On sparse countably infinite Steiner triple systems

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Abstract

We give a general construction for Steiner triple systems on a countably infinite point set and show that it yields $2^{\aleph_0}$ nonisomorphic systems all of which are uniform and r-sparse for all finite $r \geq 4$.

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1 Introduction

The problem of determining the values of v for which there exists an r-sparse Steiner triple system of order v appears to be extremely difficult, and a definitive answer is only known for the case $r = 4$ [12]. In this paper we show that it is possible to construct $2^{\aleph_0}$ nonisomorphic Steiner triple systems on a point set of cardinality $\aleph_0$, all of which are r-sparse for every $r \geq 4$. These systems are also uniform, meaning that the cycle graphs formed from each pair of points are isomorphic. We begin with some basic definitions and results, an explanation of the terminology, and the background to the problem.

A *(finite) Steiner triple system of order* v is a pair $(V, \mathcal{B})$ where V is a finite set of v elements (the *points*) and $\mathcal{B}$ is a collection of 3-element subsets (the *blocks*) of V such that each 2-element subset of V is contained in exactly one block of $\mathcal{B}$. It is well known that a Steiner triple system of order v (briefly STS(v)) exists if and only if $v \equiv 1$ or $3 \pmod 6$ [13]. Such values of v are called *admissible*. If we replace the requirement that V be finite by the requirement that V have cardinality $\aleph_0$, then the the resulting pair $(V, \mathcal{B})$ is called a *countably infinite Steiner triple system*, CIST for short. Such systems are not new and previous investigations are given in [1, 6, 10, 11, 14].

For any two points a and b in a Steiner triple system $(V, \mathcal{B})$, finite or countably infinite, the *cycle graph* $G_{a,b}$ is defined as follows. The vertex set of $G_{a,b}$ is $V \setminus \{a, b, a * b\}$, where $x * y$ denotes the third point in the block containing the pair $\{x, y\}$. The edge set of $G_{a,b}$ is the set of pairs $\{i, j\}$ such that either $\{i, j, a\}$ is a block or $\{i, j, b\}$ is a block. In the finite case, it is well known that $G_{a,b}$ is a set of disjoint cycles $\{C_{n_1}, C_{n_2}, \ldots, C_{n_r}\}$, where $n_1 + n_2 + \ldots + n_r = v - 3$. Moreover each n_i is even with $n_i \geq 4$. In the infinite case, besides finite cycles, $G_{a,b}$ may have components that are infinite two-way paths such as that formed by taking the vertex set $\mathbb{Z}$ and edge set $\{\{i, i+1\} : i \in \mathbb{Z}\}$; we will refer to such a path in $G_{a,b}$ as an *infinite cycle*.

A *configuration* (or *partial Steiner triple system*) is a set of triples, taken from a point set V, which has the property that every pair of distinct elements occurs in at most one triple. A configuration is called *finite* if it has a finite set of triples, otherwise it is called *infinite*. A finite configuration having b triples and p points is called a (b, p)-configuration. If $\mathcal{C}$ is a configuration, we denote its set of points by $P(\mathcal{C})$. The *degree* of a point in a configuration is the number of triples that contain that point (the terminology is only defined if this number is finite). If the cycle $C = (\ldots, x, y, z, \ldots)$ is a component of a cycle graph $G_{a,b}$ from a Steiner triple system, then C can be regarded as a configuration $\mathcal{C}$ since it corresponds either to blocks $\{\ldots, \{a, x, y\}, \{b, y, z\}, \ldots\}$ or to blocks $\{\ldots, \{b, x, y\}, \{a, y, z\}, \ldots\}$. If the cycle has finite length n then the corresponding configuration is an $(n, n+2)$-configuration. For $n > 4$, other $(n, n+2)$-configurations can arise in Steiner triple systems apart from those generated by finite components of cycle graphs. In this paper we will be particularly concerned with $(n, n+2)$-configurations; such configurations are of special interest because of the following result proved in [4].

