Anti-Pasch optimal packings with triples

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Abstract

It is shown that for $v \neq 6, 7, 10, 11, 12, 13$ there exists an optimal packing with triples on v points that contains no Pasch configurations. Furthermore, for all $v \equiv 5 \pmod 6$ there exists a pairwise balanced design of order v, whose blocks are all triples apart from a single quintuple, and that has no Pasch configurations amongst its triples.

AMS classification: 05B07.
Keywords: Optimal packing with triples; Maximum partial triple system; Steiner triple system; Pasch configuration.

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1 Introduction

The background to this paper is the anti-Pasch problem for Steiner triple systems. A Steiner triple system of order v, STS(v), is a pair $(V, \mathcal{B})$ where V is a set of v elements (called *points*) and $\mathcal{B}$ is a set of 3-element subsets of V (called *blocks* or *triples*) with the property that each 2-element subset of V is contained in exactly one block. An STS(v) exists if and only if $v \equiv 1$ or 3 (mod 6) [12], and such values are called *admissible*. A *Pasch configuration*, also known as a *quadrilateral*, is a set of 3-element sets on six points having the form

$$\{\{a, b, c\}, \{a, y, z\}, \{x, b, z\}, \{x, y, c\}\}.$$

The anti-Pasch conjecture, originally made by Paul Erdős [7] in a more general form, was that for all sufficiently large admissible v there exists an STS(v) that contains no Pasch configurations among its blocks. The conjecture was finally established in a series of papers [1, 9, 10, 13] culminating in [8]. So it is now known that there exists an STS(v) that contains no Pasch configurations provided v is admissible and $v \neq 7, 13$. Our current paper addresses the issue of what can be said about collections of triples when v is not admissible.

When v is not admissible, there is no STS(v). However, there will still be a maximum partial triple system, or optimal packing with triples, of order v. In the current paper we determine the anti-Pasch result for such systems. A partial triple system of order v, PTS(v), is a pair $(V, \mathcal{B})$ and is defined similarly to an STS(v), except that each 2-element subset of V is required to be contained in *at most* one block. A PTS(v) $= (V, \mathcal{B})$ for which there is no set of triples $\mathcal{B}'$ with $|\mathcal{B}'| > |\mathcal{B}|$ and $\mathcal{B} \subseteq \mathcal{B}'$ is called a *maximal* partial triple system, MPTS(v). An MPTS(v) with the largest possible set of blocks is called a *maximum* maximal partial triple system, MMPTS(v). The name is generally shortened to "maximum partial triple system". Such systems are also known as *optimal* or *maximal* packings with triples, and they give rise to optimal constant weight error-correcting codes (see [2, Section VI.40]). In a sense, these systems are as close as it is possible to get to an STS(v) when v is not admissible.

Given an MMPTS(v) $= (V, \mathcal{B})$, the set of 2-element subsets of V that do not appear in any block of $\mathcal{B}$ is called the *leave* of the system (see [2, page 553]). For $v \equiv 1$ or 3 (mod 6) an MMPTS(v) is an STS(v) and the leave is empty. For $v \equiv 0$ or 2 (mod 6) an MMPTS(v) corresponds to an STS($v + 1$) in which one point has been deleted. In these cases the leave comprises $v/2$ disjoint pairs. The more interesting case is $v \equiv 5$ (mod 6), and then it can be shown that the leave of an MMPTS(v) is a set of four pairs $\{\{a, b\}, \{b, c\}, \{c, d\}, \{d, a\}\}$, which may be represented as a 4-cycle (a, b, c, d). For $v \equiv 4$ (mod 6) an MMPTS(v) corresponds to an

MMPTS($v+1$) from which a point of its leave has been deleted. Thus an MMPTS(v) with $v \equiv 4 \pmod 6$ has a leave comprising three intersecting pairs $\{a, b\}, \{b, c\}, \{b, e\}$ and a further $(v-4)/2$ disjoint pairs covering the remaining points.

We will denote an STS(v) that contains no Pasch configurations as an APSTS(v) (anti-Pasch). Earlier papers often used the notation QFSTS(v) (quadrilateral-free) for the same property. Similarly an APMMPTS(v) denotes an anti-Pasch MMPTS(v). Anti-Pasch designs have a practical application to the construction of codes for various purposes such as erasure codes for disk arrays and regular low-density parity-check codes, see [3, 11, 15, 16] and [5, page 224].

Given an APSTS($v+1$) for $v \equiv 0$ or $2 \pmod 6$, the deletion of any point yields an APMMPTS(v). So an APMMPTS(v) exists for any $v \equiv 0$ or $2 \pmod 6$ apart possibly for $v = 6$ or 12. Up to isomorphism there is one STS(7) and two STS(13)s [14], and deletion of any single point in each case does not destroy all the Pasch configurations, so there is no APMMPTS(6) and no APMMPTS(12). Given an APMMPTS($v+1$) for $v \equiv 4$ (mod 6), the deletion of any point of its leave yields an APMMPTS(v). We will prove that an APMMPTS(v) exists for all $v \equiv 5 \pmod 6$ apart from $v = 11$, and it immediately follows that an APMMPTS(v) exists for all $v \equiv 4 \pmod 6$ apart possibly from $v = 10$. Up to isomorphism there are two MMPTS(11)s [4], and deletion of any single point of the leave in each case does not destroy all the Pasch configurations, so there is no APMMPTS(10). Hence the following result will be established.

Theorem 1.1 *There exists an anti-Pasch optimal packing with triples on v points, i.e. an APMMPTS(v), for all v except for the values $v = 6, 7, 10, 11,$ 12 and 13.*

An MMPTS(v) for $v \equiv 5 \pmod 6$ is said to be of *quintuple type* if the leave is (a, b, c, d) and the system has intersecting blocks $\{a, c, e\}$ and $\{b, d, e\}$. If these two blocks are removed from such a system and replaced by the quintuple $\{a, b, c, d, e\}$, the resulting system is a pairwise balanced design of order v having one block of size 5 and all remaining blocks of size 3. Such a design is denoted by PBD($v, \{3, 5^*\}$) and its blocks have the property that each pair of points is contained in exactly one block. The results given below produce APMMPTS(v)s of quintuple type for all $v \equiv 5 \pmod 6$ with $v \neq 11$. Furthermore, one of the two MMPTS(11)s is of quintuple type, and the associated PBD($11, \{3, 5^*\}$) has no Pasch configurations. So we also establish the following result.

Theorem 1.2 *There exists an anti-Pasch PBD($v, \{3, 5^*\}$) for all $v \equiv 5$ (mod 6).*

In the next section, two constructions are presented. These enable us to prove that for $v \equiv 5 \pmod 6$ with $v \neq 11$ there exists an APMMPTS(v). The first construction produces an APMMPTS(v) for $v = 18s + 5$ or $v = 18s - 1$ with $s \geq 3$ from three anti-Pasch Steiner triple systems. The second construction produces an APMMPTS(v) for $v = 18s + 11$ with $s \geq 4$ from three APMMPTS($6s$+5)s satisfying certain conditions. Starting with a small number of APMMPTS(v)s found by computer searches, the two constructions can be used together recursively to establish the general result given in Theorem 1.1.

