Configurations in bowtie systems

M. J. Grannell, T. S. Griggs
Department of Mathematics and Statistics
The Open University
Walton Hall
Milton Keynes MK7 6AA
UNITED KINGDOM
m.j.grannell@open.ac.uk
t.s.griggs@open.ac.uk

G. Lo Faro, A. Tripodi*
Dipartimento di Matematica e Informatica
Universitá di Messina
Contrada Papardo
Viale Ferdinando Stagno d'Alcontres, 31
98166 Messina
ITALIA
lofaro@unime.it
atripodi@unime.it

Abstract

There are ten configurations of two bowties that can arise in a bowtie system. We determine a basis for configurations of two bowties in both balanced and general bowtie systems. We also determine the avoidance spectrum for the three most compact configurations of two bowties.

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1 Introduction

Let $X = (V, E)$ be the graph with vertex set $V = \{x, a, b, c, d\}$ and edge set $E = \{xa, xb, xc, xd, ab, cd\}$. Such a graph is called a *bowtie* and will be represented

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throughout this paper by the notation $ab - x - cd$. The vertex x is called the *centre* of the bowtie and the other vertices are called *endpoints*. A decomposition of the complete graph K_n into subgraphs isomorphic to X is called a *bowtie system of order* n and denoted by BTS(n). An elementary counting argument shows that a necessary condition for the existence of a BTS(n) is $n \equiv 1$ or $9 \pmod{12}$. In a BTS(n), if every vertex of the complete graph K_n occurs the same number of times as the centre of a bowtie, then the bowtie system is said to be *balanced*, otherwise the system is said to be *unbalanced*. A necessary condition for the existence of a balanced BTS(n) is $n \equiv 1 \pmod{12}$.

It is easy to see that, given a BTS(n), by regarding each of the two triangles of every bowtie as separate entities, we have a Steiner triple system STS(n). We call this the *associated* Steiner triple system of the bowtie system. Conversely, if $n \equiv 1$ or $9 \pmod{12}$, it is also true that the triangles of every STS(n) can be amalgamated to form bowties. This is a consequence of the fact that the block intersection graph of every Steiner triple system is Hamiltonian, see for example [2, section 13.6]. If $n \equiv 1 \pmod{12}$, there exists a cyclic STS(n), see also [2, section 7.2], and this system will have an even number of full orbits. It is then immediate that we can amalgamate triangles from pairs of orbits to form a balanced BTS(n). Hence the necessary conditions for both BTS(n) and balanced BTS(n) given above are also sufficient.

In this paper we begin the study of configurations in bowtie systems. Here a configuration is a small collection of bowties which may occur in a bowtie system. We will mainly be concerned with configurations consisting of two bowties and the first task is to identify all non-isomorphic possibilities. This is straightforward. A configuration of two bowties in a BTS(n) yields a configuration of four triples in the associated Steiner triple system. These have been enumerated, [4], [2, section 13.1]; there are 16 in all. So we only have to examine them in order to determine which can be obtained from two bowties. There are ten such configurations of two bowties which are illustrated in Figure 1. In this figure each triangle of a bowtie is represented by a path on three vertices and, in each case, one bowtie is represented by solid lines and the second by dashed lines. The intersection of two solid lines (or two dashed lines) is the centre of the bowtie and the other four points are the end points. The ten configurations are each labelled $\hat{C}_i$ for some value of i, $1 \leq i \leq 16$, to reflect the fact that the bowtie with that label gives the configuration C_i in the standard listing of configurations of four triples in Steiner triple systems as given in [4], [2, section 13.1]. We also give the pairs of bowties using $ab - x - cd$ notation.

