_4 = 6$. The defect graph has 46 edges and total valence 92. There are at most nine vertices of valence 8 $(= 3+3+2)$ and, in such a case, there will be at most two of valence 4 $(= 3 + 1)$. Thus vertices of valence 4 and 8, of which there are at most 11, can contribute at most 80 to the total valence. There cannot be a vertex of valence 16 since $16 + (11 \times 4 + 5 \times 12) > 92$, so there is at least one vertex of valence 12 at which four K_4s meet and these K_4s have 12 other vertices. These 12 other vertices each need either an incident K_2 or a K_3 and the total number of available K_2 and K_3 vertices is only 11, a contradiction.

Case 5. If $e = 2$, then $5g_2 + 3g_3 = 18$. There are two possibilities: either $g_2 = 3$, $g_3 = 1$, $g_4 = 5$ or $g_2 = 0$, $g_3 = 6$, $g_4 = 3$. The first alternative has a defect graph of 36 edges and total valence 72. Since there are at most six vertices of valence 4 and in such a case at most three of valence 8, we see that all remaining vertices must have valence at least 12 and so the total valence is at least $12 + (6 \times 4 + 3 \times 8 + 3 \times 12) > 72$, a contradiction. The second alternative again gives a defect graph with 36 edges and total valence 72. Since all vertices then have valences at least 8, there are at most nine vertices and so there are exactly nine vertices of valence 8. But a vertex of valence 8 requires either two K_4s and one K_3, or four K_3s, and there are at most three vertices of the former kind and at

most one vertex of the latter kind. So this possibility is also rejected.
Case 6. If $e = 1$, then $5g_2 + 3g_3 = 22$. We find that $g_2 = 2$, $g_3 = 4$, $g_4 = 2$. The defect graph has 26 edges and total valence 52. All vertices have valences 4 or 8, and there are at least six vertices of the K_4s that appear singly. So we need a K_2 at each of these vertices, and this is impossible.
Case 7. Finally, if $e = 0$, we have $5g_2 + 3g_3 = 26$. One alternative would appear to be $g_2 = 1$, $g_3 = 7$, but this gives $g_4 = -1$, an obvious impossibility. So $g_2 = 4$, $g_3 = 2$, $g_4 = 1$, and the defect graph has 16 edges and total valence 32. Thus it consists of eight vertices of valence 4. The four vertices of the K_4 must each coincide with a vertex of a distinct K_2. The remaining four vertices of the K_2s cannot be identified as a single vertex otherwise we would have formed another quintuple. The only other possibility is that these four vertices are identified in two pairs, each of which requires a further vertex from one of the two K_3s. The remaining two vertices of the defect graph must each come from identification of two pairs of vertices from the K_3s. But this results in a repeated edge in the defect graph.
So we have established that $f = 5$ is impossible and have the following theorem.

Theorem 3.2. $g_1^{(5)}(20t + 9) \neq g_0 + 5$.

We next prove that for $t \geq 3$, $g_1^{(5)}(20t + 9) \leq g_0 + 6$. To do this, note that for $t \geq 3$, there is a covering of all $\binom{20t+9}{2}$ pairs by one block of length 17 and a set of $20t^2 + 17t - 10$ quintuples [2, 3]. Using the fact that $g_1^{(5)}(17) = 20$ [4], replace the block of length 17 by 20 blocks to give

$$g_1^{(5)}(20t + 9) \leq 20t^2 + 17t + 10 = g_0 + 6.$$

Thus we have the final result of this section.

Theorem 3.3. *For $t \geq 3$, $g_1^{(5)}(20t + 9)$ is either $g_0 + 4$ or $g_0 + 6$. The value $g_0 + 4$ is only possible if the conditions of Theorem 3.1 are met.*

4 The case $v = 20t + 17$

For $t = 0$, we have $g_1^{(5)}(17) = 20$ (cf. [4]). The values of $g_1^{(5)}(37)$, $g_1^{(5)}(57)$ and $g_1^{(5)}(77)$ are unknown, and we restrict ourselves to the case $t > 3$.