2 Constructions

Our constructions depend on the cycle structure of STS(v) and MMPTS(v) designs. For such a design $(V, \mathcal{B})$, define the double neighbourhood of $x, y \in V$ (with $x \neq y$) as

$$N(x, y) = \{\{z, w\} : \{x, z, w\} \in \mathcal{B} \text{ or } \{y, z, w\} \in \mathcal{B}, \text{ and } \{z, w\} \cap \{x, y\} = \emptyset\}.$$

A double neighbourhood $N(x, y)$ can be represented as a graph $G(x, y)$ by taking the pairs of $N(x, y)$ as edges. In the case of an STS(v) the graph $G(x, y)$ is 2-regular and so it is the union of simple cycles, each of even length at least four. We refer to these as the cycles on the pair $\{x, y\}$, or as the $\{x, y\}$ cycles. In the case of an MMPTS(v) with $v \equiv 5 \pmod 6$, if the pair $\{x, y\}$ lies in the leave, so that the leave has the form (x, y, z, w), then the points z and w have degree one in $G(x, y)$, and therefore the graph contains a path with end points z and w, which we refer to as the path on the pair $\{x, y\}$, or as the $\{x, y\}$ path. If this path has length $v - 3$ (i.e. it has $v - 2$ vertices) then there will be no cycles on $\{x, y\}$, but if its length is less than $v - 3$, there will also be cycles on $\{x, y\}$. In all cases, if there is a cycle of length four then the corresponding four blocks form a Pasch configuration, and so an APSTS(v) or an APMMPTS(v) cannot give rise to a cycle of length four on a pair of points $\{x, y\}$.

For a positive integer n denote the set $\{0, 1, \ldots, n-1\}$ by N. If $a, b \in N$, define the *difference* $d = |a - b| \pmod n$ to be the minimum of $(a - b) \pmod n$ and $(b - a) \pmod n$, so that $d \in \{0, 1, \ldots, \lfloor \frac{n}{2} \rfloor\}$. Now suppose that $\mathcal{S} = (V, \mathcal{B})$ is an STS($n + 2$) or an MMPTS($n + 2$) on the point set $V = \{A, B\} \cup N$. If $\{A, a, b\} \in \mathcal{B}$ with $a, b \in N$ then we say that A has an associated difference $d = |a - b| \pmod n$ in $\mathcal{S}$ and that d is a difference associated with A. The set of all differences associated with A in $\mathcal{S}$ is denoted by D^A. Note that a block $\{A, B, x\}$ does not generate a difference. The set of all differences associated with B in $\mathcal{S}$ is defined in a similar fashion and is denoted by D^B.

We will need to combine three STS(n+2)s or three MMPTS(n+2)s. For n a positive integer and for $i = 0, 1, 2$, we will denote the set $\{0_i, 1_i, \ldots, (n-1)_i\}$ by N_i. Now suppose that for $i = 0, 1, 2$, $\mathcal{S}_i = (V_i, \mathcal{B}_i)$ is an STS($n+2$) or an MMPTS($n+2$), where $V_i = \{A, B\} \cup N_i$. Then the sets of associated differences D_i^A and D_i^B are formed as described above as subsets of N (not N_i), so that $d \in D_i^A$ if and only if there exists a block $\{A, a_i, b_i\} \in \mathcal{B}_i$ such that $|a-b| \equiv d \pmod{n}$. If $D_i^A \cap D_j^A = \emptyset$ and $D_i^B \cap D_j^B = \emptyset$ for $i, j = 0, 1, 2$, with $i \neq j$, then we say that $\mathcal{S}_0, \mathcal{S}_1, \mathcal{S}_2$ have *different differences* with respect to $\{A, B\}$.

We are now in a position to describe our two constructions. We will often write triples or pairs without set brackets {} or commas when no confusion is likely to arise.

Construction 1. Suppose that for $i = 0, 1, 2$, $\mathcal{S}_i = (V_i, \mathcal{B}_i)$ is an APSTS($n+2$) (so $n \equiv 1$ or $5 \pmod 6$) on the point set $V_i = \{A, B\} \cup N_i$, with $AB0_i \in \mathcal{B}_i$. Suppose also that $\mathcal{S}_0, \mathcal{S}_1, \mathcal{S}_2$ have different differences with respect to $\{A, B\}$. Then an APMMPTS($3n+2$), say $\mathcal{S}$, can be formed on the point set $V = \{A, B\} \cup N_0 \cup N_1 \cup N_2$ with block set $\mathcal{B}$ containing the following triples:

- *Horizontal blocks*: All triples from $\mathcal{B}_0 \cup \mathcal{B}_1 \cup \mathcal{B}_2$, except for the three triples $AB0_i$, $i = 0, 1, 2$.
- *Vertical blocks*: All triples $x_0 y_1 z_2$ where $x_0 \in N_0$, $y_1 \in N_1$, $z_2 \in N_2$ and $x + y + z \equiv 0 \pmod{n}$, except for the triple $0_0 0_1 0_2$.
- *Mixed blocks*: The two triples $A0_0 0_1$ and $B0_0 0_2$.

The points and triples from $\mathcal{S}_i$ will be said to be at *level* i, so that A and B are common to all three levels. Note that there are no blocks of $\mathcal{B}$ containing the pairs $AB, A0_2, B0_1$ and $0_1 0_2$.

We start by proving that $\mathcal{S}$ is an MMPTS($3n+2$). Clearly the pairs covered by the horizontal and vertical blocks are all distinct. Each of the six pairs appearing in the mixed blocks lies in one of the deleted triples $AB0_0$, $AB0_1$, $AB0_2$ or $0_0 0_1 0_2$. So the blocks of $\mathcal{B}$ do not contain a repeated pair. The total number of blocks in $\mathcal{B}$ is

$$3\left(\frac{(n+2)(n+1)}{6} - 1\right) + (n^2 - 1) + 2 = \frac{3n^2 + 3n - 2}{2},$$

which is the number of blocks in an MMPTS($3n+2$). Hence $\mathcal{S}$ is an MMPTS($3n+2$) with leave $(A, 0_2, 0_1, B)$. In fact the design is of quintuple type since the blocks containing the pairs $A0_1$ and $B0_2$ have a common third point, namely 0_0. It remains to prove that $\mathcal{S}$ is anti-Pasch, and to do this we consider two cases.

Case (a) Consider the possibility of a Pasch configuration P that does not contain either of the mixed blocks.

If P were formed from four distinct vertical blocks, it would have the form $P = \{x_0y_1z_2, x_0u_1v_2, w_0y_1v_2, w_0u_1z_2\}$ where $x+y+z \equiv 0$, $x+u+v \equiv 0$, $w+y+v \equiv 0$ and $w+u+z \equiv 0 \pmod{n}$. But since n is odd, these four equivalences give $x = w, y = u, z = v$, a contradiction. So P must contain a horizontal block.

If P contains two horizontal blocks from the same level then five of the six points of P, and hence all six of the points of P lie at that level, contradicting the fact that each $\mathcal{S}_i$ is anti-Pasch. The remaining possibilities are that P contains just one horizontal block, or that P has two (or three) horizontal blocks from different levels.

