Even-cycle systems with prescribed automorphism groups

M. J. Grannell, T. S. Griggs and G. J. Lovegrove
Department of Mathematics and Statistics
The Open University
Walton Hall, Milton Keynes
MK7 6AA, U.K.
(m.j.grannell@open.ac.uk
t.s.griggs@open.ac.uk
graham.lovegrove@virgin.net)

Abstract

It is shown that for any finite group Γ, there exists a $2k$-cycle system whose full automorphism group is isomorphic to Γ. Furthermore the minimal order of such a system is at most $8k\gamma \log_2 \gamma + 1$, where $\gamma = |\Gamma|$.

AMS Subject Classifications: primary 05B30, secondary 05C51.

Keywords: Automorphism group, Cycle system.

1 Introduction

In 1975, E. Mendelsohn [11, 12] proved that for any finite group Γ there exists a Steiner triple system with full automorphism group isomorphic to Γ. The order of the Steiner triple system in Mendelsohn's construction is less than $\gamma^{c\gamma}$, where $\gamma = |\Gamma|$ and c is a certain positive constant. In this paper, it is first proved that for any finite group, Γ, there exists a 4-cycle system whose full automorphism group is isomorphic to Γ, and moreover our construction yields a 4-cycle system of order at most $16\gamma\log_2\gamma + 1$ (Theorem 2.1). The result is then extended to $2k$-cycle systems with the corresponding bound $8k\gamma\log_2\gamma + 1$ (Theorem 3.3). These results were first presented in the third author's doctoral thesis [10].

An m-cycle system of order n, denoted by $m\mathrm{CS}(n)$, is a decomposition of the complete graph on n vertices, K_n, into cycles of length m. The vertices of K_n form the point set of the system. The order $n \geq m$ is *m-admissible* if n is odd and $\frac{1}{2}n(n-1)$ is divisible by m. It has been shown [1, 4, 9, 13] that an $m\mathrm{CS}(n)$ exists for $n > 1$ if and only if n is m-admissible. In this paper we shall be almost exclusively interested in cycles of even length $2k$.

A *subsystem* of an $m\mathrm{CS}(n)$ is an $m\mathrm{CS}(n')$ with $n' \leq n$, all of whose m-cycles are also cycles of the $m\mathrm{CS}(n)$. A *proper* subsystem of an $m\mathrm{CS}(n)$ is a subsystem of order n' with $n' < n$.

Following Mendelsohn, we shall use a result of Frucht.

Theorem 1.1 ([7]) *For any given finite group Γ there exists a finite simple graph whose full automorphism group is isomorphic to Γ.*

In the course of the proofs we will work with sets of differences. If $\mathbb{Z}'_n$ is a copy of $\mathbb{Z}_n$ on the points $\{0', 1', \ldots, (n-1)'\}$ then the (*mixed*) *difference* between u and v' for $u, v \in \mathbb{Z}_n$ is defined to be $v - u \bmod n$. When we speak of *labelling* a point $u \in \mathbb{Z}_n$, we mean replacing it by the point $u' \in \mathbb{Z}'_n$. To save excessive use of brackets, compound terms such as $(u+1)'$ may be written as $u + 1'$.

2 4-cycle systems

The necessary and sufficient condition for the existence of a 4-cycle system of order $n > 1$ is that $n \equiv 1 \pmod 8$. Note that a 4CS(9) can therefore have no proper subsystems. Our construction requires the following building blocks.

(a) An automorphism-free 4CS(9) on the points $\{\infty\} \cup \{1, 2, \ldots, 8\}$, denoted by T.

(b) Two different symmetric decompositions of $K_{8,8}$.

By *automorphism-free* we mean that T has only the identity automorphism. We may take T to be any one of the three automorphism-free systems given in

Table 4 of [6], for example $(1,2,3,4),(1,3,5,6),(1,5,2,7),(1,8,2,\infty),(2,4,7,6),(3,6,4,8),(3,7,5,\infty),(4,5,8,\infty),(6,8,7,\infty)$. By a *symmetric* decomposition of $K_{8,8}$, we mean that if the 4-cycle (a,b',c,d') is in the decomposition, where the unprimed letters are vertices of one part and the primed letters are vertices of the other part, then the 4-cycle (a',b,c',d) is also in the decomposition. Such decompositions exist, for example:

$$\begin{array}{llll}
(a,a',b,b'), & (a,c',b,d'), & (a,e',b,f'), & (a,g',b,h'),\\
(c,a',d,b'), & (c,c',d,d'), & (c,e',d,f'), & (c,g',d,h'),\\
(e,a',f,b'), & (e,c',f,d'), & (e,e',f,f'), & (e,g',f,h'),\\
(g,a',h,b'), & (g,c',h,d'), & (g,e',h,f'), & (g,g',h,h').
\end{array}$$

The two decompositions of $K_{8,8}$ are required to be different only in the sense that there is a pair of edges that belong to the same cycle in one decomposition but not in the other. So exchanging b with c and b' with c' in the decomposition above is sufficient. We shall call one decomposition the *edge* decomposition, and the other the *non-edge* decomposition.

Construction 2.1

Let $G=(V,E)$ be a graph for which $Aut(G)=\Gamma$; Theorem 1.1 ensures that such a graph exists. For $v\in V$ put $\bar{v}=\{(v,i):1\le i\le 8\}$ and put $\bar{V}=\bigcup_{v\in V}\bar{v}$. We will construct a 4-cycle system $S(G)$ of order $8|V|+1$ on the set $\{\infty\}\cup\bar{V}$. The edges of the complete graph on this set comprise all the edges of the complete graphs K_9 on $\{\infty\}\cup\bar{v}$ for $v\in V$, and all the edges of the complete bipartite graphs $K_{8,8}$ with bipartitions $\{\bar{u},\bar{v}\}$ for all distinct pairs $u,v\in V$.

For each $v\in V$, decompose the complete graph on $\{\infty\}\cup\bar{v}$ using a copy of T. For each distinct $u,v\in V$ with $\{u,v\}\in E$, decompose the complete bipartite graph with bipartition $\{\bar{u},\bar{v}\}$, using a copy of the edge decomposition. Since the decompositions are symmetric, it will not matter which way the parts are allocated. If $\{u,v\}\notin E$, use the non-edge decomposition instead. Every edge is then in a unique 4-cycle, and $S(G)$ is a 4CS($8|V|+1$).