Theorem 1.1 *For every integer $d \geq 3$ and for every integer n satisfying $n \geq \lceil \frac{d}{2} \rceil$ there exists $v_0(n,d)$ such that for all admissible $v \geq v_0(n,d)$, every STS(v) contains an $(n, n+d)$-configuration.*

Here, the value of d is sharp. For $d = 2$, the theorem does not hold. Indeed, the the case $d = 2$ is the subject of the following conjecture of Erdös [3]:

For every integer $k \geq 4$, there exists $v_0(k)$ such that if $v > v_0(k)$ and if v is admissible, then there exists an STS(v) with the property that it contains no $(n, n+2)$-configurations for any n satisfying $4 \leq n \leq k$.

An STS(v) that contains no $(n, n+2)$-configurations for any n satisfying $4 \leq n \leq r$ is said to be r-*sparse*. Clearly, an r-sparse system is also r'-sparse for every r' satisfying $4 \leq r' \leq r$.

Results about r-sparse STS(v)s are somewhat limited. It was shown in a series of papers culminating in [12] that 4-sparse systems exist for all admissible v apart from $v = 7$ or 13. In fact there is only one $(4, 6)$-configuration, generally known as the Pasch configuration, and so such systems are usually called anti-Pasch systems. The Pasch configuration corresponds to a cycle of length 4 in a cycle graph and is also sometimes called a quadrilateral. It comprises 4 blocks of the form $\{a, b, c\}, \{x, y, c\}, \{x, b, z\}, \{a, y, z\}$. It is conjectured that 5-sparse systems exist for all admissible v apart from $v = 7, 9, 13$ and 15. Although substantial progress has been made by Fujiwara [7, 8] and by Wolfe [15, 16, 17], the proof is still incomplete. As regards 6-sparse systems, infinitely many of these are known [4, 5], but no non-trivial k-sparse systems are known for any $k \geq 7$.

The original proof of Theorem 1.1 given in [4] requires only simple modifications in order to establish the following result for CISTs.

Theorem 1.2 *For every integer $d \geq 3$ and for every integer n satisfying $n \geq \lceil \frac{d}{2} \rceil$, every CIST contains an $(n, n+d)$-configuration.*

A proof of this result is given in Section 2.

We will adopt the same definition of r-sparse for CISTs as that given above for finite STSs. In [10] it was shown that the number of pairwise nonisomorphic CISTs has cardinality $2^{\aleph_0}$. In [6] Franek constructed a family of pairwise nonisomorphic rigid CISTs, also of cardinality $2^{\aleph_0}$. Here *rigid* means that the system admits only the identity automorphism. In this paper we will show how to construct $2^{\aleph_0}$ nonisomorphic CISTs that are r-sparse for every $r \geq 4$; we will call such CISTs ∞-*sparse*. In fact each of our systems will also have the property that all of its cycle graphs $G_{a,b}$ are isomorphic; such systems are called *uniform*. In the case of STS(v)s, uniform systems arise from two infinite families of 2-transitive systems, the projective and affine systems, as well as from Hall triple systems and from Netto triple systems [2]. Apart from these families, only a finite number of uniform STS(v)s are known [9, 4].

Finally on a point of notation, we will often write blocks with set brackets and commas omitted, so that for example $\{0, 1, 3\}$ might be written as 0 1 3 or as 013 when no ambiguity is likely.

2 Results

We start this section by giving a proof of Theorem 1.2. In order to do this we first establish the following lemma.

Lemma 2.1 *If $n \geq 2$, then every STS(v) with $v \geq n+3$ and every CIST contains a configuration having n blocks and $n+3$ points.*

Proof It is easy to see that every STS(v) with $v \geq 7$ and every CIST contains a copy of the configuration $\mathcal{C} = \{\{0, 1, 2\}, \{0, 3, 4\}, \{0, 5, 6\}, \{2, 4, 6\}\}$. Note that $\mathcal{C}$ has 4 blocks and 7 points. It also contains the configuration $\{\{0, 1, 2\}, \{0, 3, 4\}, \{2, 4, 6\}\}$ which has 3 blocks and 6 points, and the configuration $\{\{0, 1, 2\}, \{0, 3, 4\}\}$ which has 2 blocks and 5 points. Thus the result certainly holds for $n = 2, 3$ and 4.