If P has just one horizontal block then this cannot contain A or B since all (non-mixed) blocks containing these points are horizontal and there would then have to be two such blocks in P. So if the sole horizontal block is at level 0 then P must contain blocks of the form $x_0y_0z_0$ and $x_0u_1v_2$. Without loss of generality, P then has blocks $y_0u_1w_2$ and $z_0v_2w_2$. But there is no horizontal or vertical block of this latter type with one point at level 0 and two points at level 2. A similar argument applies if the sole horizontal block is at level 1 or at level 2. Hence P cannot contain just one horizontal block.

Finally in Case (a) suppose that P contains two horizontal blocks from different levels. Since the two horizontal blocks must intersect, they must contain A or B. Assume first that they both contain A. If these blocks are at levels 0 and 1, they have the form Ax_0y_0 and Az_1w_1. Then, without loss of generality, P must contain two vertical blocks $x_0z_1u_2$ and $y_0w_1u_2$, where $x+z+u \equiv 0$ and $y+w+u \equiv 0 \pmod{n}$. Hence the differences $|x-y|$ and $|z-w|$ are equivalent modulo n. But $\mathcal{S}_0$ and $\mathcal{S}_1$ have different differences, so this is not possible. A similar argument applies if the two horizontal blocks are at levels 0 and 2, or at levels 1 and 2, or if A is replaced by B.

Case (b) Consider the possibility of a Pasch configuration P that contains one of the mixed blocks. There are six subcases.

1. P contains $A0_00_1$ and Ax_0y_0. Without loss of generality the other two blocks are x_00_0Z and y_00_1Z. The fourth block gives $Z \neq B$, so $Z = z_2$, contradicting the third block since there are no blocks other than $B0_00_2$ with two points at level 0 and one at level 2.

2. P contains $A0_00_1$ and Ax_1y_1. Without loss of generality the other two blocks are x_10_0Z and y_10_1Z. The third block gives $Z \neq B$, so $Z = z_2$, contradicting the fourth block since there are no blocks with two points at level 1 and one at level 2.

3. P contains $A0_00_1$ and Ax_2y_2. Without loss of generality the other two blocks are x_20_0Z and y_20_1Z. The fourth block gives $Z \neq B$, so $Z = z_0$, contradicting the third block since there are no blocks other than $B0_00_2$ with two points at level 0 and one at level 2.

4. P contains $B0_00_2$ and Bx_0y_0. Without loss of generality the other two blocks are x_00_0Z and y_00_2Z. The fourth block gives $Z \neq A$, so $Z = z_1$, contradicting the third block since there are no blocks other than $A0_00_1$ with two points at level 0 and one at level 1.

5. P contains $B0_00_2$ and Bx_1y_1. Without loss of generality the other two blocks are x_10_0Z and y_10_2Z. The fourth block gives $Z \neq A$, so $Z = z_0$, contradicting the third block since there are no blocks other than $A0_00_1$ with two points at level 0 and one at level 1.

6. P contains $B0_00_2$ and Bx_2y_2. Without loss of generality the other two blocks are x_20_0Z and y_20_2Z. The third block gives $Z \neq A$, so $Z = z_1$, contradicting the fourth block since there are no blocks with two points at level 2 and one at level 1.

It follows from the argument given in Cases (a) and (b) that the design $\mathcal{S}$ produced by Construction 1 cannot contain a Pasch configuration, and so it is an APMMPTS($3n + 2$). □

In order for Construction 1 to be of any use, it is necessary to prove that there is a ready supply of APSTS($n + 2$) systems $\mathcal{S}_0, \mathcal{S}_1, \mathcal{S}_2$ having different differences. We now show that this is the case.

Given a APSTS($n + 2$), any two points of the system determine cycles C_i of even lengths ℓ_i through these two points. Because the system has no Pasch configurations, ℓ_i cannot equal 4, so $\ell_i \geq 6$. Furthermore, $\sum_i \ell_i = n-1$. We show how to label the points of such a design with $A, B, 0, 1, \ldots, n-1$ in such a way that one block is $AB0$, all blocks Axy have $|x - y| = 1$ and all blocks Bxy have $|x-y| = 3$ or 5. These will be absolute differences and not just modulo n. We start by choosing two points arbitrarily and labelling them as A and B. Then label as 0 the third point in the block containing A and B. Now consider any cycle C on the pair $\{A, B\}$. As a convention we record the cycle starting with two points lying in a block with A. Suppose that C has length ℓ. How to label the cycle depends on whether $\ell \equiv 0$ or 2 (mod 4). In each case, Table 1 gives the first four possibilities and a general formula.

In every case, a block Axy has $|x-y| = 1$ and a block Bxy has $|x-y| = 3$ or 5. For subsequent purposes we observe that these differences are absolute and not just modulo n. Having labelled the first cycle C_1 (with length ℓ_1) in this way, choose another cycle C_2 of length ℓ_2, and label it in a similar fashion but add ℓ_1 to all the labels. For a third cycle $\ell_1 + \ell_2$ is added to the labels, and so on until all the cycles, and hence all the points of the system, are labelled.

$\ell \equiv 2 \pmod 4$	
6-cycle:	$(1, 2, 5, 6, 3, 4)$
10-cycle:	$(1, 2, 5, 6, 9, 10, 7, 8, 3, 4)$
14-cycle:	$(1, 2, 5, 6, 9, 10, 13, 14, 11, 12, 7, 8, 3, 4)$
18-cycle:	$(1, 2, 5, 6, 9, 10, 13, 14, 17, 18, 15, 16, 11, 12, 7, 8, 3, 4)$
ℓ-cycle:	(pairs $1 + 4j, 2 + 4j$ for $0 \leq j \leq j^*$, followed by pairs $3 + 4(j^* - j), 4 + 4(j^* - j)$ for $1 \leq j \leq j^*$), where $j^* = (\ell - 2)/4$.
$\ell \equiv 0 \pmod 4$	
8-cycle:	$(1, 2, 7, 8, 5, 6, 3, 4)$
12-cycle:	$(1, 2, 7, 8, 11, 12, 9, 10, 5, 6, 3, 4)$
16-cycle:	$(1, 2, 7, 8, 11, 12, 15, 16, 13, 14, 9, 10, 5, 6, 3, 4)$
20-cycle:	$(1, 2, 7, 8, 11, 12, 15, 16, 19, 20, 17, 18, 13, 14, 9, 10, 5, 6, 3, 4)$
ℓ-cycle:	($1, 2$, followed by pairs $7 + 4j, 8 + 4j$ for $0 \leq j \leq j^*$, followed by pairs $5 + 4(j^* - j), 6 + 4(j^* - j)$ for $0 \leq j \leq j^*$, followed by $3, 4$), where $j^* = (\ell - 8)/4$.

Table 1. Labelling an ℓ-cycle.

