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A STEINER SYSTEM $\mathbf{S(5, 6, 108)}$

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We exhibit a Steiner system $S(5, 6, 108)$ invariant under the group $PSL_2(107)$.

1 Past

There are still relatively few parameter sets with $t > 3$ for which the existence of a Steiner system $S(t,k,v)$ is known. No systems are known with $t > 5$. Before 1976 the only known systems with $t > 3$ were the unique systems $S(5,6,12)$ and $S(5,8,24)$, with automorphism groups M_{12} and M_{24} respectively, together with their unique derived systems $S(4,5,11)$ and $S(4,7,23)$. These systems must, however, be considered as "special" if only because their automorphism groups are amongst the list of 26 exceptional groups in the classification of finite simple groups. In 1976 Denniston [1] constructed Steiner systems $S(5,6,v)$ with $v = 24, 48$ and 84, and $S(5,7,28)$. The method was to assume an automorphism group $PSL_2(p^\alpha)$ with $p^\alpha = 23, 47, 83$ and 3^3, respectively, acting on the set $GF(p^\alpha) \cup \{\infty\}$ in the usual way and assembling appropriate k-block orbits, Remarkably, all Denniston's calculations were done by hand. In a subsequent paper [2], some background is given. In particular, it is shown that no Steiner system $S(2s-1, 2s, p^\alpha+1)$ where $s \geq 2$ and $p > 2s$ is prime, can be stabilized by the larger group $PGL_2(p^\alpha)$. The possibility of a system stabilized by $PSL_2(p^\alpha)$ is also eliminated if $p \equiv 1 \pmod 4$, essentially because in this case -1 is a quadratic residue. Neither result generalizes to Steiner systems where $k - t \geq 2$ or block designs with $\lambda > 1$, so in these cases also the use of the group $PSL_2(p^\alpha)$ where $p \equiv -1 \pmod 4$ remains a possibility. However, as detailed in [2], many of these further possibilities, such as an $S(5,8,44)$ stabilized by $PSL_2(43)$, can be ruled out by further elementary argument. Others, such as an $S(7,8,20)$ stabilized by $PSL_2(19)$, are eliminated after sometimes lengthy calculations. Many years ago, the present authors attempted to construct simple $7-(20,8,\lambda)$ designs stabilized by $PSL_2(19)$ for $\lambda = 1,2,3,4$ and 5, but none exist. The remaining possibility, $\lambda = 6$, remains open but again we doubt if such a design exists.

Following Denniston's work, Mills [4] investigated the case of an $S(5,6,72)$ stabilized by $PSL_2(71)$. This system was not investigated by Denniston possibly because of the differing sizes of the short orbits which such a system must contain. Mills was successful in constructing such a system by computer. The present authors [3] made an enumeration of Steiner systems $S(5,6,24)$ stabilized by $PSL_2(23)$ and discovered that there exist precisely three pairwise non-isomorphic systems. To our knowledge, the systems described above, together with their derived Steiner 4-designs are the only known Steiner systems with $t > 3$. Perhaps the next possibilities which are worthy of investigation are $S(5,6,108)$ and $S(5,8,108)$, both stabilized by

$PSL_2(107)$. After extensive calculations we determined, several years ago, that the latter does not exist. However, in the next section, we present a realization of the former.

2 Present

Let $V = GF(107) \cup \{\infty\}$ and denote by $\binom{V}{k}$ the collection of all k-element subsets (k-blocks) of V. Let G be the group $PSL_2(107)$ of all mappings of the form $z \to (az+b)/(cz+d)$ where $a, b, c, d \in GF(107)$ and $ad-bc$ is a non-zero square. Consider orbits of G acting on $\binom{V}{5}$ and $\binom{V}{6}$. We call an orbit a full orbit if its length (number of blocks in the orbit) is g, the order of G, and a $(1/n)^{th}$ orbit if its length is g/n, where $n > 1$. There are then 182 5-block orbits, all full, and the equivalent of $3124\frac{1}{3}$ 6-block orbits made up of 2898 full orbits, 441 half orbits, 8 third orbits, and 19 sixth orbits. The 6-block orbits contain (or cover) all the blocks of 6, 3, 2 or 1 of the 5-block orbits, not necessarily all different, respectively. Those 6-block orbits which contain repeated 5-block orbits are unsuitable. In order to construct an $S(5, 6, 108)$ stabilized by $PSL_2(107)$, it is necessary to assemble the equivalent of $30\frac{1}{3}$ suitable full 6-block orbits which collectively cover all of the 5-block orbits precisely once. Even with modern computer equipment a search amongst all the 6-block orbits to find such a collection is not feasible. Simplifying assumptions need to be made. The two assumptions we made are given below.