Now suppose, inductively, that the result holds for all n satisfying $4 \leq n \leq n_0$. We show that it also holds for $n = n_0 + 1$. Select any STS(v) with $v \geq n_0 + 4$ or any CIST, say $\mathcal{S}$. By the inductive hypothesis, $\mathcal{S}$ contains a configuration $\mathcal{C}$ having n_0 blocks and $n_0 + 3$ points. The set $P(\mathcal{C})$ generates $\binom{n_0+3}{2}$ pairs of which $3n_0$ appear in the blocks of $\mathcal{C}$, so there exist pairs not appearing in a block of $\mathcal{C}$. Each of these pairs lies in a unique block of $\mathcal{S}$ and so generates a third point of $\mathcal{S}$. Either

(a) there exists a pair whose third point lies outside $P(\mathcal{C})$, or

(b) every such pair generates its third point inside $P(\mathcal{C})$.

In case (a) we add the third point and corresponding block to $\mathcal{C}$ to obtain a configuration in $\mathcal{S}$ having n_0+1 blocks and n_0+4 points.

In case (b), the points of $P(\mathcal{C})$ generate an STS(n_0+3), $\mathcal{S}_0$, contained within $\mathcal{S}$. Then $7 \leq n_0+3$ and $\mathcal{S}_0$ will contain a configuration $\mathcal{D}$ having n_0-1 blocks and n_0+2 points. Let x denote the unique point of $\mathcal{S}_0$ not lying in $P(\mathcal{D})$. The number of pairs from $P(\mathcal{D})$ which do not lie in blocks of $\mathcal{D}$ and which do not lie in blocks containing x is given by

$$\binom{n_0+2}{2} - 3(n_0-1) - \frac{n_0+2}{2} = \frac{(n_0-2)^2}{2} + 1 > 0.$$

So at least one pair from $P(\mathcal{D})$ lies in a block of $\mathcal{S}_0$ outside $\mathcal{D}$ whose third point lies in $P(\mathcal{D})$. If this block is added to $\mathcal{D}$ then we obtain a configuration $\mathcal{D}'$ having n_0 blocks and n_0+2 points. Now choose a point y of $\mathcal{S}$, not lying in $\mathcal{S}_0$, and choose any point $a \in P(\mathcal{D})$. There is a block $\{a,y,z\}$ of $\mathcal{S}$ with z not lying in $\mathcal{S}_0$. By adding this block to $\mathcal{D}'$, we form a configuration in $\mathcal{S}$ having n_0+1 blocks and n_0+4 points.

The result of the lemma now follows by induction. □

Proof of Theorem 1.2 Assume, inductively, that the result is true for $d = d_0 \geq 3$. We show that it also holds for $d = d_0+1$.

First we deal with the case when d_0 is odd and $n = \lceil \frac{d_0+1}{2} \rceil$. We must show that every CIST contains a configuration having n blocks and $3n$ points. But such a configuration is a partial parallel class having n blocks. Starting with any finite configuration $\mathcal{C}$ occurring in a CIST we may always add a parallel block to form a new configuration $\mathcal{C}'$ in the CIST. This is because two points a and b, not already occurring in $P(\mathcal{C})$, may be selected in infinitely many ways, and only finitely many of these choices will have $a*b$ in $P(\mathcal{C})$.

For all remaining cases, namely if d_0 is even and $n = \lceil \frac{d_0+1}{2} \rceil$, or if d_0 has either parity and $n > \lceil \frac{d_0+1}{2} \rceil$, we have $n-1 \geq \lceil \frac{d_0}{2} \rceil$. So, by the inductive hypothesis, every CIST contains some configuration $\mathcal{C}$ having $n-1$ blocks and $n-1+d_0$ points. The CIST must also have a block $\{x,y,z\}$ with $x \in P(\mathcal{C})$ and $y,z \notin P(\mathcal{C})$. If such a block is added to $\mathcal{C}$, then we obtain a configuration having n blocks and $n+(d_0+1)$ points. The result now follows by induction. □

We next give a general construction for CISTs.