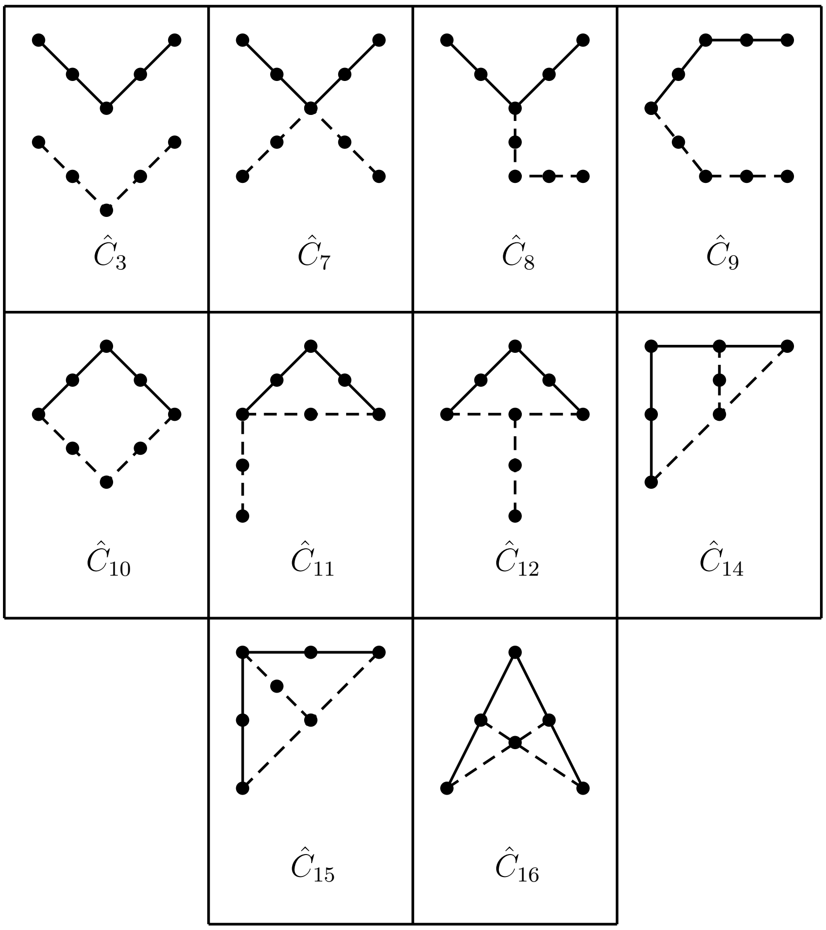

Figure 1. Configurations of two bowties

$\hat{C}_3$	$12-0-34$	$67-5-89$
$\hat{C}_7$	$12-0-34$	$56-0-78$
$\hat{C}_8$	$12-0-34$	$06-5-78$
$\hat{C}_9$	$12-0-34$	$16-5-78$
$\hat{C}_{10}$	$12-0-34$	$16-5-37$
$\hat{C}_{11}$	$12-0-34$	$35-1-67$
$\hat{C}_{12}$	$12-0-34$	$13-5-67$
$\hat{C}_{14}$	$12-0-34$	$13-5-26$
$\hat{C}_{15}$	$12-0-34$	$06-5-13$
$\hat{C}_{16}$	$12-0-34$	$13-5-24$

The structure of the rest of this paper is as follows. In the next section we derive equations which connect the number of occurrences of bowtie configurations in a BTS(n). We then specialize the results to balanced bowtie systems. In the same vein as with configurations in Steiner triple systems, bowtie configurations may be *constant*, meaning that for every given admissible value of n, they occur the same number of times in every bowtie system of that order, or *variable*. In fact no bowtie configuration consisting of two bowties is constant for general BTS(n) and only one, $\hat{C}_7$, is constant for balanced BTS(n).

Given $m \geq 1$, denote by $\mathcal{B}_m$ the set of all configurations consisting of m bowties. A set $\mathcal{S} = \{S_1, S_2, \ldots, S_k\} \subseteq \cup_{i=1}^{m} \mathcal{B}_i$ is called a *spanning set* for $\mathcal{B}_m$ if, for each n, the number of occurrences of any $X \in \mathcal{B}_m$ in a BTS(n) can be expressed as a linear combination of the number of occurrences of the configurations S_j, $1 \leq j \leq k$, where the coefficients are functions of n. A *basis* for $\mathcal{B}_m$ is a spanning set of minimal cardinality. Analogous definitions apply to balanced bowtie systems and to Steiner triple systems. For configurations of four triples in Steiner triple systems, any constant configuration, together with any variable configuration (such as the Pasch configuration C_{16}) forms a basis. We will show that a basis for $\mathcal{B}_2$ (that is, for configurations of two bowties) consists of one constant and six variable configurations for general bowtie systems, and one constant and five variable configurations for balanced bowtie systems. In both cases the constant configuration is that comprising a single bowtie; we denote this configuration by $\hat{C}_0$. The number of occurrences of $\hat{C}_0$ in a BTS(n) is $n(n-1)/12$.