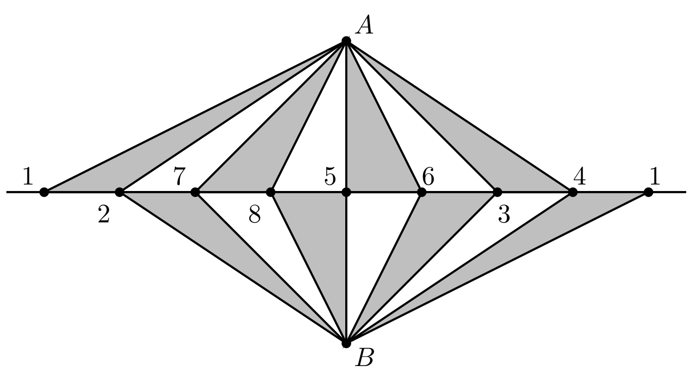

Figure 1. The case of an 8-cycle.

Example 1. As an example, in the 8-cycle case the blocks with A and B are

$$A12,\ B27,\ A78,\ B85,\ A56,\ B63,\ A34,\ B41.$$

Figure 1 shows this situation. □

We will define a *generic* labelling of an APSTS($n+2$) to be a labelling of its points by A, B and the elements of N with the following properties.

(i) One block is labelled $AB0$,

(ii) every block labelled Axy with $x, y \in N$ has $|x-y| = 1$ (absolute value, not just modulo n),

(iii) every block labelled Bxy with $x, y \in N$ has $|x-y| = 3$ or 5 (absolute values, not just modulo n),

(iv) each $\{A, B\}$ cycle is labelled with a subset of consecutive integers from N.

We have just shown that every APSTS($n+2$) has a generic labelling.

Using a generic labelling, any APSTS($n+2$) can be represented on the point set $\{A, B, 0_0, 1_0, \ldots, (n-1)_0\}$, with a block $AB0_0$, $D_0^A = \{1\}$ and $D_0^B \subseteq \{3, 5\}$. Let $\mathcal{S}_0$ denote such a system. By reversing the roles of A and B in a generic labelling, we can represent any APSTS($n+2$) on the point set $\{A, B, 0_1, 1_1, \ldots, (n-1)_1\}$ with a block $AB0_1$, $D_1^A \subseteq \{3, 5\}$ and $D_1^B = \{1\}$. Let $\mathcal{S}_1$ denote such a system. By applying the mapping $x \to 2x \pmod{n}$ (with A and B fixed) to a generic labelling we can represent any APSTS($n+2$) on the point set $\{A, B, 0_2, 1_2, \ldots, (n-1)_2\}$ with a block $AB0_2$, $D_2^A = \{2\}$ and $D_2^B \subseteq \{6, 10\}$. Let $\mathcal{S}_2$ denote such a system. If $n \geq 17$, the differences $1, 2, 3, 5$ are distinct modulo n so $D_i^A \cap D_j^A = \emptyset$ for $i \neq j$, and the differences $1, 3, 5, 6, 10$ are distinct modulo n so $D_i^B \cap D_j^B = \emptyset$ for $i \neq j$. Hence $\mathcal{S}_0, \mathcal{S}_1$ and $\mathcal{S}_2$ have different differences, and these systems may be used in Construction 1.

Example 2. As an example for $n = 17$, there is a APSTS(19) with a pair of points giving a 10-cycle and a 6-cycle. This system can be used to generate $\mathcal{S}_i$ for each $i = 0, 1, 2$. In $\mathcal{S}_0$ take the cycles as

$(1_0, 2_0, 5_0, 6_0, 9_0, 10_0, 7_0, 8_0, 3_0, 4_0)$ and $(11_0, 12_0, 15_0, 16_0, 13_0, 14_0)$.

In $\mathcal{S}_1$ take the cycles as

$(2_1, 5_1, 6_1, 9_1, 10_1, 7_1, 8_1, 3_1, 4_1, 1_1)$ and $(12_1, 15_1, 16_1, 13_1, 14_1, 11_1)$.

In $\mathcal{S}_2$ take the cycles as

$(2_2, 4_2, 10_2, 12_2, 1_2, 3_2, 14_2, 16_2, 6_2, 8_2)$ and $(5_2, 7_2, 13_2, 15_2, 9_2, 11_2)$. □

Since there exists a APSTS(v) for every admissible $v \geq 19$, we may now state the following result.

Theorem 2.1 *If* $v \equiv 5$ *or* $17 \pmod{18}$ *and* $v \geq 17$ *then there exists an anti-Pasch MMPTS*(v) *of quintuple type.*

Proof. For $v = 17, 23, 35$ and 41 the result follows from a computer search and the designs are given in [6]. For $s \geq 3$, Construction 1 may be employed using an APSTS$(6s+1)$ to give an APMMPTS$(18s-1)$, and using an APSTS$(6s+3)$ to give an APMMPTS$(18s+5)$. □

Remark. Construction 1 cannot be used to give an APMMPTS(v) for $v = 17, 23, 35$ or 41 because there is no APSTS(7), no APSTS(13), and the procedure described for obtaining three APSTS(v)s having different differences requires $v \geq 19$, and it therefore fails for $v = 9, 15$.

Construction 2. Suppose that for $i = 0, 1, 2$, $\mathcal{S}_i = (V_i, \mathcal{B}_i)$ is an APMMPTS$(n+2)$ with $n \equiv 3 \pmod 6$ on the point set $V_i = \{A, B\} \cup N_i$, such that $\mathcal{S}_0, \mathcal{S}_1$ and $\mathcal{S}_2$ have different differences with respect to $\{A, B\}$, and their leaves are respectively (A, a_0, b_0, B), (A, c_1, d_1, B) and (A, e_2, f_2, B). Suppose also that $c - d \equiv f - e \pmod n$. Let δ denote the difference $|c-d|$ modulo n, and let g be such that $g + c + e \equiv g + d + f \equiv 0 \pmod n$. Assume that

(i) $\delta \notin D_0^A \cup D_0^B$, and

(ii) there are no blocks $g_0 x_0 (x+\delta)_0 \in \mathcal{B}_0$ (where $x + \delta$ is taken modulo n).

Then an APMMPTS$(3n+2)$, say $\mathcal{S}$, can be formed on the point set $V = \{A, B\} \cup N_0 \cup N_1 \cup N_2$ with block set $\mathcal{B}$ containing the following triples:

- *Horizontal blocks*: All triples from $\mathcal{B}_0 \cup \mathcal{B}_1 \cup \mathcal{B}_2$.
- *Vertical blocks*: All triples $x_0 y_1 z_2$ where $x_0 \in N_0$, $y_1 \in N_1$, $z_2 \in N_2$ and $x + y + z \equiv 0 \pmod n$, except for the two triples $g_0 c_1 e_2$ and $g_0 d_1 f_2$.
- *Mixed blocks*: The four triples $A c_1 e_2$, $B d_1 f_2$, $g_0 c_1 d_1$ and $g_0 e_2 f_2$.

The points and triples from $\mathcal{S}_i$ will be said to be at *level* i, so that A and B are common to all three levels. Note that there are no blocks of $\mathcal{B}$ containing the pairs AB, Aa_0, Bb_0 and $a_0 b_0$.

