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SOME RECENT RESULTS ON CYCLIC STEINER QUADRUPLE SYSTEMS - A SURVEY

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1 Introduction and Enumeration Results

The necessary and sufficient condition for the existence of a Steiner quadruple system of order v (SQS(v)), namely $v \equiv 2$ or $4 \pmod 6$, was established by Hanani [14] in 1960 and is now well known. Such a v will be called admissible. In comparison, the problem of determining those values of v for which cyclic Steiner quadruple systems exist is still unresolved. A survey of results on Steiner quadruple systems was given by Lindner and Rosa [18] in 1978. However, at that time no infinite families of cyclic Steiner quadruple systems were known. A great deal of the work on the subject has been developed recently, much of it in response to some of the questions posed in Lindner and Rosa's paper.

It is an aim of this paper to provide a unified account of this work, present some of the conjectures which have been made, and discuss some of the problems which remain. We will denote a cyclic Steiner quadruple system of order v by CSQS(v) and represent it in the usual way as the union of 4-block orbits under the action of the cyclic group $C_v = \langle i \to i+1 \pmod v \rangle$. Orbits will be called full, half or quarter respectively according as the. number of distinct 4-blocks which they contain is $v, v/2$ or $v/4$,

Firstly, enumeration results have been obtained for all admissible $v \leq 22$. In the table below $n(v)$ denotes the number of distinct CSQS(v) and $N(v)$ the number of non-isomorphic CSQS(v).

v	$n(v)$	$N(v)$	Reference
8	0	0	Guregova and Rosa [13]
10	1	1	Barrau [1]
14	0	0	Guregova and Rosa [13]
16	0	0	Guregova and Rosa [13]
20	152	29	Phelps [20]
22	210	21	Diener [6]

2 Methods of construction for CSQS(v)

The first infinite families of CSQS(v) were constructed by Phelps [19]. By exploiting the structure of the groups PGL($2, q^k$) the following theorems were obtained.

Phelps Theorem 1: If there exists an SQS($q+1$) with q a prime power then there exists a CSQS(q^2+1) containing the SQS($q+1$) as a subdesign.

Theorem 2: If there exists a CSQS($q+1$) with q a prime power then there exists a CSQS(q^k+1) for all $k > 0$.

In [2], Cho gave a doubling construction for certain CSQS(v).

Cho Theorem 1: If there exists a CSQS(v) with $v \equiv 2$ or $10 \pmod{12}$ then there exists a CSQS($2v$). Moreover if the CSQS(v) comprises n orbits then there exists at least 2^n pairwise distinct CSQS($2v$).

The basis of Cho's construction is as follows:

(a) From each full orbit in the CSQS(v), choose any block, say $\{w, x, y, z\}$, $0 \le w < x < y < z < v$. Form the orbits under C_{2v} of the blocks $\{w, x, y, z\}$, $\{w, x, y+v, z+v\}$, $\{w, y, x+v, z+v\}$ and $\{w, z, x+v, y+v\}$.

(b) From each half orbit in the CSQS(v), choose any block, say $\{w, x, w+v/2, x+v/2\}$, $0 \le w < x < w+v/2$. Form the orbits under C_{2v} of the blocks $\{w, x, w+v/2, x+v/2\}$ and $\{w, x+v/2, x+v, w+3v/2\}$.

(c) The orbits formed in (a) and (b) are all full and the CSQS($2v$) is completed by adjoining all half orbits and the quarter orbit of 4-blocks under C_{2v}.

A problem with Cho's construction is that it can not be re-applied because the CSQS($2v$) contains the quarter orbit. In [4], Colbourn and Colbourn devised a method which in certain circumstances overcomes the problem. The basic idea is to omit part of the cyclic Steiner quadruple system, including the quarter orbit, apply Cho's construction to the remainder and then complete the doubled system with another appropriate cyclic Steiner quadruple system. To formalize the discussion we need some further definitions.

From any given 4-block w, x, y, z, $0 \le w < x < y < z < v$, a cyclically ordered difference quadruple $\langle x-w, y-x, z-y, w-z+v \rangle$ may be formed which is characteristic of the orbit from which the 4-block is drawn, Similarly a cyclically ordered difference triple may be formed from a 3-block. Then, since each 4-block contains four 3-blocks, each difference quadruple gives rise to four difference triples. If $m|v$, define an m-beheaded CSQS(v), denoted by CSQS($v, -m$), as a collection of 4-block orbits with the property that any 3-block whose difference triple entries have a common factor v/m does not occur in the system but all other 3-blocks occur exactly once. We note that in this definition it is not in fact necessary for v to be admissible. However, the significance of the definition is that if there exists a CSQS($v, -m$) and a CSQS(m) then there exists a CSQS(v).

