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A NOTE ON THE STEINER SYSTEMS S(5,6,24)

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In his paper [1] R.H.F. Denniston constructs several new Steiner systems. In particular he constructs two non-isomorphic $S(5,6,24)$ systems invariant under $PSL_2(23)$. It is the purpose of this note to prove that there are precisely three non-isomorphic $S(5,6,24)$ systems invariant under $PSL_2(23)$.

We set up $GF(23) \cup \{\infty\}$ and allow $PSL_2(23)$ to act on this in the usual way as the group of transformations of the form $z \to (az+b)/(cz+d)$, where $ad-bc$ is a non-zero square. From a given 6-block $\{x_1, \ldots, x_6\}$ drawn from $GF(23) \cup \{\infty\}$ we may produce an orbit (transitivity class) by forming all images of this block under $PSL_2(23)$. An orbit will be called full if its cardinality is $|PSL_2(23)|$, a half orbit if its cardinality is $\frac{1}{2}|PSL_2(23)|$, and so on. We observe that the set of all 6-blocks will give rise to the equivalent of $\binom{24}{6}/|PSL_2(23)| = 22\frac{1}{6}$ full orbits.

Each 6-block in an orbit generates six 5-blocks. An orbit will be called suitable if all these derived 5-blocks are distinct. Similarly, two suitable orbits will be called compatible if all the derived 5-blocks of the two orbits are distinct. The triple homogeneity of the PSL group shows that to determine suitability or compatibility we only need consider those 5-blocks containing $\{\infty, 0, 1\}$.

Any $S(5,6,24)$ system invariant under $PSL_2(23)$ will be isomorphic to a system built from orbits as described above. The orbits forming such a system must all be suitable, pairwise compatible, and have total cardinality the equivalent of $\binom{24}{5}/|PSL_2(23)| = 1\frac{1}{6}$ full orbits. To find the precise number of non-isomorphic systems we firstly investigate orbits for cardinality and suitability. Each of the 3-blocks listed below will, with the addition of $\{\infty, 0, 1\}$, generate a distinct orbit of 6-blocks. The total cardinality of the collection is the equivalent of $22\frac{1}{6}$ full orbits as required.

(a) Suitable Orbits:

   (i) Half orbits. H1:$\{2,3,12\}$, H2:$\{2,3,14\}$, H3:$\{2,5,6\}$, H4:$\{2,5,8\}$, H5:$\{2,5,10\}$, H6:$\{2,5,14\}$, H7:$\{2,5,15\}$, H8:$\{2,5,17\}$, H9:$\{2,5,18\}$, H10:$\{2,6,19\}$, H11:$\{3,4,11\}$, H12:$\{3,4,16\}$.

   (ii) Sixth orbits. S1:$\{2,3,13\}$, S2:$\{2,6,10\}$, S3:$\{2,8,14\}$, S4:$\{3,7,10\}$, S5:$\{3,7,21\}$.

(b) Unsuitable Orbits:

   (i) Full orbits. F1:$\{2,3,5\}$, F2:$\{2,3,6\}$, F3:$\{2,3,7\}$, F4:$\{2,3,8\}$, F5:$\{2,3,9\}$, F6:$\{2,3,10\}$, F7:$\{2,3,11\}$, F8:$\{2,3,15\}$, F9:$\{2,3,18\}$, F10:$\{2,3,19\}$, F11:$\{2,5,11\}$, F12:$\{2,5,19\}$), F13:$\{2,6,8\}$, F14:$\{2,6,14\}$.

   (ii) Half orbits. H13:$\{2,3,4\}$, H14:$\{2,5,7\}$.

   (iii) Third orbit. T1:$\{(3,4,9\}$.

The table below gives the pairwise compatibility of suitable orbits.

It is easy to see that we can construct just six realisations of $S(5,6,24)$ from these orbits, namely a) H1, H6, S4. b) H2, H5, S5. c) H9, H12, S1. d) H9, H11, S1. e) H1, H6, S5. f): H2, H5, S4.

The systems a) and c) are those given in [1]. We note that the mapping $z \to -z$ provides isomorphisms of i) a) and b), ii): c) and d), iii) e) and f). It remains to establish that a), c) and e) are non-isomorphic. To do this we count the quadrilaterals of the derived Steiner triple systems $S(2,3,21)$ obtained by deleting $\infty, 0, 1$ from those 6-blocks containing all these three elements. A quadrilateral is a set of four 3-blocks of a Steiner triple system whose union has cardinality 6. The numbers of quadrilaterals given in [1] for a) and c) were 15 and 24 respectively. However, we find that the numbers are a) 16, c) 27, e) 18. It follows that there are precisely three non-isomorphic $S(5,6,24)$ systems invariant under $PSL_2(23)$ rather than just two as remarked in [1].

	H2	H3	H4	H5	H6	H7	H8	H9	H10	H11	H12	S1	S2	S3	S4	S5
H1					√									√	√	√
H2				√									√		√	√
H3												√		√	√	√
H4												√	√		√	√
H5												√		√	√	√
H6												√	√		√	√
H7												√	√		√	√
H8												√		√	√	√
H9										√	√	√			√	√
H10												√				
H11												√	√	√		
H12												√	√	√		
S1													√	√	√	√
S2														√	√	√
S3															√	√
S4																

√ indicates compatibility.

Blank entry indicates incompatibility.

Our final comment concerns the validation of our results. It is difficult to make it feasible for the reader to verify computational results such as these. In an effort to combat the problem, our computation of orbits, suitability, compatibility and quadrilateral calculations have each been checked by at least two independently written computer programs in each case. We have checked by hand that our list of orbits is complete and that the quadrilaterals generated by the computer programs are correct.

REFERENCES

[1]. R.H.F. Denniston, *Some new 5-designs*, B.L.M.S., 8 (1976), 263–267.

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