The 4CS(9)s on the vertex sets $\{\infty\}\cup\bar{v}$ will be called the *vertex subsystems*. A 4-cycle of a $K_{8,8}$ representing an edge (respectively a non-edge) of G will be called an *edge* (respectively a *non-edge*) *cycle*. We make the additional requirement that the decompositions using T on distinct vertex sets $\{\infty\}\cup\bar{u}$ and $\{\infty\}\cup\bar{v}$ are done in such a way that if, for example, $((u,1),(u,2),(u,3),(u,4))$ is a cycle in the first vertex subsystem, then $((v,1),(v,2),(v,3),(v,4))$ is a cycle in the second, and similar requirements are applied to the edge and non-edge decompositions. In particular, and because T is automorphism-free, this ensures that the only isomorphism between two vertex subsystems is of the form $(u,i) \mapsto (v,i)$ for $i = 1, 2, \ldots, 8$, with ∞ fixed.

We will show that the 4-cycle system constructed in this manner has full automorphism group Γ. The following lemma is required.

Lemma 2.1 *If two subsystems of an $mCS(n)$ have more than one point in common, then their common m-cycles also form a subsystem.*

Proof If a and b are distinct common points in the two subsystems, then the unique m-cycle containing the edge $\{a, b\}$, must be a cycle of both subsystems. Therefore the common cycles also form an m-cycle system.

$\square$

Theorem 2.1 *The 4-cycle system S obtained from a graph with full automorphism group Γ by Construction 2.1 has itself full automorphism group Γ. Moreover, if $|\Gamma| = \gamma$, then there exists such an S with no more than $16\gamma log_2\gamma + 1$ vertices if Γ is non-cyclic, and $24\gamma + 1$ vertices otherwise.*

Proof Firstly we show that the only 4CS(9) subsystems of S are the vertex subsystems. Every point of S except ∞ is in exactly one vertex subsystem, and ∞ is in all vertex subsystems. Suppose S contains a 4CS(9), P, that is not a vertex subsystem. Then P intersects more than one vertex subsystem. It cannot intersect any vertex subsystem in more than one point since otherwise, by Lemma 2.1, it would be the whole vertex subsystem because a 4CS(9) has no proper subsystems. Therefore any two distinct points of P must be in different vertex subsystems. Hence any edge of P is either in an edge cycle or a non-edge cycle. However, every edge cycle and every non-edge cycle contains two points of the same vertex subsystem, contradicting our previous assumption. Therefore the only 4CS(9) subsystems of S are vertex subsystems.

This implies that any automorphism of S maps one vertex subsystem onto another. Moreover, for any automorphism ϕ of S, if the image of the point (u,i) is (v,j), $u, v \in V$, $\{i, j \in \{1, 2, \ldots, 8\}\}$, then $i = j$ because the 4CS(9) T of which each vertex subsystem is a copy has no automorphisms other than the identity. Thus we can put $\phi((u,i)) = (\psi(u), i)$ where ψ is a bijection on V. Now we show that ψ is an automorphism of G.

For any pair $u, v \in V$, the edges of S of the form $\{(u,i),(v,j)\}$, $i,j \in \{1,2,\ldots,8\}$ are either all in edge cycles or all in non-edge cycles by construction, according as $\{u,v\}$ is an edge of G or not. Moreover, since the edge decomposition and the non-edge decomposition differ in at least one cycle, we can tell whether $\{u,v\}$ is an edge of G by examining the cycles containing these edges of S. If $\{u,v\}$ is an edge of G, then all the edges $\{(u,i),(v,j)\}$, $i,j \in \{1,2,\ldots,8\}$ of S are in edge cycles, therefore all the edges $\{\phi((u,i)),\phi((v,j))\} = \{(\psi(u),i),(\psi(v),j)\}$, $i,j \in \{1,2,\ldots,8\}$ of S are also in edge cycles of S, and so $\{\psi(u),\psi(v)\}$ is an edge of G. Similarly if $\{u,v\}$ is not an edge of G, then neither is $\{\psi(u),\psi(v)\}$. Thus ψ is an automorphism of G.

Conversely, for any automorphism ψ of G an automorphism ϕ of S can be defined by $\phi((v,i)) = (\psi(v),i)$. We show that it is an automorphism of S. The mapping ϕ is clearly one to one on S, and maps any vertex subsystem to another. We have further to show that ϕ maps edge decompositions to edge decompositions and non-edge decompositions to non-edge decompositions. That all the edges $\{(u,i),(v,j)\}$, $i,j \in \{1,2,\ldots,8\}$ of S are in edge cycles of S implies that $\{u,v\}$ is an edge of G. But $\{u,v\}$ is an edge of G implies $\{\psi(u),\psi(v)\}$ is an edge of G, and $\{\psi(u),\psi(v)\}$ is an edge of G implies that all the edges $\{\phi((u,i)),\phi((v,j))\} = \{(\psi(u),i)),(\psi(v),j)\}$, $i,j \in \{1,2,\ldots,8\}$ are in edge cycles of S. Similarly for non-edge cycles. Thus ϕ is an automorphism of S.

There is therefore a one to one correspondence between automorphisms of S and automorphisms of G. In order to show that we have an isomorphism of the automorphism groups, we need to show that the composition of two automorphisms of G gives rise to the composition of the corresponding automorphisms of S. If ψ_1 and ψ_2 are automorphisms of G, and $\phi_1((v,i)) = (\psi_1(v),i)$, $\phi_2((v,i)) = (\psi_2(v),i)$ for all $v \in V$, $i \in \{1,2,\ldots,8\}$, then $\phi_2\phi_1((v,i)) = \phi_2((\psi_1(v),i)) = (\psi_2\psi_1(v),i)$ for all $v \in V$, $i \in \{1,2,\ldots,8\}$. Thus, since the identity on G maps to the identity on S, the correspondence provides an isomorphism of the automorphism groups.

Further results of Frucht [7, 8] have shown that if $|\Gamma| = \gamma$ and a minimal generator set for Γ is of size ν, then the graph G may be taken to have $2\gamma\nu$ vertices if Γ is non-cyclic, and 3γ otherwise. Since the group $(Z_2)^\nu$ is the smallest group with ν generators, $\nu \leq \log_2 \gamma$ for any group. Thus it follows that the resulting 4-cycle system S has no more than $16\gamma \log_2 \gamma + 1$ vertices if Γ is non-cyclic, and $24\gamma + 1$ otherwise. $\square$

Babai and Goodman [2, 3], among others, have proved results concerning the minimum orders and numbers of edges for graphs with a given automorphism group. These may allow further reduction of the order of 4-cycle systems with a given automorphism group in particular cases.