Construction 2.1

Take finite (possibly empty) sets A_i, for $i = 1, 2, \ldots$, that are mutually disjoint, and take a further disjoint set $B = \{b_1, b_2, b_3, \ldots\}$. For each i, let $\mathcal{C}_i$ be a (possibly null) configuration defined on the points of A_i. A CIST can be formed on the point set $V = (\bigcup_i A_i) \cup B$ with block set $\mathcal{B}$ as follows. First place in $\mathcal{B}$ all the blocks of all the configurations $\mathcal{C}_i$; we will call these blocks *horizontal*. All remaining blocks will be described as *vertical*. The vertical blocks are formed in stages, and it may be helpful to consider the partial ordering of the points given by the list $b_1, A_1, b_2, A_2, b_3, A_3, b_4, \ldots$. To make this ordering more precise, we define $\phi(x) = 2i - 1$ if $x = b_i$ and $\phi(x) = 2i$ if $x \in A_i$. Also, if $x \in \bigcup_i A_i$ then x will be called an A-point and if $x \in B$ then x will be called a B-point.

At the first stage, list all pairs of points from $\{b_1\} \cup A_1$ that are not already in a block. If there are no such pairs, put $k_1 = 0$ and go to the second stage. Otherwise, suppose these pairs are $p_1, p_2, \ldots, p_{k_1}$ and put $i_1 = 1$. Form the vertical blocks $p_i \cup \{b_{i_1+i}\}$ for $i = 1, 2, \ldots, k_1$.

At the second stage, list all pairs of points from $\{b_1, b_2\} \cup A_1 \cup A_2$ that are not already in a block. If there are no such pairs, put $k_2 = k_1$ and go to the third stage. Otherwise, suppose these pairs are $p_{k_1+1}, p_{k_1+2}, \ldots, p_{k_2}$ and take i_2 to be the maximum value of i such that b_i already appears in a block, or in a pair p_j for some j in the range $k_1 + 1 \leq j \leq k_2$. Then form the vertical blocks $p_{k_1+i} \cup \{b_{i_2+i}\}$ for $i = 1, 2, \ldots, k_2 - k_1$.

We continue in this fashion. At stage r, list all pairs of points from $\{b_1, b_2, \ldots, b_r\} \bigcup_{i=1}^{r} A_i$ that are not already in a block. If there are no such pairs, put $k_r = k_{r-1}$ and go to the next stage. Otherwise, suppose these pairs are $p_{k_{r-1}+1}, p_{k_{r-1}+2}, \ldots, p_{k_r}$ and take i_r to be the maximum value of i such that b_i already appears in a block, or in a pair p_j for some j in the range $k_{r-1} + 1 \leq j \leq k_r$. Then form the vertical blocks $p_{k_{r-1}+i} \cup \{b_{i_r+i}\}$ for $i = 1, 2, \ldots, k_r - k_{r-1}$.

Since every uncovered pair is eventually allocated to a unique triple, the resulting set of horizontal and vertical blocks, $\mathcal{B}$, forms a CIST on the point set V. Note two particular features. First, no horizontal block contains a point from B whereas each vertical block contains at least one point from B. Second, given any two distinct vertical blocks $\mathbf{b}_1, \mathbf{b}_2$, the B-point in $\mathbf{b}_1$ having the largest ϕ value is different from the B-point in $\mathbf{b}_2$ having the largest ϕ value. $\square$

A particularly simple system created by the foregoing construction is obtained when $A_i = \emptyset$ for each i. One may then take the blocks to be:

$$1\ 2\ 3, \quad 1\ 4\ 5, \quad 2\ 4\ 6, \quad 3\ 4\ 7, \quad 2\ 5\ 8, \quad 3\ 5\ 9, \quad 1\ 6\ 10, \quad 3\ 6\ 11,$$

$$5\ 6\ 12, \quad 1\ 7\ 13, \quad 2\ 7\ 14, \quad 5\ 7\ 15, \quad 6\ 7\ 16, \quad 1\ 8\ 17, \quad \ldots$$

The proof that this particular CIST is ∞-sparse is also extremely simple. Let $r \geq 4$ and choose any n blocks where $4 \leq n \leq r$. Then the largest point in each block is different. Together with the other two points from the block with the smallest largest point, these form a set S of $n+2$ points. Now consider the block with the second smallest largest point. At least one of the other two points is not in S. Hence the CIST contains no $(n, n+2)$-configuration and is r-sparse for every $r \geq 4$. This design has many of the features that we wish to capture, although we also wish to construct many designs with these features. In order to do this we examine (b, p)-configurations that are suitable for use as the horizontal blocks in the construction.

