Large sets of large sets of Steiner triple systems of order 9

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Abstract

We describe a direct method of partitioning the 840 Steiner triple systems of order 9 into 120 large sets. The method produces partitions in which all of the large sets are isomorphic and we apply the method to each of the two non-isomorphic large sets of STS(9).

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1 The problem

The following facts are well-known. We recall them to explain the problem in which we are interested and to introduce notation. Up to isomorphism, the Steiner triple system of order 9 (STS(9) = AG(2,3)) is unique. It contains 12 triples which partition into four parallel classes. A convenient representation of an STS(9) is by a 3×3 array

$$\begin{array}{ccc} a & b & c \\ d & e & f \\ g & h & i \end{array}$$

where the rows, columns, forward diagonals and back diagonals form the triples of the four parallel classes. We use this representation throughout the paper. The number of distinct triples which are subsets of a fixed base set V of cardinality 9 is $\binom{9}{3} = 84$. A partition of these 84 triples into 7 STS(9)s is called a *large set* and will be denoted by LS(9). Up to isomorphism, there are precisely two LS(9)s, [1], see also [2].

The first of these, type $\mathcal{A}$, has automorphism group of order 42 and is generated from the STS(9)

$$\begin{array}{ccc} \infty & \infty' & 0 \\ 1 & 2 & 4 \\ 5 & 6 & 3 \end{array}$$

by the mapping $i \mapsto i + 1 \pmod 7$ with both ∞ and ∞' as fixed points.

The second, type $\mathcal{B}$, has automorphism group of order 54. It consists of the STS(9)

$$\begin{array}{ccc} \infty & A & B \\ 0 & 2 & 4 \\ 3 & 1 & 5 \end{array}$$

together with 6 further systems generated from the STS(9)

$$\begin{array}{ccc} \infty & 0 & 1 \\ A & 2 & 5 \\ 3 & B & 4 \end{array}$$

by the permutation (∞)(A B)(0 1 2 3 4 5).

The automorphism group of an STS(9) has order 432. Hence the number of distinct STS(9)s which exist on a base set V is $9!/432 = 840$. A natural question to ask is whether these 840 distinct STS(9)s can be partitioned into 120 LS(9)s; a *large set of large sets* or *super large set* SLS(9). The question first seems to have been considered by Mathon & Street, [3]. By computational means they constructed an SLS(9) in which 119 of the LS(9)s are isomorphic to type $\mathcal{B}$ and the remaining LS(9) is isomorphic to type $\mathcal{A}$. They remark that they "have no reason to believe that this partition is unique". In this paper we describe a straightforward method to construct an SLS(9). We use the method to produce two partitions, in one of which every LS(9) is isomorphic to type $\mathcal{A}$ and in the other every LS(9) is isomorphic to type $\mathcal{B}$. Hence neither partition is isomorphic to that given by Mathon & Street.

2 The solution

Let $\mathcal{L} = \{S_i : i = 0, 1, \ldots, 6\}$, where each S_i is an STS(9), be a large set defined on a base set V. Choose four distinct points $a, b, c, d \in V$. Let G be the symmetric group $\mathcal{S}_5$ of all permutations of the set $V \setminus \{a, b, c, d\}$. For each $p \in G$, construct the large set $p\mathcal{L} = \{pS_i : i = 0, 1, \ldots, 6\}$. Since G has order 120, we have the required partition if and only if all 840 systems are distinct. We deduce a necessary and sufficient condition on the four points a, b, c, d which ensures this and show that in both large sets of types $\mathcal{A}$ and $\mathcal{B}$ it is possible to find four such points. Note that all large sets in such a partition are necessarily isomorphic.

Suppose that $p_1 S_i = p_2 S_j$ where $0 \leq i, j \leq 6$ and $p_1, p_2 \in G$. Then by applying the inverse permutation p_2^{-1}, $pS_i = S_j$, where $p = p_2^{-1} p_1$. Now consider the distribution of the points a, b, c, d in the systems $S_0, S_1, \ldots, S_6$. Four of these must have the following structure

$$\begin{array}{ccc} a & b & c \\ d & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{array} \qquad \begin{array}{ccc} a & b & d \\ c & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{array} \qquad \begin{array}{ccc} a & c & d \\ b & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{array} \qquad \begin{array}{ccc} b & c & d \\ a & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{array}$$

In the other three systems, the six pairs ab, ac, ad, bc, bd, cd occur in different triples, i.e. the structure of the STS(9) is

$$\begin{array}{ccc} a & b & \cdot \\ c & * & * \\ \cdot & * & \cdot \end{array}$$

where the point d can only occur in the three positions marked with an asterisk. Suppose that in these three remaining systems the point d occupies these three positions. Equivalently, the structure of the three STS(9)s is

$$\begin{array}{ccc} a & b & \cdot \\ c & d & \cdot \\ \cdot & \cdot & \cdot \end{array} \qquad \begin{array}{ccc} a & d & \cdot \\ b & c & \cdot \\ \cdot & \cdot & \cdot \end{array} \qquad \begin{array}{ccc} a & c & \cdot \\ d & b & \cdot \\ \cdot & \cdot & \cdot \end{array}$$

Since a, b, c, d are fixed points, if $pS_i = S_j$ then it follows that $i = j$ and p is the identity permutation, i.e. $p_1 = p_2$. Thus all 840 STS(9)s are distinct and we have an SLS(9).

Conversely, if one of these last three structures is repeated then there will exist a permutation p such that $pS_i = S_j$ for some $i \neq j$ and we do not have the required partition.

Finally, in order to construct an SLS(9) in which each LS(9) is of type $\mathcal{A}$ (respectively type $\mathcal{B}$) it remains to identify four points having the property described. With the representations of the large sets as given above it is easily verified that we can choose $\{a, b, c, d\} = \{0, 1, 2, 4\}$ for LS(9) of type $\mathcal{A}$ and $\{a, b, c, d\} = \{\infty, 0, 2, 4\}$ for LS(9) of type $\mathcal{B}$.

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