3 Even-cycle systems

In this section we seek to generalize Theorem 2.1. In order to do this we will first prove Theorems 3.1 and 3.2 which are stated below.

Theorem 3.1 *For every* $k \geq 2$*, there exists an automorphism-free* $2k\,CS(4k+1)$ *with no proper subsystems.*

Theorem 3.2 *For each* $k \geq 2$ *there exists at least two different decompositions of* $K_{4k,4k}$ *into* $2k$*-cycles which are symmetric in the sense that if the* $2k$*-cycle* $(x_0, x'_1, x_2, x'_3, \ldots, x_{2k-2}, x'_{2k-1})$ *is in the decomposition, where the unprimed letters are vertices of one part and the primed letters are vertices of the other part, then the* $2k$*-cycle* $(x'_0, x_1, x'_2, x_3, \ldots, x'_{2k-2}, x_{2k-1})$ *is also in the decomposition.*

We shall prove these results in the following subsections. The generalization of Theorem 2.1 is then as follows.

Theorem 3.3 *For every* $k \geq 2$*, and for any group* Γ*, there exists a* $2k$*-cycle system* S *having full automorphism group* Γ*. Moreover, if* $|\Gamma| = \gamma$*, then there exists such an* S *with no more than* $8k\gamma log_2\gamma + 1$ *vertices if* Γ *is non-cyclic, and* $12k\gamma + 1$ *vertices otherwise.*

Proof Reviewing Construction 2.1, and the proof of Theorem 2.1, it should be clear that the construction and the result extend to $2k$-cycle systems provided only that there is a $2k$CS$(4k+1)$ that is automorphism-free and has no subsystems and there are two different symmetric decompositions of $K_{4k,4k}$ into $2k$-cycles. Theorems 3.1 and 3.2 provide the required existence results. $\square$

3.1 Proof of Theorem 3.1

We give the next three lemmas for general m-cycle systems before specializing to even-cycle systems.

Lemma 3.1 *If the set of fixed points of any automorphism of a mCS(n) has size greater than one, then it forms a subsystem.*

Proof If a and b are fixed points then the unique cycle containing the edge $\{a, b\}$ is fixed, so the other points in this cycle are also fixed. Thus the complete graph on the fixed points is decomposed into m-cycles. $\square$

Lemma 3.2 *A cyclic mCS($2m + 1$) has no proper subsystems if $m > 3$.*

Proof An $m\mathrm{CS}(2m + 1)$ has $2m + 1$ cycles, and there are no short cyclic orbits of m-cycles modulo $2m+1$, so a cyclic $m\mathrm{CS}(2m+1)$ has just one orbit. Now suppose that S is a cyclic $m\mathrm{CS}(2m + 1)$ with a proper subsystem. Let T be a minimal proper subsystem of S of order t. Then $m \leq t < 2m + 1$.

Since S has only one orbit, every cycle of S is a translate of every other cycle, and so every cycle of S is in a translate of T. Since T is minimal, T does not share any cycles with any of its translates, since otherwise, by Lemma 2.1 the intersection of two translates of T would be a smaller subsystem than T.

Every point of T is in precisely m cycles of S, and is in precisely $(t-1)/2$ cycles of T and of each translate in which it occurs. Each point of T is in the same number of translates of T. Let this number be r. Then $m = r(t-1)/2$. Putting $t = 2q + 1$, since $t \geq m \geq 3$, we have $2q + 1 \geq qr \geq 3$, so either $r = 1$, $r = 2$, or $q = 1$ and $r = 3$.

If $r = 1$, then T is the whole of S. If $q = 1$ and $r = 3$, then $m = 3$, and the $m\mathrm{CS}(2m + 1)$ is the unique Steiner triple system of order 7, and every 3-cycle is a proper subsystem.

If $r = 2$, then $t = m + 1$, but then m must be even since t must be odd to be admissible. But t cannot be admissible because then m must divide $\frac{1}{2}(m + 1)m$. Therefore for $m > 3$ a cyclic $m\mathrm{CS}(2m + 1)$ has no proper subsystems. $\square$

Table 4 of [6] establishes Theorem 3.1 for the case $k = 2$. To prove it for $k > 2$, we shall modify the following construction of Buratti and Del Fra for cyclic m-cycle systems .

Lemma 3.3 ([5]) *Let $B = (b_1, b_2, \ldots, b_m)$ be the m-cycle defined by:*

$$b_i = \begin{cases} i(-1)^{i+1} & \textit{for } i < \frac{m}{2}, \\ i(-1)^i & \textit{for } i \geq \frac{m}{2}. \end{cases}$$

The translates of B modulo $2m+1$ generate a cyclic mCS($2m+1$).

□

As an example, the 6CS(13) constructed by this method has the following cycles.

$$\begin{array}{lll} (1,-2,-3,4,-5,6), & (2,-1,-2,5,-4,-6), & (3,0,-1,6,-3,-5), \\ (4,1,0,-6,-2,-4), & (5,2,1,-5,-1,-3), & (6,3,2,-4,0,-2), \\ (-6,4,3,-3,1,-1), & (-5,5,4,-2,2,0), & (-4,6,5,-1,3,1), \\ (-3,-6,6,0,4,2), & (-2,-5,-6,1,5,3), & (-1,-4,-5,2,6,4), \\ (0,-3,-4,3,-6,5). & & \end{array}$$

It is convenient in this construction to number the positions in the cycle from 1 to m, but the cycles themselves from 0 to $2m$. Cycle B is numbered 0 and the cycle obtained from B by adding i is numbered i. Observe that for sufficiently large m, and $1 \leq i < \frac{m}{2} - 2$, and for $\frac{m}{2} \leq i \leq m-2$, the difference between the i^{th} and $(i+2)^{\text{th}}$ entries of B is ± 2. Consequently, considering alternate positions in a cycle, the same pair of points occurs several times in the system, for instance in the 6CS(13) example above, the points -3 and -5 occur in the third and fifth positions in the zeroth cycle, and in the sixth and fourth positions in the fourth cycle. This feature may be exploited to produce a different mCS($2m+1$) by exchanging the intervening points; in our example the 4 in cycle 0 may be exchanged with the -1 in cycle 4. Each edge still occurs exactly once in the new system, but the new system is no longer cyclic. In fact it is automorphism-free and has no proper subsystems as we show in Lemma 3.4 below. Construction 3.1 below gives a general method which, for $k > 4$, produces an automorphism-free $2k$CS($4k+1$) with no proper subsystems. But first we deal with the special cases $k = 3$ and $k = 4$ that do not quite fit this general pattern.

