PRESTON
POLYTECHNIC
RESEARCH
NOTE

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August 1978

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In a recent survey paper, Lindner and Rosa [2, page 178], asked the question whether the known S-cyclic Steiner quadruple system of order 20, SQS(20), is the unique cyclic system for this order. In this paper we show that it is not.

We recap that the known cyclic SQS(20) was constructed by Jain [1] and contains the following quadruples:

$$\begin{array}{lll} (i,i+1,i+3,i+4), & (i,i+1,i+5,i+16), & (i,i+1,i+6,i+7), \\ (i,i+1,i+8,i+13), & (i,i+1,i+9,i+12), & (i,i+2,i+5,i+17), \\ (i,i+2,i+6,i+8), & (i,i+2,i+7,i+9), & (i,i+3,i+7,i+16), \\ (i,i+1,i+2,i+11), & (i,i+2,i+4,i+12), & (i,i+3,i+6,i+13), \\ (i,i+3,i+9,i+14), & (i,i+4,i+8,i+14), & i=0,1,2\ldots,19, \end{array}$$

and $(i,i+5,i+10,i+15)$, where all additions are performed modulo 20. This Steiner system is also S-cyclic; indeed it is known to be the unique S-cyclic SQS(20).

From each block (x,y,z,w), $0 \le x < y < z < w < v$, of an SQS(v) a difference quadruple (a,b,c,d) with $a = y - x, b = z - y, c = w - z$, and $d = x - w + v$ can be calculated. A difference quadruple (a,b,c,d) is said to be symmetric if and only if

(i) $a = c$, or

(ii) $b = d$, or

(iii) $a = b$ and $c = d$, or

(iv) $a = d$ and $b = c$.

A cyclic SQS(v) all of whose blocks have symmetric difference quadruples is said to be S-cyclic.

We list below the quadruples of a non-symmetric cyclic SQS(20).

$$\begin{array}{lll} (i,i+1,i+3,i+15), & (i,i+1,i+5,i+7), & (i,i+2,i+5,i+13), \\ (i,i+5,i+17,i+19), & (i,i+13,i+15,i+19), & (i,i+7,i+15,i+18), \\ (i,i+1,i+4,i+17), & (i,i+1,i+8,i+9), & (i,i+4,i+9,i+13), \\ (i,i+1,i+2,i+11), & (i,i+2,i+4,i+12), & (i,i+3,i+6,i+13), \\ (i,i+3,i+9,i+14), & (i,i+4,i+8,i+14), & i=0,1,2\ldots,19, \end{array}$$

and $(i,i+5,i+10,i+15)$, where all additions are performed modulo 20.

It remains to prove our claim that this system is indeed different from the one constructed by Jain and not just a rearrangement. We note that both systems are fixed by a dihedral group, D_{20}, though the action of the group in the two cases is different. This implies that if the systems are isomorphic then each is stabilised by two such dihedral groups, and this seems very unlikely. To be certain we can calculate invariants of the two systems. Since both quadruple systems are transitive, all the derived Steiner triple systems in each are isomorphic. Thus we can count the number of quadrilaterals in a derived Steiner triple system in each of the two cases. A quadrilateral is a family of four blocks of a triple system which contain only six points. We find that for the system constructed by Jain each derived triple system contains 24 quadrilaterals, while for the new system the corresponding number is 6.

REFERENCES

[1] R. K. Jain, *On cyclic Steiner quadruple systems*, M.Sc. thesis, McMaster University, Hamilton, Ontario (1971).
[2] C. C. Lindner and A. Rosa, *Steiner quadruple systems - a survey*, Discrete Math. 21 (1978), 147–181.

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