In the final section, we discuss avoidance results relating to the three most compact configurations of two bowties, namely $\hat{C}_{14}, \hat{C}_{15}$ and $\hat{C}_{16}$ which have respectively 7, 7 and 6 vertices.

2 Formulae

There are four equations which connect the number of occurrences of the various configurations of two bowties and we now derive these. In order to do this we will consider pairs (B, T) where B is a bowtie of the BTS(n) and T is one of the triangles of a second bowtie. Let the number of occurrences of the configuration $\hat{C}_i$ be c_i.

1. Suppose that one of the points of the triangle T coincides with the centre of the bowtie B. The number of such pairs (B, T) is the number of bowties in the system $(n(n-1)/12)$ multiplied by the number of triangles from other bowties which contain the centre of B $((n-5)/2)$. Such pairs occur four times in configuration $\hat{C}_7$ and once in each of configurations $\hat{C}_8$ $\hat{C}_{11}$ $\hat{C}_{15}$. Hence we have the equation

$$4c_7 + c_8 + c_{11} + c_{15} = n(n-1)(n-5)/24. \qquad (1)$$

2. Suppose that two of the points of the triangle T coincide with endpoints of the bowtie B. These endpoints will of necessity be in different triangles of the bowtie. The number of such pairs (B, T) is therefore $(n(n-1)/12) \times 4$ and they occur once in configurations $\hat{C}_{11}$ and $\hat{C}_{12}$, twice in $\hat{C}_{14}$, three times in $\hat{C}_{15}$ and

four times in $\hat{C}_{16}$. Hence we have the equation

$$c_{11} + c_{12} + 2c_{14} + 3c_{15} + 4c_{16} = n(n-1)/3. \tag{2}$$

3. Suppose that the triangle T intersects the bowtie B in just one endpoint. The number of these pairs is $(n(n-1)/12) \times 4 \times ((n-1)/2 - 3)$ and they occur twice in configurations $\hat{C}_8$, $\hat{C}_9$, $\hat{C}_{11}$, $\hat{C}_{12}$ and $\hat{C}_{14}$ and four times in configuration $\hat{C}_{10}$. Hence we have the equation

$$c_8 + c_9 + 2c_{10} + c_{11} + c_{12} + c_{14} = n(n-1)(n-7)/12. \tag{3}$$

4. Finally, suppose that T is disjoint from B. The number of such pairs is slightly more involved to count. The total number of triangles in a bowtie system BTS(n), i.e. the number of blocks in the associated Steiner triple systems STS(n), is $n(n-1)/6$. Of these, $(n-1)/2$ will contain the centre of the bowtie B and $4(n-3)/2 - 4$ will contain either one or two endpoints. Hence the total number of disjoint triangles is $(n(n-1)/6 - (n-1)/2 - (4(n-3)/2 - 4)) = (n-7)(n-9)/6$. The pair of a bowtie and a disjoint triangle occurs four times in configuration $\hat{C}_3$, once in configurations $\hat{C}_8$ and $\hat{C}_{12}$ and twice in configuration $\hat{C}_9$. This gives the equation

$$4c_3 + c_8 + 2c_9 + c_{12} = n(n-1)(n-7)(n-9)/72. \tag{4}$$

So for $\mathcal{B}_2$ (the ten configurations of two bowties) in a BTS(n) there is a spanning set of cardinality 7 consisting of the constant configuration $\hat{C}_0$ and $10 - 4 = 6$ of the configurations in $\mathcal{B}_2$. As we will see later, 7 is also the cardinality of a basis. However, it remains to decide what is the most natural basis. For four-line configurations in Steiner triple systems, where a basis consists of a constant configuration and a variable configuration, the natural variable candidate is the Pasch configuration on 6 points. This is the configuration having the least number of points and initially it seems appropriate to adopt a similar approach here. Configuration $\hat{C}_{16}$ has 6 points, configurations $\hat{C}_{14}$ and $\hat{C}_{15}$ have 7 points, and configurations $\hat{C}_{10}$, $\hat{C}_{11}$ and $\hat{C}_{12}$ have 8 points; six configurations in all. But equation (2) gives a relationship between the number of occurrences of five of these six configurations. Therefore in place of $\hat{C}_{11}$ we choose $\hat{C}_7$. The reason for replacing $\hat{C}_{11}$ is that the other five configurations have the property that at least two of the endpoints of each bowtie are also endpoints of the other. The reason for introducing $\hat{C}_7$ is that, as we show below, this is a constant configuration in balanced bowtie systems. The equations now become