Lemma 3.4 *There exists an automorphism-free 6CS(13) with no proper subsystems.*

Proof As indicated above, we modify the system S produced by the construction of Lemma 3.3 for $m = 6$. The entry 4 at position 4 in the 0^{th} cycle is exchanged with the entry -1 at position 5 in the 4^{th} cycle. The resulting system S' has the following 6-cycles.

$$\begin{array}{lll}
(1,-2,-3,-1,-5,6), & (2,-1,-2,5,-4,-6), & (3,0,-1,6,-3,-5), \\
(4,1,0,-6,-2,-4), & (5,2,1,-5,4,-3), & (6,3,2,-4,0,-2), \\
(-6,4,3,-3,1,-1), & (-5,5,4,-2,2,0), & (-4,6,5,-1,3,1), \\
(-3,-6,6,0,4,2), & (-2,-5,-6,1,5,3), & (-1,-4,-5,2,6,4), \\
(0,-3,-4,3,-6,5). & &
\end{array}$$

Any proper subsystem of S' must contain either the 0^{th} cycle or the 4^{th} cycle, since otherwise the same subsystem would be present in S which, by Lemma 3.2, is known to have no proper subsystems. But if the 0^{th} cycle is in a subsystem of S' then, by considering non-adjacent points in this cycle and locating the corresponding edges in other cycles, it is easy to show that all the other cycles must also be in the subsystem. The same argument applies to the 4^{th} cycle. So S' has no proper subsystems.

The differences between alternate pairs in the cycles of S are $4,6,2,2,6,5$. In the 0^{th} cycle of S' the corresponding differences are $4,1,2,6,6,5$, and in the 4^{th} cycle of S' they are $4,6,3,2,1,5$. In particular, the alternate pair $[6,-1]$ occurs three times in S', in the 0^{th}, 8^{th}, and 11^{th} cycles respectively, whereas no other alternate pair occurs with frequency greater than two. Consequently, either the points 6 and -1 are both fixed or they are transposed by any automorphism of S'. If they are fixed then the automorphism is the identity by Lemma 3.1. Otherwise, by considering the triples of consecutive points $-1,-5,6$ in the 0^{th} cycle, $6,5,-1$ in the 8^{th} cycle, and $6,4,-1$ in the 11^{th} cycle, the point -5 must either be fixed or mapped to either 5 or 4. But by considering the 2^{nd} cycle, which contains the edge $\{-1,6\}$ and also -5, if 6 and -1 are transposed, then -5 must be mapped to 3, a contradiction. Hence the only automorphism of S' is the identity. $\square$

Lemma 3.5 *There exists an automorphism-free* $8CS(17)$ *with no proper subsystems.*

Proof We modify the system S produced by the construction of Lemma 3.3 for $m = 8$. The entry 6 at position 6 in the 0^{th} cycle is exchanged with the entry -3 at position 7 in the 4^{th} cycle. The resulting system S' has the following 8-cycles.

$$\begin{array}{ll}
(1,-2,3,4,-5,-3,-7,8), & (2,-1,4,5,-4,7,-6,-8),\\
(3,0,5,6,-3,8,-5,-7), & (4,1,6,7,-2,-8,-4,-6),\\
(5,2,7,8,-1,-7,6,-5), & (6,3,8,-8,0,-6,-2,-4),\\
(7,4,-8,-7,1,-5,-1,-3), & (8,5,-7,-6,2,-4,0,-2),\\
(-8,6,-6,-5,3,-3,1,-1), & (-7,7,-5,-4,4,-2,2,0),\\
(-6,8,-4,-3,5,-1,3,1), & (-5,-8,-3,-2,6,0,4,2),\\
(-4,-7,-2,-1,7,1,5,3), & (-3,-6,-1,0,8,2,6,4),\\
(-2,-5,0,1,-8,3,7,5), & (-1,-4,1,2,-7,4,8,6),\\
(0,-3,2,3,-6,5,-8,7). &
\end{array}$$

The method used in Lemma 3.4 can be used to prove that S' has no proper subsystems. The differences between alternate pairs in the cycles of S are $2,6,8,2,2,2,8,7$. In the 0^{th} cycle of S' the corresponding differences are $2,6,8,7,2,6,8,7$, and in the 4^{th} cycle of S' they are $2,6,8,2,7,2,1,7$. From these differences it can be seen that, except for the alternate pairs $[4,6]$, $[6,8]$ and $[-1,-3]$, each alternate pair with difference 2 occurs four times in S', whereas these pairs each occur only three times in S'. No other alternate pair occurs three times. Therefore any automorphism can only permute these pairs, and because the point 6 is common to two of them it is fixed by all automorphisms. By Lemma 3.1, an automorphism that fixes more than one point is the identity, so any non-trivial automorphism transposes 4 with 8 and -1 with -3. Thus the 6^{th} cycle, which contains the edge $\{-1,-3\}$, is stabilized. However, this cycle contains 4 but not 8, so the only automorphism is the identity. □

We now deal with the general case.

Construction 3.1
For $k>4$, let S denote the cyclic $2k\mathrm{CS}(4k+1)$ constructed by Lemma 3.3. Put $k=2t$ for k even and $k=2t+1$ for k odd. Then a new system S' may be constructed as follows.

In the case $k=2t$, exchange the $(2t+2)^{\text{th}}$ point of the 0^{th} cycle with the $(4t-3)^{\text{th}}$ point of the $(2t+2)^{\text{th}}$ cycle. In the case $k=2t+1$, exchange the $(2t+4)^{\text{th}}$ point of the 0^{th} cycle with the $(4t-1)^{\text{th}}$ point of the $(2t+2)^{\text{th}}$ cycle.

Lemma 3.6 *The system S' produced by Construction 3.1 for $k>4$ is a $2k$CS$(4k+1)$.*

Proof In order to show that S' is a $2k$CS$(4k+1)$ it must be shown that it is a decomposition of K_{4k+1} into $2k$-cycles. Since S' differs from the $2k$CS$(4k+1)$ S in only two cycles we merely have to show that the two new cycles contain the same edges as the originals, and that the points in each cycle are distinct.

We shall deal first with the case when k is even, $k = 2t$, $t > 2$. The $(2t+1)^{\text{th}}$, $(2t+2)^{\text{th}}$, and $(2t+3)^{\text{th}}$ points of the 0^{th} cycle of the original system S are $-(2t+1)$, $2t+2$, and $-(2t+3)$ respectively, by reference to the expression for the points of the cycle B given in Lemma 3.3.