The main result in [4] is:
Colbourn and Colbourn Theorem 1: If there exists a CSQS($2v, -2m$) with $v \equiv m \pmod 2$ then there exists a CSQS($4v, -4m$).
The theorem is of great importance and has wide ranging applications which are dealt with in Section 4.

Finally in this section we announce that we have recently obtained product constructions for cyciic Steiner quadruple systems which with an appropriate parameter and minimal modification yield the constructions of Cho and of Colbourn and Colbourn.

In [12], the following results are proved:
Grannell and Griggs Theorem 1: If there exists a CSQS(u) and a CSQS(v), the latter being composed entirely of full orbits then there exists a CSQS($uv, -2u$).

Theorem 2: If there exists a CSQS(u) and a CSQS($v, -m$) with $v \equiv m$ (mod 2) and the latter being composed entirely of full orbits then there exists a CSQS($uv, -um$).

Theorem 3: If there exists a CSQS(u), a CSQS($2u$) and a CSQS(v) with $v \equiv 2$ or $10 \pmod{12}$ then there exists a CSQS(uv).

Theorem 4: If there exists a CSQS(u), a CSQS($2u$) and a CSQS($v, -m$) with $v, m \equiv 0 \pmod 2$ and the latter being composed entirely of full and half orbits then there exists a CSQS($uv, -um$).

Theorems 3 and 4 with $u = 2$ in effect give Cho's and Colbourn and Colbourn's constructions respectively.

3 R-cyclic and S-cyclic Steiner quadruple systems

CSQS(v) which admit additional automorphisms are also of interest particularly those which are stabilized by the mapping $i \to v - i$, i.e. which are also reverse Steiner quadruple systems. We will call these R-cyclic Steiner quadruple systems and denote them by RCSQS(v). An easy argument shows that no RCSQS(v) may contain a half orbit and hence that $v \equiv 2, 4, 10$ or 20 (mod 24) is a necessary condition. It has been conjectured that this is also sufficient, but little work appears to have been done on RCSQS(v).

Of more importance both from an historical point of view and because of their use with Colbourn and Colbourn's construction are symmetric cyclic or S-cyclic Steiner quadruple systems, denoted by SCSQS(v). These are defined as CSQS(v) in which each orbit contributing to the system is stabilized by the mapping $i \to v - i$. The present authors [9] and Diener [7] independently obtained the following results on the structure of SCSQS(v).

Grannell and Griggs/Diener Theorem 1: If there exists an SCSQS(v) then $v = 2n$ or $4n$ where the prime factors of n are all $\equiv 1$ or 5 (mod 12). Moreover, if m is even and $m|v$ then the SCSQS(v) contains an SCSQS(m) as a subdesign.

We conjecture that the necessary condition given in the theorem is also sufficient. Until recently, most known CSQS(v) were also S-cyclic. However, a non S-cyclic CSQS(26) is given (although with a misprint) in a paper by Fitting [8] of 1915.

Constructional techniques for SCSQS(v) have been investigated in two papers [16], [17] by Köhler. The basis for this work is as follows: For a given v, not necessarily admissible, define a graph $H^*(v)$ where the set of vertices is

$$\{\{x, y, z\} : x, y, z \in \{1, 2, \ldots, v-3\} \setminus \{v/2\},\ x, y, z \text{ unequal},\ \ x + y + z = v\}$$

and the set of edges is defined by the relation (which it is easily verified is both irreflexive and symmetric) that $\{x, y, z\}$ is joined to $\{x', y', z'\}$ if there is some ordering X, Y, Z of x, y, z such that $\{x', y', z'\} = \{X, X+Y, v-2X-Y\}$. In the case where $v \equiv 2, 4, 10$ or 20 (mod 24) careful analysis of the difference quadruples and the difference triples they contain leads to the graph $H^*(v)$ where each vertex represents two difference triples and each edge represents a difference quadruple. Hence

Köhler Theorem 1: There exists an SCSQS(v) if and only if $H^*(v)$ contains a 1-factor.
(This result is in fact slightly stronger than that given originally by Köhler [16] and is due to Diener [7]. The basic idea of the work goes back to Fitting [8].

Köhler proceeds to prove various properties of the graph $H^*(v)$. In particular it is not necessarily connected. In the case where $v = 2p$ with p a prime $\equiv 1$ or 5 (mod 12), $H^*(v)$ consists of two components. In one of these components, $H_2^*(v)$, the elements of each set which comprises a vertex are all even and in the other component, $H_1^*(v)$, at least one element (and therefore precisely two elements) of each set comprising a vertex are odd. Now it can be verified that a 1-factor of $H_1^*(v)$ may be obtained by selecting the set of difference quadruples $\langle a, b, a, v-(2a+b)\rangle$ with $a = 1, 3, \ldots, (v-8)/2$ and $b = 2, 4, \ldots, (v-2a-4)/2$. In addition the graph $H_2^*(v)$, which is also denoted by $H(v)$ in [16], can easily be seen to be isomorphic to the graph $H^*(p)$ which leads to:

Theorem 2: There exists an SCSQS($2p$) with p a prime $\equiv 1$ or $5 \pmod{12}$ if and only if $H^*(p)$ contains a 1-factor.