$$\begin{aligned}
c_3 &= n(n-1)(n-12)(n-13)/288 - c_7 + c_{10} - c_{15} - c_{16} \\
c_8 &= n(n-1)(n-13)/24 - 4c_7 + c_{12} + 2c_{14} + 2c_{15} + 4c_{16} \\
c_9 &= n(n-1)(n-9)/24 + 4c_7 - 2c_{10} - c_{12} - c_{14} + c_{15} \\
c_{11} &= n(n-1)/3 - c_{12} - 2c_{14} - 3c_{15} - 4c_{16}
\end{aligned}$$

Turning now to balanced bowtie systems, another equation may be deduced. Each point will be the centre of $(n-1)/12$ bowties and hence

$$c_7 = n \times (n-1)/12 \times ((n-1)/12 - 1)/2 = n(n-1)(n-13)/288. \tag{5}$$

Thus for balanced bowtie systems, by substituting the formula for c_7 from equation (5) the equations become

$$\begin{aligned}
c_3 &= n(n-1)(n-13)^2/288 + c_{10} - c_{15} - c_{16} \\
c_8 &= n(n-1)(n-13)/36 + c_{12} + 2c_{14} + 2c_{15} + 4c_{16} \\
c_9 &= n(n-1)(n-10)/18 - 2c_{10} - c_{12} - c_{14} + c_{15} \\
c_{11} &= n(n-1)/3 - c_{12} - 2c_{14} - 3c_{15} - 4c_{16}
\end{aligned}$$

It remains to show that the above spanning sets are indeed bases as claimed. In order to do this we will use the following six examples of balanced BTS(25)s. Each is cyclic and consists of two orbits. Also given is the vector $(c_3, c_7, c_8, c_9, c_{10}, c_{11}, c_{12}, c_{14}, c_{15}, c_{16})$ specifying the number of occurrences of each configuration.

A. $\{(i+1),(i+3)-i-(i+7),(i+21),$
$(i+5),(i+13)-i-(i+6),(i+15): \ i \in \mathbb{Z}_{25}\}.$
$(325, 25, 225, 425, 25, 175, 25, 0, 0, 0).$

B. $\{(i+1),(i+3)-i-(i+4),(i+11),$
$(i+5),(i+13)-i-(i+6),(i+15): \ i \in \mathbb{Z}_{25}\}.$
$(325, 25, 275, 400, 25, 125, 25, 25, 0, 0).$

C. $\{(i+1),(i+3)-i-(i+6),(i+15),$
$(i+5),(i+13)-i-(i+4),(i+11): \ i \in \mathbb{Z}_{25}\}.$
$(275, 25, 275, 500, 0, 100, 25, 0, 25, 0).$

D. $\{(i+2),(i+24)-i-(i+14),(i+18),$
$(i+5),(i+13)-i-(i+6),(i+15): \ i \in \mathbb{Z}_{25}\}.$
$(300, 25, 250, 475, 25, 125, 0, 0, 25, 0).$

E. $\{(i+1),(i+3)-i-(i+7),(i+11),$
$(i+12),(i+20)-i-(i+10),(i+16): \ i \in \mathbb{Z}_{25}\}.$
$(325, 25, 350, 350, 50, 50, 50, 0, 0, 25).$

F. $\{(i+1),(i+3)-i-(i+5),(i+13),$
$(i+4),(i+11)-i-(i+6),(i+15): \ i \in \mathbb{Z}_{25}\}.$
$(350, 25, 250, 350, 50, 150, 50, 0, 0, 0).$

From this data we construct the following 6×6 matrix M whose rows represent the configurations $\hat{C}_0$, $\hat{C}_{10}$, $\hat{C}_{12}$, $\hat{C}_{14}$, $\hat{C}_{15}$ and $\hat{C}_{16}$, and columns represent the BTS(25)s A, B, C, D, E and F. The entries give the number of occurrences of each configuration in each system, divided by 25 for simplicity.