Firstly, we limited our search to the short, i.e. half, third and sixth 6-block orbits. There is no specific mathematical reason for this but the search is narrowed substantially, and we observe that Denniston's $S(5, 6, 84)$ is constructed entirely from short orbits. The second assumption was based on an analysis of different types of orbits. Consider the 5-block orbits. Define an orbit to be of type A if it is also orbit under $PGL_2(107)$. This is equivalent to requiring that the orbit is stabilized by the mapping $z \to -z$ or, again equivalently, by the mapping $z \to 1/z$. There are in fact 26 such orbits. Next define an orbit to be of type B if it is not stabilized by $z \to -z$ but contains a block of the form $\{\infty, -1, 0, 1, a\}$. Observe that by applying the mapping $z \to -1/z$ this orbit also contains the block $\{\infty, -1, 0, 1, -1/a\}$ Clearly, type B orbits occur in pairs, each the negative of the other under the mapping $z \to -z$. In what follows, if $\mathbf{b}$ is a type B orbit then its negative will be denoted by $\mathbf{b}'$. There are 12 pairs of type B orbits. Now consider the

6-block orbit O_1 generated by $\{\infty, -1, 0, 1, a, -a\}$, where $\{\infty, -1, 0, 1, a\}$ is the generator of a type B orbit, $\mathbf{b}$. Since O_1, also covers the 5-block orbit generated by $\{\infty, -1, 0, 1, -a\}$, O_1 contains $\mathbf{b}'$. Furthermore, the 5-block $\{\infty, -1, 1, a, -a\}$ generates a type A orbit $\mathbf{a}$ because it is also stabilized by $z \to -z$. O_1 is itself stabilized by both the mappings $z \to \pm a/z$, one of which is a member of $PSL_2(107)$. Hence, O_1 is a half orbit, we say of type ABB′. Similarly, considering the orbit O_2 generated by $\{\infty, -1, 0, 1, a, 1/a\}$, this is stabilized by both mappings $z \to (az-1)/(z-a)$ and $z \to (z-a)/(az-1)$ one of which is again a member of $PSL_2(107)$. It also contains the 5-block orbits generated by $\{\infty, -1, 0, 1, a\}$, $\{\infty, -1, 0, 1, 1/a\}$ and $\{\infty, 0, 1, a, 1/a\}$. Denoting the first of these by $\mathbf{b}$, the second is $\mathbf{b}'$, both of type B. The third, $\mathbf{a}$, is of type A since it stabilized by $z \to 1/z$. Hence, O_2 is of type ABB′. O_2 is a different orbit from O_1, because otherwise O_1 would contain a repeated 5-block. This would imply that O_1 was unsuitable but this in turn is impossible because it contains three distinct 5-block orbits. Hence, an effective way of covering all type B orbits is to ensure that for each pair $\mathbf{b}, \mathbf{b}'$ either the orbit O_1 or O_2 is included as a potential orbit of the Steiner system $S(5, 6, 108)$. With the two simplifying assumptions the computer was able to complete the system in a matter of hours, although it still remains to check how many non-isomorphic solutions are obtainable.

Listed below is a collection of generators for 66 6-blocks orbits which under the action of the group $PSL_2(107)$ acting in the normal way, collectively give the 18,578,196 blocks of a Steiner system $S(5, 6, 108)$. Set brackets and the elements $\infty, 0, 1$ are omitted for brevity. The first 12 orbits are those of type ABB′.

Half orbits (type ABB′).

38	69	106	20	87	106	37	70	106
45	62	106	14	93	106	12	95	106
32	75	106	44	63	106	15	92	106
29	78	106	27	80	106	5	102	106

Half orbits (others).

51	69	95	29	43	70	7	90	95
21	30	95	6	104	105	25	68	95
40	46	60	46	49	73	17	41	50
22	33	84	32	36	96	68	78	105
18	70	99	17	30	104	20	42	82
21	45	48	27	45	73	60	64	95
36	50	88	12	59	66	28	92	98
73	92	95	8	33	104	50	83	96
23	26	63	35	78	97	45	98	102
20	23	32	17	95	100	17	45	105
62	93	95	63	83	91	15	27	84
51	75	82	38	56	95	17	78	99
6	34	82	28	76	95	17	19	91
5	19	95	28	48	63	26	39	72
24	31	86	13	32	95			

Third orbits.

51	87	92	48	66	79	47	67	100
23	34	94						

Sixth orbits.

24	50	93	19	46	101	13	75	98
10	33	95	6	64	90	2	54	106

It is difficult for the reader to check the results for herself. We have checked our own computations by using the triple homogeneity of the automorphism group. It is sufficient to consider all blocks of the form $\{\infty, 0, 1, a, b, c\}$ and then to confirm that the resulting triples $\{a, b, c\}$ form a Steiner triple system, $STS(105)$. The checking program is completely distinct from the programs which compute and search for the system. A similar checking program is given in Denniston's paper [1] and the reader may utilize this if required.

3 Future

It seems reasonable from the combined results to conjecture that there exists a Steiner system $S(5, 6, p + 1)$ invariant under the group $PSL_2(p)$ whenever $p \equiv -1 \pmod 4$ and is prime and $p + 1$ satisfies the admissibility conditions $\binom{6-i}{5-i} \mid \binom{p+1-i}{5-i}$ for $i = 0, 1, 2, 3$ and 4. The next such cases to consider are $p + 1 = 132, 168, 192$ and 228. It may be possible to construct these using the same or similar assumptions to those given in this paper in order to restrict the field of search. However, what is really required is a mathematical theory to identify precisely which 6-block orbits must be chosen to construct the Steiner system so that an infinite class may be obtained.

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