The $(4t-4)^{\text{th}}$, $(4t-3)^{\text{th}}$, and $(4t-2)^{\text{th}}$ points of the $(2t+2)^{\text{th}}$ cycle of S are obtained by adding $2t+2$ to the corresponding points of the 0^{th} cycle, and so working modulo $8t+1$, these are

$$\begin{array}{lclcl} (4t-4)+2t+2 & = & 6t-2 & = & -(2t+3), \\ -(4t-3)+2t+2 & & & = & -(2t-5), \text{ and} \\ 4t-2+2t+2 & = & 6t & = & -(2t+1) \end{array}$$

respectively. Note from this that the $(2t+1)^{\text{th}}$ point of the 0^{th} cycle of S is equal to the $(4t-2)^{\text{th}}$ point of the $(2t+2)^{\text{th}}$ cycle, and that the $(2t+3)^{\text{th}}$ point of the 0^{th} cycle of S is equal to the $(4t-4)^{\text{th}}$ point of the $(2t+2)^{\text{th}}$ cycle. Hence, if the $(2t+2)^{\text{th}}$ point of the 0^{th} cycle is exchanged with the $(4t-3)^{\text{th}}$ point of the $(2t+2)^{\text{th}}$ cycle, the new cycles contain the same edges as before.

It is also necessary to check that the 0^{th} cycle of S does not contain the point $-(2t-5)$, and that the $(2t+2)^{\text{th}}$ cycle does not contain the point $2t+2$. In the first case this is because if $-(2t-5)$ were in the 0^{th} cycle, it would be the $(2t-5)^{\text{th}}$ point. But referring to Lemma 3.3, the $(2t-5)^{\text{th}}$ point of the cycle is $2t-5$, and $-(2t-5) \not\equiv 2t-5 \pmod{8t+1}$. In the second case, we observe that the $(2t+2)^{\text{th}}$ cycle of S contains the point $2t+2$ only if the 0^{th} cycle contains the point 0. But this is not the case. Hence for the case $k = 2t$ with $t > 2$, S' is a $2k$CS$(4k+1)$.

Now we deal with the case when k is odd, $k = 2t+1$, $t \geq 2$. The $(2t+3)^{\text{th}}$, $(2t+4)^{\text{th}}$, and $(2t+5)^{\text{th}}$ points of the 0^{th} cycle of the original system S are $-(2t+3)$, $2t+4$, and $-(2t+5)$ respectively, by reference to the expression for the points of the cycle B given in Lemma 3.3.

The $(4t-2)^{\text{th}}$, $(4t-1)^{\text{th}}$, and $(4t)^{\text{th}}$ points of the $(2t+2)^{\text{th}}$ cycle of S are obtained by adding $2t+2$ to the corresponding points of the 0^{th} cycle, and so working modulo $8t+5$, these are

$$\begin{array}{lclcl} 4t-2+2t+2 & = & 6t & = & -(2t+5), \\ -(4t-1)+2t+2 & & & = & -(2t-3), \text{ and} \\ 4t+2t+2 & = & 6t+2 & = & -(2t+3) \end{array}$$

respectively. Note from this that the $(2t+3)^{\text{th}}$ point of the 0^{th} cycle of S is equal to the $(4t)^{\text{th}}$ point of the $(2t+2)^{\text{th}}$ cycle, and that the $(2t+5)^{\text{th}}$ point of the 0^{th} cycle of S is equal to the $(4t-2)^{\text{th}}$ point of the $(2t+2)^{\text{th}}$ cycle. Hence, if the $(2t+4)^{\text{th}}$ point of the 0^{th} cycle is exchanged with the $(4t-1)^{\text{th}}$ point of the $(2t+2)^{\text{th}}$ cycle, the new cycles contain the same edges as before.

Again it is necessary to check that the 0^{th} cycle of S does not contain the point $-(2t-3)$, and that the $(2t+2)^{\text{th}}$ cycle does not contain the point $2t+4$. In the first case this is because if $-(2t-3)$ were in the 0^{th} cycle, it would be the $(2t-3)^{\text{th}}$ point. But referring to Lemma 3.3, the $(2t-3)^{\text{th}}$ point of the cycle is $2t-3$. Also, we observe that the $(2t+2)^{\text{th}}$ cycle of S contains the point $2t+4$ only if the 0^{th} cycle contains the point $(2t+4)-(2t+2)=2$. But this is not the case. Hence for the case $k=2t+1$ with $t \geq 2$, S' is a $2k\text{CS}(4k+1)$. $\square$

Lemma 3.7 *The system S' produced by Construction 3.1 for $k>4$ has no proper subsystems.*

Proof We first deal with the case $k=2t$. Any proper subsystem of S' must contain either the 0^{th} cycle or the $(2t+2)^{\text{th}}$ cycle, since otherwise the same subsystem would be present in the original cyclic system S, which has no proper subsystems by Lemma 3.2. If a subsystem contains the 0^{th} cycle then it contains the points 1 and $-(4t-1)$. These appear in adjacent positions in the $(2t+2)^{\text{th}}$ cycle, and so this cycle must also be in the subsystem. Similarly, if a subsystem contains the $(2t+2)^{\text{th}}$ cycle then it contains the points $2t$ and $-(2t+1)$ which are adjacent in the 0^{th} cycle, and so this cycle must also be present in the subsystem. Hence if a subsystem contains either of these cycles then it must contain both. But if a proper subsystem of S' contains both these cycles, then by restoring the exchanged points to their original places, we could produce a proper subsystem of S, contradicting Lemma 3.2. Therefore S' has no proper subsystems.

The argument for $k=2t+1$ is similar. If a subsystem of S' contains the 0^{th} cycle then it contains the points 1 and $-(4t+1)$ which are adjacent in the $(2t+2)^{\text{th}}$ cycle, so this cycle must also be in the subsystem. The reverse case is slightly more complicated. If a subsystem contains the $(2t+2)^{\text{th}}$ cycle then it contains the points $-(2t+3)$ and $-(2t+1)$ which are adjacent in the $(2t)^{\text{th}}$ cycle. So the $(2t)^{\text{th}}$ cycle must also be in the subsystem. But this cycle contains the points $4t+2$ and $-(4t+1)$ which are adjacent in the 0^{th} cycle and which is therefore also in the subsystem. So, as in the previous case, if a subsystem contains either of these cycles, then it must contain both and the rest of the argument is as before. $\square$

We shall call a pair of alternate points in a cycle an *alternate pair*. For the sake of clarity, alternate pairs will be shown in square brackets, for example $[a, b]$. In order to prove that S' has the trivial automorphism group, we examine the frequency of each possible alternate pair in S'. It is convenient to do this first for the cyclic system S where it is only necessary to look at the 0^{th} cycle of S.