We start by proving that $\mathcal{S}$ is an MMPTS$(3n+2)$. Clearly the pairs covered by the horizontal and vertical blocks are all distinct. Each of the 12 pairs appearing in the mixed blocks either lies in the leave of $\mathcal{S}_1$ or $\mathcal{S}_2$, or in one of the deleted triples $g_0 c_1 e_2$, $g_0 d_1 f_2$. So the blocks of $\mathcal{B}$ do not contain a repeated pair. The total number of blocks in $\mathcal{B}$ is

$$3\left(\frac{(n+2)(n+1)-8}{6}\right) + (n^2 - 2) + 4 = \frac{3n^2 + 3n - 2}{2},$$

which is the number of blocks in an MMPTS($3n+2$). Hence $\mathcal{S}$ is an MMPTS($3n+2$) with leave (A, a_0, b_0, B). It remains to prove that $\mathcal{S}$ is anti-Pasch, and to do this we consider two cases.

Case (a) Consider the possibility of a Pasch configuration P that does not contain any of the four mixed blocks.

The argument that eliminates this possibility is identical with that given for the corresponding case in Construction 1.

Case (b) Consider the possibility of a Pasch configuration P that contains one of the mixed blocks. Altogether there are thirteen subcases and to save endlessly writing "(mod n)" we state once that all the congruences are taken modulo n. Consider first a possible Pasch configuration P containing the point A. There are three possibilities for the two blocks containing A.

1. Ac_1e_2, Ax_0y_0. Without loss of generality the other two blocks are x_0c_1Z and y_0e_2Z. The third block shows that $Z \neq B$. Examining the third block gives two possibilities. If $Z = d_1$ then $x_0 = g_0$ and the fourth block is $y_0d_1e_2$, which gives $y+d+e \equiv 0$, so that $|x-y| \equiv |g+d+e| \equiv |e-f| \equiv \delta$. Thus the block Ax_0y_0 gives $\delta \in D_0^A$, a contradiction. The other possibility is that $Z = z_2$, in which case the fourth block is $y_0e_2z_2$, which implies that $y_0 = g_0$ and $z_2 = f_2$. Then the third block gives $x+c+z \equiv 0$, so that $|x-y| \equiv |g+c+f| \equiv |c-d| \equiv \delta$. Thus also in this case the block Ax_0y_0 gives $\delta \in D_0^A$, a contradiction.

2. Ac_1e_2, Ax_1y_1. Without loss of generality the other two blocks are x_1c_1Z and y_1e_2Z. The fourth block shows that $Z \neq B$ and so $Z = z_0$, where $y+e+z \equiv 0$. Then the third block gives $z_0 = g_0$, so $y+e+g \equiv 0$ and hence $y_1 = c_1$, a contradiction.

3. Ac_1e_2, Ax_2y_2. Without loss of generality the other two blocks are x_2c_1Z and y_2e_2Z. The third block shows that $Z \neq B$ and so $Z = z_0$, where $x+c+z \equiv 0$. Then the fourth block gives $z_0 = g_0$, so $x+c+g \equiv 0$ and hence $x_2 = e_2$, a contradiction.

Consider next a possible Pasch configuration P containing the point B. There are three possibilities for the two blocks containing B.

4. Bd_1f_2, Bx_0y_0. Without loss of generality the other two blocks are x_0d_1Z and y_0f_2Z. The third block shows that $Z \neq A$. Examining the third block gives two possibilities. If $Z = c_1$ then $x_0 = g_0$ and the fourth block is $y_0c_1f_2$, which gives $y+c+f \equiv 0$, so that $|x-y| \equiv |g+f+c| \equiv |c-d| \equiv \delta$. Thus the block Bx_0y_0 gives $\delta \in D_0^B$, a contradiction. The other possibility is that $Z = z_2$, in which case the

fourth block is $y_0 f_2 z_2$, which implies that $y_0 = g_0$ and $z_2 = e_2$. Then the third block gives $x + d + z \equiv 0$, so that $|x - y| \equiv |g + d + e| \equiv |e - f| \equiv \delta$. Thus also in this case the block Bx_0y_0 gives $\delta \in D_0^B$, a contradiction.

5. Bd_1f_2, Bx_1y_1. Without loss of generality the other two blocks are x_1d_1Z and y_1f_2Z. The fourth block shows that $Z \neq A$ and so $Z = z_0$, where $y+f+z \equiv 0$. Then the third block gives $z_0 = g_0$, so $y+f+g \equiv 0$ and hence $y_1 = d_1$, a contradiction.

6. Bd_1f_2, Bx_2y_2. Without loss of generality the other two blocks are x_2d_1Z and y_2f_2Z. The third block shows that $Z \neq A$ and so $Z = z_0$, where $x + d + z \equiv 0$. Then the fourth block gives $z_0 = g_0$, so $x + d + g \equiv 0$ and hence $x_2 = f_2$, a contradiction.

It follows from the arguments above that there can be no Pasch configurations involving either of the two mixed blocks containing A and B if condition (i) is satisfied. So we next examine the possibility of a Pasch configuration containing one of the other two mixed blocks. First we deal with the case of a possible Pasch configuration P containing both of these mixed blocks.

7. Suppose that P has blocks $g_0c_1d_1, g_0e_2f_2$. The pair c_1e_2 lies in a triple with A and the pair d_1f_2 lies in a triple with B. So suppose the other two blocks of P are c_1f_2Z and d_1e_2Z. Then $Z = z_0$ and $c+f+z \equiv d+e+z \equiv 0$. This gives $c-d \equiv e-f$, but we already have $c - d \equiv f - e$, and since n is odd these give $c = d$, a contradiction.

Next consider a possible Pasch configuration P containing just the one mixed block $g_0c_1d_1$. Without loss of generality there are three possibilities.

8. Suppose that P has blocks $g_0c_1d_1, g_0x_0y_0, x_0c_1Z, y_0d_1Z$. The third block shows that $Z \neq A, B$ and since $Z \neq d_1$, we must have $Z = z_2$. So the third and fourth blocks give $x \equiv -(c + z)$ and $y \equiv -(d + z)$. Now the second block may be written as $g_0(-(c + z))_0(-(d + z))_0$ which has the form $g_0w_0(w+\delta)_0$ because $|c+z-(d+z)| = |c-d| \equiv \delta$. But this contradicts the supposition (ii) that there are no such blocks.

9. Suppose that P has blocks $g_0c_1d_1, g_0x_1y_2, x_1c_1Z, y_2d_1Z$. The fourth block shows that $Z \neq A, B$ and so $Z = z_0$. But then the third block gives $z_0 = g_0$ and $x_1 = d_1$, a contradiction.

10. Suppose that P has blocks $g_0c_1d_1, g_0x_1y_2, x_1d_1Z, y_2c_1Z$. The fourth block shows that $Z \neq A, B$ and so $Z = z_0$. But then the third block gives $z_0 = g_0$ and $x_1 = c_1$, a contradiction.

Finally, consider a possible Pasch configuration P containing just the one mixed block $g_0e_2f_2$. Without loss of generality there are three possibilities.

11. Suppose that P has blocks $g_0e_2f_2, g_0x_0y_0, x_0e_2Z, y_0f_2Z$. The third block shows that $Z \neq A, B$ and since $Z \neq f_2$, we must have $Z = z_1$. So the third and fourth blocks give $x \equiv -(e+z)$ and $y \equiv -(f+z)$ Now the second block may be written as $g_0(-(e+z))_0(-(f+z))_0$ which has the form $g_0w_0(w+\delta)_0$ because $|e+z-(f+z)| = |e-f| \equiv \delta$. But this contradicts the supposition (ii) that there are no such blocks.