$$M = \begin{bmatrix} 2 & 2 & 2 & 2 & 2 & 2 \\ 1 & 1 & 0 & 1 & 2 & 2 \\ 1 & 1 & 1 & 0 & 2 & 2 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

If any one of $c_0, c_{10}, c_{12}, c_{14}, c_{15}$ or c_{16} was expressible as a linear combination of the others, then there would be a non-zero solution $\mathbf{x}$ of the equation $\mathbf{x}M = 0$. But the matrix M is clearly of full rank and so no such solution exists. Hence the configurations form a basis for $\mathcal{B}_2$ in balanced bowtie systems.

In the general case, where unbalanced systems are permitted, we add to the existing six systems a seventh system G which is unbalanced. A suitable system G may be obtained from E by replacing the two bowties $1, 3 - 0 - 7, 11$ and $3, 22 - 12 - 7, 24$ by $0, 11 - 7 - 12, 24$ and $0, 1 - 3 - 12, 22$. The configuration vector for system G is $(326, 27, 333, 357, 51, 59, 49, 0, 0, 23)$. A 7×7 matrix M^* may be then constructed in a similar way to M, adding an additional row (representing $\hat{C}_7$) and an additional column (representing system G). It is easy to show that this matrix has full rank. Hence the configurations $\hat{C}_0$, $\hat{C}_7$, $\hat{C}_{10}$, $\hat{C}_{12}$, $\hat{C}_{14}$, $\hat{C}_{15}$ and $\hat{C}_{16}$ form a basis for general bowtie systems.

Remark We have chosen to use cyclic systems for the above arguments because the bowties can be described succinctly. Up to isomorphism there are only three cyclic BTS(13)s [3], so although it might seem more natural to use systems of order 13, the greater availability of cyclic BTS(25)s makes these systems an easier choice.

3 Avoidance

In this section we show that the three most compact configurations, $\hat{C}_{14}, \hat{C}_{15}$ and $\hat{C}_{16}$, are avoidable in both balanced and unbalanced BTS(n) for every admissible $n > 9$; in fact we will show that they are simultaneously avoidable. Our proof is divided into four cases each of which uses a specific 3-GDD (group divisible design). The four 3-GDDs employed are known to exist (see [1, section IV 4.1]), and the notation employed to describe them is also explained there. We also use a balanced bowtie decomposition of the complete tripartite graph $K_{6,6,6}$ and a bowtie decomposition of $K_{4,4,4}$. These are given in the Appendix, along with some specific bowtie systems of small orders. We explain Case 1 in detail; the other cases are similar.

Case 1. Take a 3-GDD of type 2^t, where $t = 3s$ or $3s + 1$ and $s \geq 1$. Inflate each point to six points and add a single new point ∞. On each inflated group of 12 points augmented with the ∞ point, place a BTS(13). Replace each of the original blocks of the GDD by a copy of $K_{6,6,6}$, and decompose each of these copies into bowties. The resulting collection of bowties forms a BTS($12t + 1$). If all the BTS(13)s used are balanced and all the $K_{6,6,6}$ decompositions are balanced, then the resulting BTS($12t+1$) is balanced. If any one of the BTS(13)s is unbalanced (but all other decompositions are balanced) then the resulting BTS($12t+1$) is unbalanced. Thus this construction facilitates the production of both balanced and unbalanced systems of orders $36s+1$ and $36s+13$ for $s \geq 1$.

Consider next the possible intersections of bowties produced by this construction. Two bowties from the same inflated group will intersect as determined by the corresponding BTS(13). Two bowties from different inflated groups will intersect in at most one point, namely ∞. A bowtie from an inflated group and a bowtie from a $K_{6,6,6}$ will intersect in at most two points. Two bowties from the same $K_{6,6,6}$ will intersect as determined by the corresponding decomposition. Finally, two bowties from different $K_{6,6,6}$s will intersect in at most two points. Since $\hat{C}_{14}, \hat{C}_{15}$ and $\hat{C}_{16}$ each have their two bowties intersecting in three or four points, avoidance of these configurations in the BTS$(12t+1)$ may be controlled by using appropriate BTS(13)s and $K_{6,6,6}$ decompositions. If the BTS(13)s and the $K_{6,6,6}$ decompositions avoid these configurations, then the resulting BTS$(12t+1)$ will also avoid them. In the Appendix we give a $K_{6,6,6}$ decomposition into bowties with $c_{14} = c_{15} = c_{16} = 0$; this decomposition is also balanced (each of the 18 vertices appears precisely once as a bowtie centre). Also in the Appendix we give both balanced and unbalanced BTS(13)s having $c_{14} = c_{15} = c_{16} = 0$. Thus we can assert the existence of both balanced and unbalanced BTS(n)s avoiding $\hat{C}_{14}, \hat{C}_{15}$ and $\hat{C}_{16}$ for $n \equiv 1, 13 \pmod{36}$ $(n > 1)$.