Since $\binom{20t+17}{2} = 200t^2 + 330t + 136$, we have $D(2, 5, 20t + 17) \leq 20t^2 + 33t + 13$. If this bound were met, the defect graph would have six edges and total valence 12, which is impossible, as pointed out in Section 3. So we have the following result.

Lemma 4.1. $D(2, 5, 20t + 17) \leq 20t^2 + 33t + 12$.

We proceed as in Section 3. Since $200t^2 + 330t + 136$ pairs must be covered,

$$g_1^{(5)}(v) \geq 20t^2 + 33t + 14 = g_0.$$

Again, at least 16 pairs cannot be covered by quintuples and so at least three shorter blocks are required. So put $g_1^{(5)}(v) = g_0 + f$, where $f \geq 1$, and define e by

$$g_5 = 20t^2 + 33t + 12 - e.$$

Then $g_2 + g_3 + g_4 = g_1^{(5)} - g_5 = g_0 + f - g_5 = 2 + f + e$. As before, $10g_1^{(5)}(v) - \binom{20t+17}{2} - 4(g_2 + g_3 + g_4) = 5g_2 + 3g_3$. But this expression is also equal to $140 + 10f - 136 - 4e - 4f - 8 = 6f - 4e - 4$. This gives the next lemma.

Lemma 4.2. $5g_2 + 3g_3 = 6f - 4e - 4$.

This is the same formula as in Lemma 3.2, and so all the discussion of cases in Section 3 may be repeated and we obtain the following theorem.

Theorem 4.1. $g_1^{(5)}(20t + 17) \geq g_0 + 4$, *with equality if and only if it is possible to have* $D(2, 5, 20t + 17) = 20t^2 + 33t + 12$, *with a defect graph on eight vertices consisting of two quadruples and four pairs.*

The discussion of the possibility that $g_1^{(5)}(v) = g_0 + 5$ is identical with that in Section 3. Hence we have the next theorem.

Theorem 4.2. $g_1^{(5)}(20t+17) \neq g_0 + 5$.

We next prove that for $t \geq 4$, $g_1^{(5)}(20t+17) \leq g_0 + 6$. To do this, note that for $t \geq 4$, there is a covering of all $\binom{20t+17}{2}$ pairs by one block of length 17 and a set of $20t^2 + 33t$ quintuples [2, 3]. Again using the fact that $g_1^{(5)}(17) = 20$ [4], replace the block of length 17 by 20 blocks to give

$$g_1^{(5)}(20t+17) \leq 20t^2 + 33t + 20 = g_0 + 6.$$

Thus we have the final result of this section.

Theorem 4.3. *For $t \geq 4$, $g_1^{(5)}(20t+17)$ is either $g_0 + 4$ or $g_0 + 6$. The value $g_0 + 4$ is only possible if the conditions of Theorem 4.1 are met.*

5 Concluding remarks

Apart from some unresolved small cases, for $v = 20t+9$, $v = 20t+13$, and $v = 20t+17$, our results show that $g_1^{(5)}(v)$ takes one of two specific values. Clearly it would be desirable to determine which of the two. The unresolved small cases are $v = 29, 33, 37, 49, 57$ and 77.

For $v = 37$, take a resolvable $(28, 63, 9, 4, 1)$-BIBD; this has nine resolution classes. Take nine new points, adjoining them to the blocks of the resolution classes, a different point for each class, to form 63 quintuples. Then add the 12 triples of a Steiner triple system on the nine new points. This gives a covering of pairs from 37 points in $75 = g_0 + 8$ blocks, so $g_1^{(5)}(37) \leq g_0 + 8$.

For $v = 57$ or 77, there is a covering of all pairs by one block of length 9 and all other blocks of length 5, i.e. quintuples [2, 3]. Replacing the block of length 9 by the 12 triples of a Steiner triple system on 9 points gives $g_1^{(5)}(57) \leq 168 = g_0 + 8$ and $g_1^{(5)}(77) \leq 301 = g_0 + 8$.

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