When $p \equiv 5 \pmod{12}$, Köhler proceeds further with the analysis. By using the group $\langle i \to ai + b \pmod{p}\rangle$ with $a, b \in GF(p),\ \ a \neq 0$, in order to reduce the number of orbits required, another graph $B(p)$ is obtained as follows:
Let $F = GF(p) \setminus \{0, 1, (p-1)/2, p-2, p-1\}$. If $\alpha \in F$ define

$$\bar{\alpha} = \{\alpha, 1/\alpha, -\alpha/(\alpha+1), -\alpha-1, -1/\alpha-1, \alpha/(\alpha+1)-1\}.$$

Then it follows that for $\alpha, \beta \in F$ either $\bar{\alpha} = \bar{\beta}$ or $\bar{\alpha} \cap \bar{\beta} = \emptyset$. Take as the set of vertices of $B(p)$, the set $\{\bar{\alpha} : \alpha \in F\}$ and join $\bar{\alpha}$ and $\bar{\beta}$ by an edge if there exists $\alpha \in \bar{\alpha}$ and $\beta \in \bar{\beta}$ with $\alpha = \beta + 1$ or $\alpha = \beta - 1$. The following two theorems are proved.

Theorem 3: If $B(p)$ contains a 1-factor then there exists an SCSQS($2p$).

Theorem 4: If $B(p)$ is bridgeless with $p \equiv 53$ or $77 \pmod{120}$ then there exists an SCSQS($2p$).

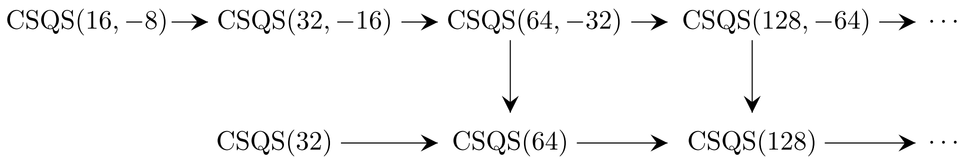
These results enabled Köhler to construct SCSQS(v) for $v = 26, 34, 50, 58, 74, 82, 106, 178, 202, 226, 274, 298, 346, 394, 466, 586$ and 634. Using similar techniques Cho [3] has constructed SCSQS(v) for other values of v not necessarily covered by the theorems.

Enumeration results for R-cyclic and S-cyclic Steiner quadruple systems are few. The unique CSQS(10) is also S-cyclic. For $v = 20$, there exist 16 distinct RCSQS(v) which partition into 4 isomorphism classes (Phelps [20]). Of these, 4, a11 within a single isomorphism class, are also S-cyclic (Jain [15]). The present authors [11] enumerated SCSQS(26). There are 87 distinct SCSQS(26) in 18 isomorphism classes.

4 Existence Results for CSQS(v)

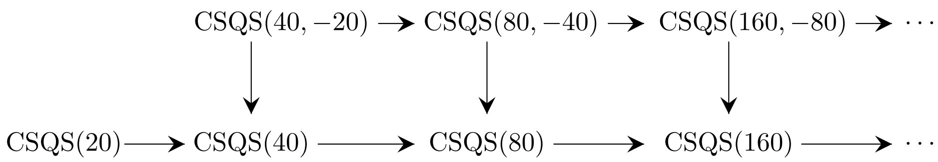
In this section we apply the known methods of construction, particularly those devised by Cho, Colbourn and Colbourn, and ourselves, to various "small" systems constructed by hand or with the aid of a computer search to give an account of the known spectrum of CSQS(v).

(a) In [4], Colbourn and Colbourn give a CSQS($16, -8$). Together with a CSQS(32), two examples of which were constructed by the present authors [10], all CSQS(2^n) for $n \geq 5$ may be obtained using the following constructional scheme. The details appear in [4].



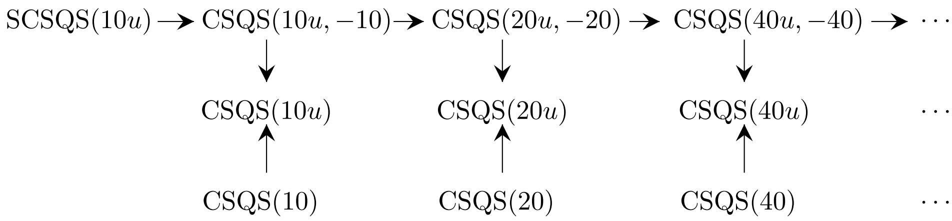
Theorem A: CSQS(2^n) exist for $n = 2$ and $n \geq 5$. CSQS(2^3) and CSQS(2^4) do not exist.