Case 2. Take a 3-GDD of type 2^t4^1, where $t = 3s$ and $s \geq 1$. Inflate by a factor six and proceed as in Case 1, except that one inflated group (augmented with ∞) requires the placement of a BTS(25). For this we use the balanced BTS(25) with $c_{14} = c_{15} = c_{16} = 0$ given in the Appendix. We are thus able to produce both balanced and unbalanced BTS(n)s avoiding $\hat{C}_{14}, \hat{C}_{15}$ and $\hat{C}_{16}$ for $n \equiv 25 \pmod{36}$ with $n \geq 61$. The Appendix also contains an unbalanced BTS(25) with $c_{14} = c_{15} = c_{16} = 0$. So, combining with the results of Case 1, for each $n \equiv 1 \pmod{12}$ $(n > 1)$, we have both a balanced and an unbalanced BTS(n) avoiding $\hat{C}_{14}, \hat{C}_{15}$ and $\hat{C}_{16}$.

Case 3. This case and Case 4 deal with $n \equiv 9 \pmod{12}$, and the systems produced are necessarily unbalanced. For Case 3 take a 3-GDD of type 3^t5^1, where $t = 2s$ and $s \geq 2$. Inflate by a factor four, proceeding as before but employing a bowtie decomposition of $K_{4,4,4}$ and an (unbalanced) BTS(21) on the inflated group of size 21 (including ∞). Both these designs are given in the Appendix and they each have $c_{14} = c_{15} = c_{16} = 0$. We are thereby able to produce BTS(n)s avoiding $\hat{C}_{14}, \hat{C}_{15}$ and $\hat{C}_{16}$ for $n \equiv 21 \pmod{24}$ with $n \geq 69$. The case of $n = 45$ is also given in the Appendix.

Case 4. Take a 3-GDD of type 6^t8^1, where $t \geq 3$. Inflate by a factor four and proceed as in Case 3, but placing BTS(25)s and a BTS(33) as appropriate on the inflated groups augmented with ∞. A BTS(33) with $c_{14} = c_{15} = c_{16} = 0$ is given in the Appendix. We are thereby able to produce BTS(n)s avoiding $\hat{C}_{14}, \hat{C}_{15}$ and $\hat{C}_{16}$ for $n \equiv 9 \pmod{24}$ with $n \geq 105$. The cases of $n = 57$ and $n = 81$ are also given in the Appendix. So, combining with the results of Case 3, for each $n \equiv 9 \pmod{12}$ $(n > 9)$, we have a BTS(n) avoiding $\hat{C}_{14}, \hat{C}_{15}$ and $\hat{C}_{16}$.

We can now state the following result.

Theorem 3.1. *For each* $n \equiv 1 \pmod{12}$ *(*$n > 1$*), there exists both a balanced and an unbalanced BTS(n) simultaneously avoiding* $\hat{C}_{14}, \hat{C}_{15}$ *and* $\hat{C}_{16}$*. For each* $n \equiv 9 \pmod{12}$ *(*$n > 9$*), there exists a (necessarily unbalanced) BTS(n) simultaneously avoiding* $\hat{C}_{14}, \hat{C}_{15}$ *and* $\hat{C}_{16}$*.*

It remains only to consider the case $n = 9$. It was shown in [3] that, up to isomorphism, there are precisely 12 distinct bowtie systems of order 9. Since the Steiner triple system of order 9 has no Pasch (C_{16}) configuration, all 12 of these systems avoid $\hat{C}_{16}$. However, only one avoids $\hat{C}_{14}$, and none avoid $\hat{C}_{15}$. Numbering them 1 to 12 in the order in which they are listed on pages 156-7 of [3], the configuration vectors $(c_3, c_7, c_8, c_9, c_{10}, c_{11}, c_{12}, c_{14}, c_{15}, c_{16})$ are as follows.