Lemma 2.2 *If $b > 4$ and $p \leq b+2$ then any (b, p)-configuration has a point of degree at least 3.*

Proof There are $3b$ occurrences of points, so the average degree is

$$\frac{3b}{p} \geq \frac{3b}{b+2} = 3 - \frac{6}{b+2} > 2 \text{ since } b > 4.$$

□

Lemma 2.3 *For every $n \geq 3$ there exists a connected configuration having $3n$ points all of degree 2, and consequently $2n$ blocks.*

Proof For $n = 3$ the connected configuration $\{abc, def, ghi, adg, beh, cfi\}$ has $3n$ points all of degree 2. Suppose inductively that we have a connected configuration $\mathcal{D}$, with $3m$ points all of degree 2, where $m \geq 3$. There exist two points of $P(\mathcal{D})$, say a, b, that do not lie together in a block of $\mathcal{D}$. Replace a by a new point a_1 in one of the two blocks containing a, and replace a by another new point a_2 in the other block containing a. Carry out a similar replacement of b by new points b_1 and b_2. This may disconnect the configuration, but we can assume that a_1 remains in the same component as b_1. Now introduce a new point c and two new blocks a_1cb_2 and b_1ca_2. The resulting configuration

$\mathcal{D}'$ has $3m+3$ points, all of degree 2, and is connected. The result follows by induction. $\square$

Starting with $\mathcal{D}_1 = \{abc, def, ghi, adg, beh, cfi\}$ and applying the construction from the proof of Lemma 2.3, we may form an infinite sequence $\mathcal{D}_1, \mathcal{D}_2, \mathcal{D}_3, \ldots$ of connected configurations, where $\mathcal{D}_i$ has $3(i+2)$ points all of degree 2. Clearly no $\mathcal{D}_i$ can contain any other $\mathcal{D}_j$ as a sub-configuration.

Lemma 2.4 *For each $i \geq 1$, the configuration $\mathcal{D}_i$ does not contain any (b,p)-sub-configuration having $b \geq 4$ and $p \leq b+2$.*

Proof This follows trivially from Lemma 2.2 if $b > 4$. If $b = 4$, the only possibility is the $(4,6)$ Pasch configuration which has all points of degree 2 and so cannot be a sub-configuration of any $\mathcal{D}_i$. $\square$

Theorem 2.1 *Suppose that each of the configurations $\mathcal{C}_i$ used in Construction 2.1 is taken from $\{\mathcal{D}_0, \mathcal{D}_1, \mathcal{D}_2, \mathcal{D}_3, \ldots\}$, where $\mathcal{D}_0$ is the configuration consisting of a single point and no blocks. Then the resulting CIST has the following properties.*
(i) The only connected finite configurations in the CIST having all points of degree at least 2, are precisely those configurations $\mathcal{C}_i$ which are not copies of $\mathcal{D}_0$.
(ii) The CIST is ∞-sparse.
(iii) Each cycle graph $G_{a,b}$ comprises an infinite number of infinite cycles and so the CIST is uniform.

Proof We show first that any finite configuration containing a vertical block has a point of degree 1. So suppose that $\mathcal{C}$ is a finite configuration containing a vertical block. In Construction 2.1, every vertical block contains a B-point. Let β be the B-point with the largest ϕ value in all the blocks of $\mathcal{C}$. No horizontal block contains β, and no two vertical blocks can contain β since they would then have the same B-point with the largest ϕ value. So the point β has degree 1 in $\mathcal{C}$.

It follows that any finite configuration $\mathcal{C}$ in the CIST which has all points of degree at least 2 must contain only horizontal blocks. Consequently the only connected finite configurations with all points of degree at least 2 are those configurations $\mathcal{C}_i$ which are not copies of $\mathcal{D}_0$.