12. Suppose that P has blocks $g_0e_2f_2, g_0x_1y_2, x_1e_2Z, y_2f_2Z$. The third block shows that $Z \neq A, B$ and so $Z = z_0$. But then the fourth block gives $z_0 = g_0$ and $y_2 = e_2$, a contradiction.

13. Suppose that P has blocks $g_0e_2f_2, g_0x_1y_2, x_1f_2Z, y_2e_2Z$. The third block shows that $Z \neq A, B$ and so $Z = z_0$. But then the fourth block gives $z_0 = g_0$ and $y_2 = f_2$, a contradiction.

It follows from Cases (a) and (b) that the design $\mathcal{S}$ produced by Construction 2 cannot contain a Pasch configuration, and so it is an APMMPTS$(3n+2)$. □

In order for Construction 2 to be of any use, it is necessary to prove that there is a ready supply of APMMPTS$(n+2)$ systems $\mathcal{S}_0, \mathcal{S}_1, \mathcal{S}_2$ having the appropriate properties. We will show that this is the case, but we will do so in stages.

Given an APMMPTS$(n+2)$ with $n \equiv 3 \pmod 6$ and with the leave (X, α, β, Y), the pair $\{X, Y\}$ determines a path of even length p (i.e. having an odd number of vertices) having the form $(\beta, \ldots, \alpha)$. If $p < n-1$ there will also be cycles C_i having even lengths ℓ_i on the pair $\{X, Y\}$. Because the system has no Pasch configurations, ℓ_i cannot equal 4, so $\ell_i \geq 6$. Furthermore, $p + \sum_i \ell_i = n-1$. Note that the design is of quintuple type if and only if the path is of length $p = 2$, in which case the path is (β, γ, α) where γ is the point forming blocks $X\beta\gamma$ and $Y\alpha\gamma$.

Our earlier definition of a generic labelling of an APSTS$(n+2)$ can be modified for APMMPTS$(n+2)$s with $n \equiv 3 \pmod 6$ and leave (X, α, β, Y). A *generic* labelling of such an APMMPTS$(n+2)$ is a labelling of its points by A, B and the elements of N with the following properties.

(i) The leave is labelled $(A, 0, 1, B)$,

(ii) every block labelled Axy with $x, y \in N$ has $|x-y| = 1$ (absolute value, not just modulo n),

(iii) every block labelled Bxy with $x, y \in N$ has $|x-y| = 2$, 3 or 5 (absolute values, not just modulo n),

(iv) the $\{A, B\}$ path and each $\{A, B\}$ cycle (if any) is labelled with a subset of consecutive integers from N.

Every APMMPTS$(n+2)$ with $n \equiv 3 \pmod 6$ has a generic labelling. To see this consider first the path. How to label this depends whether its length p has $p \equiv 0$ or $2 \pmod 4$. In each case Table 2 gives the first four cases and a general formula. As a convention we record the path starting with two points lying in a block with A, so the first point represents β and the last point represents α.

$p \equiv 2 \pmod 4$

$p = 2$:	$(1, 2, 0)$
$p = 6$:	$(1, 2, 5, 6, 4, 3, 0)$
$p = 10$:	$(1, 2, 5, 6, 9, 10, 8, 7, 4, 3, 0)$
$p = 14$:	$(1, 2, 5, 6, 9, 10, 13, 14, 12, 11, 8, 7, 4, 3, 0)$
$p \equiv 2$:	(pairs $1 + 4j, 2 + 4j$ for $0 \le j \le j^*$, followed by pairs $4(j^* - j), 4(j^* - j) - 1$ for $0 \le j \le j^* - 1$, followed by 0), where $j^* = (p - 2)/4$.

$p \equiv 0 \pmod 4$

$p = 4$:	$(1, 2, 4, 3, 0)$
$p = 8$:	$(1, 2, 5, 6, 8, 7, 4, 3, 0)$
$p = 12$:	$(1, 2, 5, 6, 9, 10, 12, 11, 8, 7, 4, 3, 0)$
$p = 16$:	$(1, 2, 5, 6, 9, 10, 13, 14, 16, 15, 12, 11, 8, 7, 4, 3, 0)$
$p \equiv 0$:	(pairs $1 + 4j, 2 + 4j$ for $0 \le j \le j^*$, followed by pairs $4 + 4(j^* - j), 3 + 4(j^* - j)$ for $0 \le j \le j^*$, followed by 0), where $j^* = (p - 4)/4$.

Table 2. Labelling the path.

Every block Axy in the path has $|x - y| = 1$, and every block Bxy in the path has $|x - y| = 2$ or 3. The cycles (if any) can then be labelled as described previously in connection with Construction 1, adding an appropriate constant to all the labels for each cycle so that every block Axy in every cycle has $|x - y| = 1$ and every block Bxy in every cycle has $|x - y| = 3$ or 5. Thus in the complete labelling, the leave is labelled $(A, 0, 1, B)$ and every block Axy has $|x - y| = 1$ and every block Bxy has $|x - y| = 2$, 3 or 5, so that $D^A = \{1\}$ and $D^B \subseteq \{2, 3, 5\}$. Furthermore the path and each cycle is labelled with a subset of consecutive integers from N.

Now suppose that we have an APMMPTS$(n+2)$ with $n \equiv 3 \pmod 6$ that is generically labelled and has the additional property, which we call *property* G, that there is some point $g \neq A, B$ for which there are no blocks

$\{g, x, x+4\}$. Form a copy $\mathcal{S}_0$ of this system by appending the suffix 0 to all the points other than A and B. Form $\mathcal{S}_1$ by applying the mapping $x \to 4x \pmod n$ (with A and B fixed) to any generically labelled APMMPTS(n+2) and then appending the suffix 1 to all the points other than A and B. The leave of $\mathcal{S}_1$ is then $(A, 0_1, 4_1, B)$ so, in the notation of Construction 2, $c = 0$ and $d = 4$. Form $\mathcal{S}_2$ from any generically labelled APMMPTS($n+2$) as follows. First exchange A and B; this can be achieved by taking the first two points in the path and in each cycle to be in a block with B, the second and third points with A, and so on. Then apply the mapping $x \to 4x + \lambda \pmod n$ (with A and B fixed and λ a constant specified below), finally append the suffix 2 to all the points other than A and B. The constant λ is chosen as follows. The leave of $\mathcal{S}_2$ is $(A, (4+\lambda)_2, \lambda_2, B)$ so, in the notation of Construction 2, $e \equiv 4 + \lambda$ and $f \equiv \lambda$. We wish to have blocks $g_0 c_1 e_2$ and $g_0 d_1 f_2$, so we require $g + c + e \equiv g + d + f \equiv 0$, and this can be achieved by setting $\lambda \equiv -(g+4) \pmod n$. This choice gives $e \equiv -g$ and $f \equiv -(g+4)$. Again in the notation of Construction 2, $\delta = 4$. As regards the sets of differences associated with A and B, we have $D_0^A = \{1\}$, $D_0^B \subseteq \{2,3,5\}$, $D_1^A = \{4\}$, $D_1^B \subseteq \{8,12,20\}$, $D_2^A \subseteq \{8,12,20\}$, $D_2^B = \{4\}$. If $n \geq 27$, the differences $1,2,3,4,5,8,12,20$ are all distinct modulo n, and then the systems $\mathcal{S}_0, \mathcal{S}_1$ and $\mathcal{S}_2$ have different differences with respect to $\{A, B\}$, $\delta \notin D_0^A \cup D_0^B$, and there are no blocks $g_0 x_0 (x+\delta)_0$. So the three labelled systems are suitable for use in Construction 2.