(b) Colbourn and Phelps [5] constructed a CSQS($40, -20$). Again, together with one of the known CSQS(20)s a similar constructional scheme to the one above may be adopted.



Theorem B: CSQS($2^n \cdot 5$) exist for $n \geq 1$.

(c) It is clear that what is really required are appropriate "starter" systems to which Colbourn and Colbourn's construction may be applied. These are provided by the S-cyclic Steiner quadruple systems. From the theorem on the structure of these systems, if m is even and $m|v$ then by removing all those orbits which contribute blocks of the SCSQS(m) subdesign from the SCSQS(v), a CSQS($v, -m$) is obtained. In particular if $m = 10$ and $u = v/m$ is odd we have the following:



using the CSQS($2^n \cdot 5$) from the previous theorem. An SCSQS(50) is given by Köhler [16], and Cho [3] has constructed SCSQS($10u$) for $u = 13, 17, 25, 29$ and 37.

Theorem C: CSQS($2^n \cdot 5u$) exist for $n \geq 1$ and $u = 5, 13, 17, 5^2, 29$ and 37.

(d) The results given in Theorems A, B and C appear to be the only known infinite families, the orders of whose members increase by a factor of 2, which are totally determined. In many others there are gaps. Using our Theorem 3, let $u = 2^n w$ for some $n \geq 1$ and $w = 5$ or $5u$ for any of the values of u given in Theorem C. Then for any $v = 2x$, with x odd, for which a CSQS(v) exists, it follows that a CSQS($2^{n+1}wx$) exists i.e. only the existence of a CSQS($2wx$) is left undetermined. Values of x for which CSQS($2x$) are known are $x = 5, 11, 13, 17, 19$ (Colbourn and Phelps [5]), $x = 5^2, 29, 37, 41, 7^2$ (Cho [3]) as well as other values greater than 50 constructed by Köhler [16] and listed towards the end of Section 3.

Theorem D: If there exists a CSQS($2x$) with x odd then there exist CSQS($2^n \cdot 5x$) and CSQS($2^n \cdot 5ux$) for $u = 5, 13, 17, 5^2, 29$ and 37 and $n \geq 2$.

(e) The existence of a CSQS($2x$) with x odd also implies the existence of another infinite family. Immediately from Cho's construction there exists a CSQS($4x$) and also, by removing the quarter orbit, a CSQS($4x, -4$). Applying Colbourn and Colbourn's construction one can then produce CSQS($2^n x, -2^n$) for all $n \geq 2$ and when $n \geq 5$ these systems can be completed using the CSQS(2^n) from Theorem A. Again a gap appears i.e. the existence of a CSQS($2^3 x$) and a CSQS($2^4 x$) is left unresolved.

Theorem E: If there exists a CSQS($2x$) with x odd then there exists CSQS($2^n x$) for $n = 2$ and $n \geq 5$.

(f) There are other values of v for which CSQS(v) are known to exist, These include $v = 2^n \cdot 7$ for $n = 2$ and $n \geq 5$, constructed as in (e) from a CSQS(28). Phelps' theorems enable further systems to be constructed and there are applications of our Theorem 3 in addition to that given above such as the construction of a CSQS($2^n \cdot 5xy$) for $n \geq 3$ from a CSQS($2x$) and CSQS($2y$) with x and y odd. Also Cho [3] has constructed CSQS(v) for $v = 88, 92$ and 124.

5 Concluding Remarks

The evidence of the results in the previous section would seem to support the conjecture that CSQS(v) exist for all admissible v apart from $v = 8, 14$ and 16. The only values less than 100 for which the existence of a CSQS(v) is unresolved are $v = 46, 56, 62, 70, 86$ and 94.

A major problem with the methods known at present appears to be that the non-existence of CSQS(8), CSQS(14) and CSQS(16) frustrate the construction of certain systems of higher order. To overcome this, it would be useful to have a generalisation of Cho's construction which in effect "doubles" the quarter orbit. If $v = 4m$ then starting from a CSQS($4m, -4$), a CSQS($8m, -8$) can be constructed but not completed. However, it seems not unlikely that the removal of a relatively small number of orbits from the CSQS($8m, -8$) may result in completion being possible.

Other major lines of investigation would appear to be the construction af CSQS($2x$) with x odd and the development of more general recursive constructions. Finally, the question of whether the known necessary conditions for R-cyclic and S-cyclic Steiner quadruple systems are also sufficient remains open.

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