1.	$(0,1,0,0,6,0,0,0,8,0)$	2.	$(0,1,0,0,2,4,0,4,4,0)$
3.	$(0,1,0,0,2,4,0,4,4,0)$	4.	$(0,1,0,0,2,4,0,4,4,0)$
5.	$(0,1,0,0,2,4,0,4,4,0)$	6.	$(0,1,0,0,0,6,0,6,2,0)$
7.	$(0,0,0,0,2,7,0,1,5,0)$	8.	$(0,0,0,0,2,7,0,1,5,0)$
9.	$(0,0,0,0,0,9,0,3,3,0)$	10.	$(0,0,0,0,0,9,0,3,3,0)$
11.	$(0,0,0,0,0,9,0,3,3,0)$	12.	$(0,0,0,0,0,9,0,3,3,0)$

Finally we remark that the proof of Theorem 3.1 given above may be adjusted, by choosing suitable BTS(13)s and BTS(25)s, to avoid precisely one, or precisely two, of the configurations $\hat{C}_{14}$, $\hat{C}_{15}$ and $\hat{C}_{16}$. In a future paper we hope to deal with avoidance of all the other configurations of two bowties.

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Appendix

Here we list the bowtie systems and the decompositions of $K_{4,4,4}$ and $K_{6,6,6}$ referenced in Section 3. Each is given with its configuration vector $\mathbf{v} = (c_3, c_7, c_8, c_9, c_{10}, c_{11}, c_{12}, c_{14}, c_{15}, c_{16})$, the last three entries of which are zero in each case.

- A balanced BTS(13) with point set $\mathbb{Z}_{13}$.
  Develop the starter $1, 4 - 0 - 2, 7$ by successively adding 1 modulo 13 to give 13 bowties.
  $\mathbf{v} = (0, 0, 13, 13, 0, 39, 13, 0, 0, 0)$.

- An unbalanced BTS(13) with point set $\mathbb{Z}_{13}$.
  $6, 8 - 0 - 9, 10$; $0, 4 - 1 - 2, 5$; $0, 7 - 2 - 3, 6$;
  $0, 12 - 3 - 1, 8$; $5, 8 - 4 - 6, 11$; $1, 12 - 6 - 5, 9$;
  $1, 9 - 7 - 3, 4$; $2, 4 - 9 - 8, 12$; $2, 8 - 10 - 3, 5$;
  $4, 12 - 10 - 6, 7$; $0, 5 - 11 - 3, 9$; $1, 10 - 11 - 7, 8$;
  $2, 11 - 12 - 5, 7$.
  $\mathbf{v} = (2, 2, 8, 10, 4, 36, 16, 0, 0, 0)$.

- An (unbalanced) BTS(21) with point set $\mathbb{Z}_{21}$.
  Develop the following 5 starters by successively adding 3 modulo 21 to give 35 bowties.
  $2, 5 - 0 - 9, 20$; $3, 13 - 0 - 6, 14$; $0, 7 - 1 - 4, 14$;
  $2, 8 - 1 - 6, 10$; $4, 6 - 2 - 14, 19$.
  $\mathbf{v} = (105, 14, 133, 189, 14, 91, 49, 0, 0, 0)$.

- A balanced BTS(25) with point set $\mathbb{Z}_{25}$ (system A from Section 2).
  Develop the two starters $1, 3 - 0 - 7, 21$ and $5, 13 - 0 - 6, 15$ by successively adding 1 modulo 25 to give 50 bowties.
  $\mathbf{v} = (325, 25, 225, 425, 25, 175, 25, 0, 0, 0)$.

- An (unbalanced) BTS(25).
  In the balanced BTS(25) (system A) replace the two bowties $1, 3 - 0 - 7, 21$ and $6, 14 - 1 - 7, 16$ by $0, 3 - 1 - 6, 14$ and $0, 21 - 7 - 1, 16$.
  The resulting vector is $\mathbf{v} = (325, 26, 224, 424, 26, 172, 28, 0, 0, 0)$.