There still remains the difficulty of finding an APMMPTS($n+2$) with $n \equiv 3 \pmod 6$ that can be generically labelled in such a way that it has property G. For small values of $n \geq 27$ these are easy to find using a computer and a hill-climbing algorithm. We will prove that any system *produced* by Construction 1 has this property, and that an additional condition on the ingredients will ensure that some systems produced by Construction 2 also have this property. In preparation for this we give the following rather trivial but useful lemma.

Lemma 2.1 *Suppose that $\mathcal{S}$ is a generically labelled APMMPTS($n+2$) with $n \equiv 3 \pmod 6$. Then there exists a point $g \neq A, B$ for which there are no blocks $\{g, x, x+4\}$, if and only if there exists a point h for which there are two blocks $\{h, z, z+4\}$ and $\{h, w, w+4\}$ ($z \neq w$ and arithmetic modulo n).*

Proof. The point set of $\mathcal{S}$ has n points other than A and B, and consequently n pairs $\{x, x+4\}$. None of these pairs appear in a triple with A or B because $4 \notin D^A \cup D^B$. If two such pairs appear with a point h then $h \neq A, B$, and so there must exist a point $g \neq A, B$ for which there are no blocks $\{g, x, x+4\}$. Conversely if there is a point $g \neq A, B$ for which there are no blocks $\{g, x, x+4\}$, then some point h must occur in blocks with two distinct pairs $\{z, z+4\}$ and $\{w, w+4\}$. □

Define *property H* for a generically labelled APMMPTS($n+2$) by the requirement that there should exist a point h for which there are two blocks $\{h, z, z+4\}$ and $\{h, w, w+4\}$ ($z \neq w$ and arithmetic modulo n). The lemma shows that properties G and H are equivalent. The lemma is useful because property H is easier to establish than property G. It is advantageous to consider a stronger requirement than property H in which the points z and w are well away from $n-1$. So we define *property H** for a generically labelled APMMPTS($n+2$) by the requirement that there should exist a point h for which there are two blocks $\{h, z, z+4\}$ and $\{h, w, w+4\}$ with $z \neq w$ and $0 \leq z, w \leq n-5$. Property H* ensures that the differences of four between z and $z+4$, and between w and $w+4$, are absolute, and not just modulo n. Clearly property H* implies property G.

We now explain how an APMMPTS($3n+2$) produced by Construction 1 can be generically labelled by $A, B, 0, 1, \ldots, 3n-1$ in such a way that it has property H*, i.e. there is a point h for which there are two blocks $\{h, z, z+4\}$ and $\{h, w, w+4\}$ with $z \neq w$ and $0 \leq z, w \leq 3n-5$. Any system produced by Construction 1 comes with the (non-generic) labelling inherited from that construction, but here we specify a relabelling. Obviously no relabelling will create Pasch configurations. But the relabelling will result in the vertical blocks $x_0 y_1 z_2$ no longer satisfying the condition $x + y + z \equiv 0 \pmod{n}$. A system produced by Construction 1 will be of quintuple type. The relabelling is done in stages.

The original labelling of the constructed system has the $\{A, B\}$ path $(0_1, 0_0, 0_2)$, and has the $\{A, B\}$ cycles of system $\mathcal{S}_0$ labelled with $1_0, 2_0, \ldots, (n-1)_0$. Note that the differences given by the original labellings are absolute (not modulo n). In the relabelling, the points A and B retain their original labels. The path is relabelled as $(1, 2, 0)$ so that 0_1 is relabelled 1, 0_0 is relabelled 2, and 0_2 is relabelled 0. For the $\{A, B\}$ cycles of system $\mathcal{S}_0$, drop the suffix 0 and add 2 to all the labels, so that the cycles (and hence the points $1_0, 2_0, \ldots (n-1)_0$) are now labelled by $3, 4, \ldots, n+1$. This relabelling does not affect the differences since these are absolute differences. So, up to this point in the argument, the differences generated by the path and relabelled cycles on A are all 1, and those on B are all 2, 3 or 5. Now pick two distinct points from $3, 4, \ldots, n+1$ with an absolute difference of 4, say z and $z+4$. These lie in a block with some other point already relabelled (not A or B), say h.

Next, the $\{A, B\}$ cycles (and hence the points) of system $\mathcal{S}_1$, originally labelled with $1_1, 2_1, \ldots, (n-1)_1$, are relabelled with $n+2, n+3, \ldots, 2n$, but we carry out the relabelling by the generic method described above so that the differences on A are all 1 and those on B are all 3 or 5 (again, absolute values).

Now consider the triple containing the pair of points already relabelled as h and $2n$. This triple is one of the original vertical triples from the construction. So there is some point from $\mathcal{S}_2$, say u_2, that forms a block with the points now relabelled as h and $2n$. Assume for the moment that $u_2 \neq 0_2$ so that u_2 lies in one of the original $\{A, B\}$ cycles of $\mathcal{S}_2$. These cycles (and hence the points) of $\mathcal{S}_2$, originally labelled with $1_2, 2_2, \ldots,$ $(n-1)_2$, will be relabelled with $2n+1, 2n+2, \ldots, 3n-1$, but we again carry out the relabelling by the generic method described above so that the differences on A are all 1 and those on B are all 3 or 5 (again, absolute values). Moreover, it is possible to arrange that the point u_2 is relabelled as $2n+4$. To achieve this, take the cycle containing u_2 as the first cycle to be relabelled. Let ℓ denote the length of this cycle, so that it is relabelled with $2n+1, 2n+2, \ldots, 2n+\ell$. Thus one of the points in this cycle is relabelled as $2n+4$, and we can ensure that this point is u_2 by taking an appropriate equivalent form for listing the cycle (see Example 3 below for an example of what we mean by this). Such a relabelling results in a block $\{h, w, w+4\}$ with $w = 2n$.

In the exceptional case when $u_2 = 0_2$, there will be a point v_2 that forms a block with the points now relabelled as h and $2n-1$, and $v_2 \neq 0_2$. So in this case we relabel the points $1_2, 2_2, \ldots, (n-1)_2$, as before, but now arrange for v_2 to receive the label $2n+3$. This results in a block $\{h, w, w+4\}$ with $w = 2n-1$.