Firstly consider the case $k = 2t$, $t > 2$. The 0^{th} cycle of S is

$$(1, -2, 3, \ldots, -(2t-2), 2t-1, 2t, -(2t+1), 2t+2, \ldots, -(4t-1), 4t),$$

where all points are taken modulo $8t+1$. Except for the alternate pairs $[-(2t-2), 2t]$, $[2t-1, -(2t+1)]$, $[-(4t-1), 1]$, and $[4t, -2]$, the absolute difference between alternate pairs is 2. In a single cycle of S, there are therefore $4t-4$ alternate pairs with difference 2, two with difference $4t$, and one each with differences $4t-2$ and $4t-1$ respectively. The frequency of any alternate pair in S is equal to the frequency of pairs with the same difference in a single cycle, so for instance the alternate pair $[1, 3]$ occurs $4t-4$ times in S.

Now we consider S'. The only cycles of S' that are different from those of S are the 0^{th} and $(2t+2)^{\text{th}}$ cycles, and these are

$$(1, -2, 3, \ldots, 2t, -(2t+1), \mathbf{-(2t-5)}, -(2t+3), 2t+4, \ldots, 4t), \quad \text{and}$$
$$(2t+3, 2t, \ldots, -(2t-7), -(2t+3), \mathbf{2t+2}, -(2t+1), -(2t-3), -(2t-1)\ ),$$

where the exchanged points are shown in bold. This has the effect of replacing the alternate pairs $[2t, 2t{+}2]$ and $[2t{+}2, 2t{+}4]$ in the 0^{th} cycle by $[2t, -(2t{-}5)]$ and $[-(2t-5), 2t+4]$, and the pairs $[-(2t-7), -(2t-5)]$ and $[-(2t-5), -(2t-3)]$ in the $(2t+2)^{\text{th}}$ cycle by $[-(2t-7), 2t+2]$ and $[2t+2, -(2t-3)]$. Thus four alternate pairs with differences 2 are replaced by alternate pairs with differences $4t-5$, $4t-1$, $4t-5$, and $4t-1$. In particular we note that in S' the alternate pairs $[2t, 2t+2]$, $[2t+2, 2t+4]$, $[-(2t-7), -(2t-5)]$ and $[-(2t-5), -(2t-3)]$ each occur with frequency $4t-5 \geq 7$, whereas every other alternate pair with difference 2 occurs with frequency $4t-4$, and no other alternate pair occurs with frequency more than two.

Secondly consider the case $k = 2t+1$, $t \geq 2$. The 0^{th} cycle of S is $(1, -2, 3, \ldots, 2t-1, -2t, -(2t+1), 2t+2, -(2t+3), \ldots, -(4t+1), 4t+2)$, where all points are taken modulo $8t+5$. Except for the alternate pairs $[2t-1, -(2t+1)]$, $[-2t, 2t+2]$, $[-(4t+1), 1]$, and $[4t+2, -2]$, the absolute difference between alternate pairs is 2. In a single cycle of S, there are therefore $4t-2$ alternate pairs with difference 2, two pairs with difference $4t+2$, and one each with difference $4t$ and $4t+1$ respectively. All alternate pairs with these differences occur with these frequencies in S.

Considering S', the only cycles of S' that are different from those of S are the 0^{th} and $(2t+2)^{\text{th}}$ cycles, and these are

$(1, -2, 3, \ldots, 2t+2, -(2t+3), \mathbf{-(2t-3)}, -(2t+5), 2t+6, \ldots, 4t+2)$, and
$(2t+3, 2t, \ldots, -(2t-5), -(2t+5), \mathbf{2t+4}, -(2t+3), -(2t-1), -(2t+1)\ )$.

This has the effect of replacing the alternate pairs $[2t+2, 2t+4]$ and $[2t+4, 2t+6]$ in the 0^{th} cycle by the pairs $[2t+2, -(2t-3)]$ and $[-(2t-3), 2t+6]$, and the pairs $[-(2t-5), -(2t-3)]$ and $[-(2t-3), -(2t-1)]$ in the $(2t+2)^{\text{th}}$ cycle by $[-(2t-5), 2t+4]$ and $[2t+4, -(2t-1)]$. Thus four alternate pairs with difference 2 are replaced by alternate pairs with differences $4t-1$, $4t+2$, $4t-1$, and $4t+2$. In particular we note that in S' the alternate pairs $[2t+2, 2t+4]$, $[2t+4, 2t+6]$, $[-(2t-5), -(2t-3)]$ and $[-(2t-3), -(2t-1)]$ each occur with frequency $4t-3 \geq 5$, whereas every other alternate pair with difference 2 occurs with frequency $4t-2$, and no other alternate pair occurs with frequency more than two.

Lemma 3.8 *The system S' produced by Construction 3.1 for $k > 4$ has only the identity automorphism.*

Proof Since by Lemma 3.1 the fixed points of any automorphism of S' form a subsystem, and since by Lemma 3.7 S' has no proper subsystems, no non-trivial automorphism has more than one fixed point. First consider the case $k = 2t$. As shown above, the four alternate pairs $[2t, 2t+2]$, $[2t+2, 2t+4]$, $[-(2t-7), -(2t-5)]$ and $[-(2t-5), -(2t-3)]$ occur with unique frequency $4t-5$. Any automorphism of S' must either permute or stabilize these pairs. Since the point $2t+2$ is common to two pairs, and $-(2t-5)$ is common to the other two, any automorphism either fixes both points or transposes them. Supposing them to be transposed, the automorphism either exchanges

$2t$ with $-(2t-7)$ and $2t+4$ with $-(2t-3)$, or exchanges $2t$ with $-(2t-3)$ and $2t+4$ with $-(2t-7)$.

The 3^{rd} cycle contains the sequence $-(2t-7), 2t, -(2t-5), 2t+2, -(2t-3), -(2t-2)$, starting at the $(2t-4)^{\text{th}}$ position. Since this is the unique cycle containing the edge $\{-(2t-5), 2t+2\}$, the automorphism preserves this cycle and reverses it. But then $-(2t-7)$ is mapped to $-(2t-2)$, and not to $2t$ or $2t+4$. Therefore the only automorphism of S' is the identity.