- An (unbalanced) BTS(33) with point set $\mathbb{Z}_{33}$. Develop the following 8 starters by successively adding 3 modulo 33 to give 88 bowties.
  $3, 19 - 0 - 17, 20$; $9, 22 - 0 - 14, 23$; $3, 30 - 1 - 5, 7$;
  $6, 27 - 1 - 10, 23$; $9, 24 - 1 - 11, 19$; $0, 1 - 2 - 17, 24$;
  $6, 14 - 2 - 7, 28$; $16, 19 - 2 - 29, 30$.
  $\mathbf{v} = (1584, 77, 638, 1056, 121, 286, 66, 0, 0, 0)$.

- An (unbalanced) BTS(45) with point set $\mathbb{Z}_{45}$. Develop the following 11 starters by successively adding 3 modulo 45 to give 165 bowties.
  $5,44-0-22,42$; $11,38-0-19,36$; $17,32-0-33,40$;
  $3,42-1-23,43$; $9,36-1-20,37$; $12,33-1-14,25$;
  $15,30-1-34,41$; $0,1-2-23,33$; $4,43-2-21,44$;
  $10,37-2-18,38$; $16,31-2-35,39$.
  $\mathbf{v}=(7395,225,1875,3015,360,525,135,0,0,0)$.

- An (unbalanced) BTS(57) with point set $\mathbb{Z}_{57}$. Develop the following 14 starters by successively adding 3 modulo 57 to give 266 bowties.
  $5,56-0-28,54$; $9,34-0-11,50$; $14,47-0-45,52$;
  $21,40-0-23,38$; $3,54-1-29,55$; $9,48-1-10,35$;
  $12,45-1-46,53$; $15,42-1-28,44$; $21,36-1-22,41$;
  $0,1-2-29,42$; $4,55-2-27,56$; $10,49-2-11,33$;
  $13,46-2-47,51$; $22,37-2-23,39$.
  $\mathbf{v}=(22230,494,4161,6517,779,779,285,0,0,0)$

- An (unbalanced) BTS(81) with point set $\mathbb{Z}_{81}$. Develop the following 20 starters by successively adding 3 modulo 81 to give 540 bowties.
  $5,80-0-40,78$; $11,74-0-37,72$; $14,71-0-69,76$;
  $15,49-0-17,68$; $20,65-0-21,52$; $25,48-0-29,56$;
  $3,78-1-41,79$; $9,72-1-38,73$; $12,69-1-70,77$;
  $15,66-1-16,50$; $18,63-1-22,53$; $21,60-1-40,62$;
  $26,49-1-27,54$; $0,1-2-41,60$; $4,79-2-39,80$;
  $10,73-2-36,74$; $13,70-2-71,75$; $16,67-2-17,48$;
  $19,64-2-23,51$; $24,50-2-28,55$.
  $\mathbf{v}=(105435,1539,12744,22248,1404,1620,540,0,0,0)$.

- An (unbalanced) decomposition of $K_{4,4,4}$ with vertex parts $\{0,1,2,3\}$, $\{4,5,6,7\}$ and $\{8,9,10,11\}$.
  $4,8-0-5,9$; $4,9-1-6,11$; $4,10-2-6,8$;
  $5,8-3-7,10$; $0,11-7-1,8$; $2,7-9-3,6$;
  $0,6-10-1,5$; $2,5-11-3,4$.
  $\mathbf{v}=(1,0,3,3,5,13,3,0,0,0)$.

- A balanced decomposition of $K_{6,6,6}$ with vertex parts $\{0,1,2,3,4,5\}$, $\{6,7,8,9,10,11\}$ and $\{12,13,14,15,16,17\}$.
  $6,12-0-9,15$; $7,14-1-10,17$; $10,12-2-11,13$;
  $6,15-3-7,16$; $8,12-4-11,15$; $6,17-5-9,14$;
  $1,13-6-2,14$; $0,13-7-2,15$; $0,14-8-2,16$;
  $2,17-9-3,12$; $0,16-10-3,13$; $0,17-11-5,16$;
  $1,11-12-5,7$; $4,9-13-5,8$; $3,11-14-4,10$;
  $1,8-15-5,10$; $1,9-16-4,6$; $3,8-17-4,7$.
  $\mathbf{v}=(22,0,43,39,13,29,7,0,0,0)$.