The constructed system is now labelled with a generic labelling, with $D^A = \{1\}$ and $D^B \subseteq \{2, 3, 5\}$, and we have two distinct blocks of the form $\{h, z, z+4\}$ and $\{h, w, w+4\}$. Consequently there must exist a point $g \neq A, B$ for which there is no block labelled $\{g, x, x+4\}$. This system, as now labelled, is suitable for use in an application of Construction 2 (with n now replaced by $3n$). It is also useful to note that the two blocks $\{h, z, z+4\}$ and $\{h, w, w+4\}$ have $0 \leq z, w \leq 3n-5$, i.e the system has property H*. As explained below, this will enable a system produced by such an application of Construction 2 to be itself used in a reapplication of Construction 2.

Example 3. To clarify the relabelling of system $\mathcal{S}_2$ described above, suppose that u_2 lies in the 6-cycle $(\ldots, x_2, u_2, y_2, \ldots)$. Note that the "standard form" for a 6-cycle given earlier is $(1, 2, 5, 6, 3, 4)$.

If the block containing the pair $\{x_2, u_2\}$ is Ax_2u_2, take the cycle in the equivalent form $(y_2, \ldots, x_2, u_2)$. Then relabel:

$$y_2 \to 2n+1, \ldots, x_2 \to 2n+3, u_2 \to 2n+4,$$

so that the cycle is now $(2n+1, 2n+2, 2n+5, 2n+6, 2n+3, 2n+4)$. Thus u_2 is relabelled as $2n+4$ and the differences on A are 1 and those on B are 3 or 5 (actually 3 in this example).

If the block containing the pair $\{x_2, u_2\}$ is Bx_2u_2, first reverse the cycle to get the equivalent form $(\ldots, y_2, u_2, x_2, \ldots)$, and then write it as

$(x_2, \dots, y_2, u_2)$. Then relabel:

$$x_2 \to 2n+1, \dots, y_2 \to 2n+3, u_2 \to 2n+4,$$

so that the cycle is again $(2n+1, 2n+2, 2n+5, 2n+6, 2n+3, 2n+4)$. Thus u_2 is relabelled as $2n+4$ and the differences on A are 1 and those on B are 3 or 5 (actually 3 in this example). □

It should be clear from this example that it is possible to relabel the points $1_2, 2_2, \dots, (n-1)_2$ of system $\mathcal{S}_2$ with the labels $2n+1, 2n+2, \dots, 3n-1$ in such a way that the $\{A, B\}$ cycles are labelled with subsets of consecutive integers, the differences on A are 1, those on B are 3 or 5, and any specified point x_2 can be relabelled with any specified label $y \in \{2n+1, 2n+2, \dots, 3n-1\}$.

Next we show that some systems produced by Construction 2 can also be given a generic relabelling with property H*. So suppose that $\mathcal{S}$ has been produced using the constituent APMMPTS($n+2$) designs $\mathcal{S}_0$, $\mathcal{S}_1$ and $\mathcal{S}_2$, and suppose also that $\mathcal{S}_0$ has property H*. Observe that the $\{A, B\}$ path and cycles of $\mathcal{S}_0$ are retained in $\mathcal{S}$. Hence $\mathcal{S}$, with its original labelling contains two blocks $\{h_0, z_0, (z+4)_0\}$ and $\{h_0, w_0, (w+4)_0\}$ with $0 \le z, w \le n-5$. In the relabelling, A and B retain their original labels and each point x_0 of $\mathcal{S}_0$ is relabelled as x. We then have two blocks of $\mathcal{S}$ labelled as $\{h, z, z+4\}$ and $\{h, w, w+4\}$, and $0 \le z, w \le 3n-5$. The relabelling does not affect the differences on the $\{A, B\}$ path and cycles since these are absolute differences. So, up to this point in the argument, the differences generated by the path and relabelled cycles on A are all 1, and those on B are all 2, 3 or 5.

We now relabel all the remaining cycles by the generic method described above, using labels $n, n+1, \dots, 3n-1$, so that the differences on A are all 1 and the differences on B are all 3 or 5 (again, absolute values). The result is that $\mathcal{S}$ is generically labelled and has property H*. Consequently the relabelled system $\mathcal{S}$ is suitable for use in a reapplication of Construction 2, taking the role of the new system $\mathcal{S}_0$; the system resulting from such a reapplication may again be reused, and so on.

We can summarise the results of this section as follows.

- Every APMMPTS($3n+2$) ($n \equiv 1$ or 5 (mod 6), $n \ge 17$) produced by Construction 1 is of quintuple type and it can be generically labelled in such a way that it has property H*.
- If there is an APMMPTS($n+2$) ($n \equiv 3$ (mod 6), $n \ge 27$) that can be generically labelled in such a way that it has property H*, then Construction 2 can be applied to yield an APMMPTS($3n+2$) that can also be generically labelled in such a way that it has property H*.

If the APMMPTS($n+2$) is of quintuple type, then so is the resulting APMMPTS($3n+2$).

3 Recursion

We now show how the results of the previous section can be used to establish that there exists an APMMPTS($6s+5$) for all $s \neq 1$. The case $s=0$ is trivial. Computer searches based on a hill-climbing algorithm deal with the values $2 \leq s \leq 7$ and $s=10$. By this method, we have constructed APMMPTS($6s+5$) designs of quintuple type

(a) for $s=2$ and 3, and

(b) for $s=4,5,6,7$ and 10, and these designs can be generically labelled to have property H*.

These designs are available from the authors [6]. The cases $s=8$ and 9 are resolved by Construction 1, using APSTS(19)s and APSTS(21)s respectively. As noted in the previous section, the resulting APMMPTS(53) and APMMPTS(59) designs are of quintuple type and can be generically labelled to have property H*.

Having established the result for $4 \leq s \leq 10$, the following lemma provides the inductive step.

Lemma 3.1 *If there exists an APMMPTS*($6s+5$) *of quintuple type that can be generically labelled to have property H* for each value s satisfying* $4 \leq s \leq M$, *where* $M \geq 10$, *then there exists an APMMPTS*($6(M+1)+5$) *of quintuple type that can be generically labelled to have property H*.*

Proof. The proof fall into two cases: if $M \equiv 0 \pmod 3$ then Construction 2 is applied inductively, otherwise Construction 1 is applied directly.

So suppose first that $M=3t$. Then $t \geq 4$ so, by the hypothesis, there is an APMMPTS($6t+5$) of quintuple type that can be generically labelled to have property H*. Applying Construction 2 with $n=6t+3$ gives an APMMPTS($6(M+1)+5$) of quintuple type that can be generically labelled to have property H*.

Next suppose that $M=3t+1$. Then $t \geq 3$ and there exists an APSTS($6t+7$). Applying Construction 1 with $n=6t+5$ gives an APMMPTS($6(M+1)+5$) of quintuple type that can be generically labelled to have property H*. Similarly if $M=3t+2$ then $t \geq 3$ and there exists an APSTS($6t+9$). Applying Construction 1 with $n=6t+7$ gives an APMMPTS($6(M+1)+5$) of quintuple type that can be generically labelled to have property H*. □

Corollary 3.1 follows immediately from the Lemma 3.1.

Corollary 3.1 *There exists an APMMPTS$(6s + 5)$ of quintuple type for all $s \neq 1$.*

As explained in the Introduction, this result establishes the truth of Theorems 1.1 and 1.2.

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