The case $k = 2t+1$, $t \geq 2$ is treated in a similar way. Suppose there exists a non-trivial automorphism of S'. The alternate pairs $[2t+2, 2t+4]$, $[2t+4, 2t+6]$, $[-(2t-5), -(2t-3)]$ and $[-(2t-3), -(2t-1)]$ are the only ones that occur with frequency $4t-3$ in S'. Therefore a non-trivial automorphism transposes $2t+4$ and $-(2t-3)$, and either exchanges $2t+2$ with $-(2t-5)$ and $2t+6$ with $-(2t-1)$ or exchanges $2t+2$ with $-(2t-1)$ and $2t+6$ with $-(2t-5)$.

However the $(6t+7)^{\text{th}}$ cycle contains the sequence $2t+6, 2t+4, -(2t-3), -2t$ starting at the $(4t+1)^{\text{th}}$ position. Since this is the unique cycle containing the edge $\{2t+4, -(2t-3)\}$, the automorphism preserves this cycle and reverses it. But then $2t+6$ is mapped to $-2t$, and not to $-(2t-1)$ or $-(2t-5)$. Therefore the only automorphism of S' is the identity.

□

Lemmas 3.4, 3.5, 3.6, 3.7 and 3.8 complete the proof of Theorem 3.1.

3.2 Proof of Theorem 3.2

Two different symmetric decompositions of $K_{4k,4k}$ into $2k$-cycles are required. We first show that there is at least one such decomposition.

Theorem 3.4 *For each $k \geq 2$ there exists a symmetric decomposition of $K_{4k,4k}$ into $2k$-cycles.*

Proof The case $k = 2$ was covered in Section 2. For $k > 2$ we present separate constructions for the residue classes modulo 4.

Case 1. For $k \equiv 0, 2 \pmod 4$, we first obtain a symmetric decomposition of $K_{2k,2k}$ into $2k$-cycles. Initially the $k-1$ differences (mod $2k$), excluding k and 0, are arranged into the sequence $1, -2, 3, -4, \ldots, (k-1)$. This has sum $\frac{k}{2}$, which can be seen by pairing consecutive terms, giving $-\frac{(k-2)}{2} + (k-1) = \frac{k}{2}$. This is developed into a sequence of k points starting with 0 by successive addition to obtain $0, 1, -1, 2, -2 \ldots, \frac{k-2}{2}, -\frac{k-2}{2}, \frac{k}{2}$.

Next take the four sequences obtained by adding 0, $\frac{k}{2}$, k and $\frac{3k}{2}$ respectively to each point:

(s1) $0, 1, -1, 2, -2, \ldots, \frac{k-2}{2}, -\frac{k-2}{2}, \frac{k}{2},$

(s2) $\frac{k}{2}, \frac{k+2}{2}, \frac{k-2}{2}, \frac{k+4}{2}, \frac{k-4}{2}, \ldots, k-1, 1, k,$

(s3) $k, k+1, k-1, k+2, k-2, \ldots, \frac{3k-2}{2}, \frac{k+2}{2}, \frac{3k}{2},$

(s4) $\frac{3k}{2}, \frac{3k+2}{2}, \frac{3k-2}{2}, \frac{3k+4}{2}, \frac{3k-4}{2}, \ldots, -1, k+1, 0.$

Now concatenate two copies of (s1), one reversed, label alternately, and join the ends to obtain the $2k$-cycle

(c1) $(0', 1, -1', \ldots, -\frac{k-2}{2}', \frac{k}{2}, \frac{k}{2}', -\frac{k-2}{2}, \ldots, -1', 1', 0)$.

Repeat this process with sequence (s3) to obtain the $2k$-cycle

(c2) $(k', k+1, k-1', \ldots, \frac{k+2}{2}', \frac{3k}{2}, \frac{3k}{2}', \frac{k+2}{2}, \ldots, k-1, k+1', k)$

The cycles (c1) and (c2) each contain two new edges with difference zero. Next concatenate (s2) with (s4) reversed, join the ends and label in both possible ways to obtain the two $2k$-cycles

(c3) $(\frac{k}{2}', \frac{k+2}{2}, \frac{k-2}{2}', \ldots, 1', k, 0', k+1, \ldots, \frac{3k-2}{2}, \frac{3k+2}{2}', \frac{3k}{2})$, and

(c4) $(\frac{k}{2}, \frac{k+2}{2}', \frac{k-2}{2}, \ldots, 1, k', 0, k+1', \ldots, \frac{3k-2}{2}', \frac{3k+2}{2}, \frac{3k}{2}')$.

The cycles (c3) and (c4) each contain two new edges with difference k.

Develop each of (c1), (c2), (c3) and (c4) cyclicly (mod $2k$) for $\frac{k}{2}$ repetitions by adding $0, 1, \ldots, \frac{k}{2}-1$ to obtain a total of $2k$ $2k$-cycles. Collectively these cycles form a symmetric decomposition of $K_{2k,2k}$ into $2k$-cycles.

This can be expanded to a symmetric decomposition of $K_{4k,4k}$ as follows. First partition $K_{4k,4k}$ into four copies of $K_{2k,2k}$ by dividing the two vertex sets into lower and upper halves: $\{0, 1, \ldots, 2k-1\}$ and $\{2k, 2k+1, \ldots, 4k-1\}$. Then decompose each copy of $K_{2k,2k}$ into $2k$-cycles.

This completes the decomposition for $k \equiv 0, 2 \pmod 4$.

Case 2. For $k \equiv 1 \pmod 4$, put $k = 4s+1$, and partition the $2k-2$ differences (mod $4k$), excluding 0, k and $2k$, into two sequences each of length $k-1$. For $k \geq 9$ these are:

$-1, 2, -3, \ldots, -(2s - 1), (2s + 1), -(2s + 2), (2s + 3), \ldots, -4s, (6s + 1),$ and
$-2s, (4s + 2), -(4s + 3), (4s + 4), \ldots, -(6s - 1), 6s, -(6s + 2), (6s + 3), \ldots$
$\ldots, -8s, (8s + 1).$
For example, the sequences for $k = 9$ are $-1, 2, -3, 5, -6, 7, -8, 13$ and $-4, 10, -11, 12, -14, 15, -16, 17$. For $k = 5$ take the sequences as $-1, 3, -4, 7$ and $-2, 6, -8, 9$.