To prove part (ii), note that the horizontal blocks contain no (b,p)-configurations with $b \geq 4$ and $p \leq b+2$. So any configuration of this type in the CIST must contain a vertical block. If $\mathcal{C}$ is such a configuration with the smallest value of b, consider the effect of removing from it a vertical block having a point of degree 1 in $\mathcal{C}$. The result is a (b', p') configuration $\mathcal{C}'$ where $b' = b - 1$ and $p' = p - 1$, $p - 2$ or $p - 3$. If $b \geq 5$ we then have $b' \geq 4$, $p' \leq b' + 2$ and $\mathcal{C}'$ having fewer blocks than $\mathcal{C}$, a contradiction. So we must have $b = 4$, $p \leq 6$, and $\mathcal{C}$ is therefore a Pasch configuration. But then $\mathcal{C}$ has no point of degree 1. It follows that the CIST cannot contain any (b, p)-configurations with $b \geq 4$ and $p \leq b + 2$, and consequently it is ∞-sparse.

To establish part (iii) suppose firstly that C is a finite cycle in $G_{a,b}$. Regarding C as a configuration $\mathcal{C}$, all points of $\mathcal{C}$ have degree at least 2 and so, by (i), $\mathcal{C}$ is one of $\mathcal{D}_1, \mathcal{D}_2, \mathcal{D}_3, \ldots$, and this is a contradiction since none of these corresponds to a finite cycle. Thus no cycle graph can contain a finite cycle and it only remains to prove that the number of cycles in any $G_{a,b}$ is infinite.

In order to do this, we first establish that given any $a, b \in V$ and any integer n, there exists a point $x \in B$ such that $\phi(x) > n$, $\phi(a * x) > \phi(x)$ and $\phi(b * x) > \phi(x)$. So, take any B-point, d, such that $d \neq a, b, a * b$. There will be at most three blocks that contain the pairs $\{a, d\}$ or $\{b, d\}$, or have d as the point with the largest ϕ value. Let the remaining blocks containing d be denoted by $\{d, w_i, x_i\}$ for $i = 1, 2, 3, \ldots$, where $\phi(w_i) < \phi(x_i)$ and $x_i \in B$ for each i. Now consider the block $\{a, x_i, a * x_i\}$. If x_i has the largest ϕ value in this block, then the block is $\{d, w_i, x_i\}$ and so $\{d, w_i\} = \{a, a * x_i\}$. But $d \neq a$ and $w_i \neq a$, a contradiction. Hence x_i is <u>not</u> the point with the largest ϕ value in $\{a, x_i, a * x_i\}$. Similarly, x_i is not the point with the largest ϕ value in $\{b, x_i, b * x_i\}$. Moreover, there is at most a finite number of points x_i for which a can be the point with the largest ϕ value in $\{a, x_i, a * x_i\}$ or for which b can be the point with the largest ϕ value in $\{b, x_i, b * x_i\}$. So, for an infinite subset $I \subseteq \mathbb{Z}^+$, we have $\phi(a * x_i) > \phi(x_i)$ and $\phi(b * x_i) > \phi(x_i)$ whenever $i \in I$. Hence, given n, there exists a B-point x for which $\phi(x) > n$, $\phi(a * x) > \phi(x)$ and $\phi(b * x) > \phi(x)$.

To complete the proof of (iii), suppose that $C_1, C_2, \ldots, C_k$ is any finite collection of cycles in $G_{a,b}$. Denote by l_i a point appearing in C_i and having the smallest ϕ value amongst the points of C_i. Take l^* to be an l_i for which $\phi(l_i) \geq \phi(l_j)$ for each $j = 1, 2, \ldots, k$. Now consider the cycle C in $G_{a,b}$ containing a point $x \in B$ for which $\phi(x) > \max\{\phi(l^*), \phi(a), \phi(b)\}$, $\phi(a * x) > \phi(x)$ and $\phi(b * x) > \phi(x)$. Let the blocks of this cycle be $axb_1, bxa_1, aa_1a_2, bb_1b_2, ab_2b_3, ba_2a_3, \ldots$. We claim that in the order in which we have written the three points in each of these blocks, the third point has the unique largest ϕ value amongst the three points. To see this, note firstly that x has been chosen so that $\phi(b_1) = \phi(a{*}x) > \phi(x) > \phi(a)$ and $\phi(a_1) = \phi(b{*}x) > \phi(x) > \phi(b)$. But then a_1 and b_1 must be B-points and so cannot be the points with the largest ϕ values in the blocks aa_1a_2 and bb_1b_2. Hence a_2 and b_2 are also B-points and $\phi(a_2) > \phi(a_1)$, $\phi(b_2) > \phi(b_1)$, and so on. It follows that the sequences $\{\phi(a_i)\}$ and $\{\phi(b_i)\}$ are strictly increasing and all of these values, together with $\phi(x)$, exceed $\phi(l^*)$. Consequently, C is not one of $C_1, C_2, \ldots, C_k$, and it follows that no finite collection of cycles can contain all the points of V. This completes the proof of (iii) and the theorem. $\square$

Corollary 2.1.1 *There are $2^{\aleph_0}$ nonisomorphic ∞-sparse and uniform CISTs*

Proof If two distinct infinite subsets of $\{\mathcal{D}_1, \mathcal{D}_2, \mathcal{D}_3, \ldots\}$ are taken to form the set $\{\mathcal{C}_i\}$ in two applications of the theorem, then the two resulting CISTs are nonisomorphic since they contain different numbers of the configurations $\mathcal{D}_i$. Since there are $2^{\aleph_0}$ such subsets, there are that many nonisomorphic ∞-sparse and uniform CISTs. $\square$

Finally we remark that an extremal case of uniformity in a finite STS(v) occurs when every one of its cycle graphs consists of a single cycle of length v; such a system is called *perfect*. Only a finite number of perfect STS(v)s are known, [9, 4]. By analogy with this definition we will call a CIST perfect if every one of its cycle graphs consists of a single infinite cycle. We have no example of such a system.

References

[1] P. J. Cameron, *An orbit theorem for Steiner triple systems,* Discrete Math. **125** (1994), 97–100.

[2] C. J. Colbourn and A. Rosa, *Triple systems*, Oxford University Press, New York 1999, ISBN 0-19-853576-7.

[3] P. Erdös, *Problems and results in combinatorial analysis,* Colloquio Internazionale sulle Teorie Combinatorie (Rome, 1973), Tomo II, pp. 3–17. Atti dei Convegni Lincei, No. 17, Accad. Naz. Lincei, Rome, 1976.

[4] A. D. Forbes, M. J. Grannell and T. S. Griggs, *On 6-sparse Steiner triple systems,* J. Combin. Theory Ser. A. **114** (2007), 235–252.

[5] A. D. Forbes, M. J. Grannell and T. S. Griggs, *Further 6-sparse Steiner triple systems,* Graphs Combin. (to appear).

[6] F. Franek, *Isomorphisms of infinite Steiner triple systems,* Ars Combin. **38** (1994), 7–25.

[7] Y. Fujiwara, *Constructions for anti-mitre Steiner triple systems,* J. Combin. Des. **13** (2005), 286–291.

[8] Y. Fujiwara, *Infinite classes of anti-mitre and 5-sparse Steiner triple systems,* J. Combin. Des. **14** (2006), 237–250.

[9] M. J. Grannell, T. S. Griggs and J. P. Murphy, *Some new perfect Steiner triple systems,* J. Combin. Des. **7** (1999), 327–330.

[10] M. J. Grannell, T. S. Griggs and J. S. Phelan, *Countably infinite Steiner triple systems,* Ars Combin. **24B** (1987), 189–216.

[11] M. J. Grannell, T. S. Griggs and J. S. Phelan. *On infinite Steiner systems,* Discrete Math. **97** (1991), 199–202.

[12] M. J. Grannell, T. S. Griggs and C. A. Whitehead, *The resolution of the anti-Pasch conjecture,* J. Combin. Des. **8** (2000), 300–309.

[13] T. P. Kirkman, *On a problem in combinations,* Cambridge and Dublin Math. J. **2** (1847), 191–204.

[14] B. S. Webb, *Orbits of infinite block designs,* Discrete Math. **169** (1997), 287–292.

[15] A. J. Wolfe, *The resolution of the anti-mitre Steiner triple system conjecture,* J. Combin. Des. **14** (2006), 229–236.

[16] A. J. Wolfe, *5-sparse Steiner triple systems of order n exist for almost all admissible n,* Electron. J. Combin. **12** (2005), #R68, 42pp (electronic).

[17] A. J. Wolfe, *Block transitive meager squares over $GF(q)$ for prime q,* preprint, 2005, 15pp.