Each sequence sums to k. This can be seen for the first sequence by pairing -1 with $-4s$, 2 with $(4s - 1)$, etc., which sum alternately to $\pm k$, giving $-ks + k(s - 1) + (2s + 1) + (6s + 1) = k$. Similarly for the second sequence, pairing $(4s + 2)$ with $(8s + 1)$, $-(4s + 3)$ with $-8s$, etc., gives $-2s + 3ks - (6s + 2) - 3k(s - 1) = k$. Each of these sequences may be developed into a sequence of k points starting with 0 by successive addition:

(s1) $0, -1, 1, -2, 2, \ldots, -(s - 1), (s - 1), -s, (s + 1), -(s + 1), (s + 2), -(s + 2), \ldots, 2s, -2s, k,$

(s2) $0, -2s, (2s + 2), -(2s + 1), (2s + 3), -(2s + 2), (2s + 4), \ldots, -(3s - 1), (3s + 1), -(3s + 1), (3s + 2), -(3s + 2), \ldots, (k - 1), -(k - 1), k.$

From sequence (s1) we construct $4k$ $2k$-cycles that additionally contain the edges with difference k. From sequence (s2) we construct $2k$ $2k$-cycles that contain the edges with difference zero, and a further $2k$ $2k$-cycles that additionally contain the edges with difference $2k$.

First construct a cycle (c1) by concatenating two copies of sequence (s1), labelling alternate points, and joining the ends:

(c1) $(0', -1, 1', \ldots, -2s, k', 0, -1', 1, \ldots, -2s', k).$

Cycle (c1) has two new edges with difference k. Develop (c1) cyclicly for $4k$ repetitions (mod $4k$) to obtain $4k$ $2k$-cycles.

Next take the four sequences obtained from sequence (s2) by adding 0, k, $2k$ and $3k$ respectively:

(s3) $0, -2s, \ldots, -(k - 1), k,$

(s4) $k, (k - 2s), \ldots, 1, 2k,$

(s5) $2k, (2k - 2s), \ldots, k + 1, 3k,$

(s6) $3k, (3k - 2s), \ldots, 2k + 1, 0.$

To construct the cycles with difference zero, first concatenate two copies of sequence (s3), one reversed, label alternately, and join the ends to obtain the cycle:

(c2) $(0', -2s, \ldots, -(k-1)', k, k', -(k-1), \ldots, -2s', 0)$

Repeat this process with sequence (s5) to form the cycle:

(c3) $(2k', (2k-2s), \ldots, (k+1)', 3k, 3k', (k+1), \ldots, (2k-2s)', 2k)$.

Develop (c2) and (c3) cyclicly for k repetitions by adding $0, 1, \ldots, (k-1)$ to obtain a total of $2k$ $2k$-cycles.

To construct the cycles with difference $2k$, concatenate sequence (s4) with sequence (s6) reversed, join the ends and label in both ways to obtain the cycles:

(c4) $(k', (k-2s), \ldots, 1', 2k, 0', (2k+1), \ldots, (3k-2s)', 3k)$,

(c5) $(k, (k-2s)', \ldots, 1, 2k', 0, (2k+1)', \ldots, (3k-2s), 3k')$.

Develop (c4) and (c5) cyclicly for k repetitions by adding $0, 1, \ldots, (k-1)$ to obtain a further $2k$ $2k$-cycles.

Collectively the cycles formed from (c1) to (c5) form a symmetric decomposition of $K_{4k,4k}$ into $2k$-cycles for $k \equiv 1 \pmod 4$.

Case 3. For $k \equiv 3 \pmod 4$, put $k = 4s+3$. As for Case 1, we decompose $K_{2k,2k}$ into $2k$-cycles, and expand this to a decomposition of $K_{4k,4k}$.

First arrange the $k-1$ differences (mod $2k$), excluding 0 and k, into the sequence $-1, 3, -5, \ldots, (k-4), -(k-2), (k+1), -(k+3), (k+5), -(k+7), \ldots, -(2k-4), (2k-2) = -2$. This sequence sums to k, which can be seen by pairing -1 with $(k+1)$, 3 with $-(k+3)$, etc., giving $k(s+1) - ks = k$. Develop this into a sequence of points starting with 0 by successive addition:

(s1) $0, -1, 2, -3, \ldots, -\frac{k-1}{2}, \frac{k+3}{2}, -\frac{k+3}{2}, \ldots, k-2, -(k-2), k$.

Concatenate two copies of (s1), one reversed, label alternately, and join the ends to obtain the $2k$-cycle:

(c1) $(0', -1, 2', \ldots, -(k-2), k', k, -(k-2)', \ldots, -1', 0)$.

This cycle contains two new edges, each with difference 0. Develop (c1) cyclicly (mod $2k$) for k repetitions by adding $0, 1, \ldots, (k-1)$ to give k $2k$-cycles. Next add k to each point of the sequence (s1) to obtain the sequence:

(s2) $k, (k-1), (k+2), \ldots, 2, 0.$

Concatenate two copies of (s2) end to end, label alternately, and join the ends to obtain the $2k$-cycle:

(c2) $(k', (k-1), (k+2)', \ldots, 2, 0', k, (k-1)', (k+2), \ldots, 2', 0).$

This cycle contains two new edges, each with difference k. Develop this (mod $2k$) for k repetitions by adding $0, 1, \ldots, (k-1)$. The cycles obtained from (c1) and (c2) form a symmetric decomposition of $K_{2k,2k}$ into $2k$-cycles which may be extended to a symmetric decomposition of $K_{4k,4k}$ as explained in Case 1.

This completes the proof of Theorem 3.4. $\square$

To complete the proof of Theorem 3.2 we must show that for each $k \geq 2$ there is a second symmetric decomposition of $K_{4k,4k}$ into $2k$-cycles. Recall that we have already provided two different decompositions of $K_{8,8}$ into 4-cycles in Section 2. For $k > 2$, consider the decomposition $\mathcal{D}$ of $K_{4k,4k}$ constructed in the proof of Theorem 3.4. Denote the vertices of the two parts of $K_{4k,4k}$ by $0, 1, \ldots, (4k-1)$ and $0', 1', \ldots, (4k-1)'$ respectively. For the cases $k \equiv 1, 3 \pmod 4$, the decomposition $\mathcal{D}$ contains a cycle with the edges $\{0, 0'\}$ and $\{k, k'\}$ but not the edge $\{1, 1'\}$. For the cases $k \equiv 0, 2 \pmod 4$ the decomposition $\mathcal{D}$ contains a cycle with the edges $\{0, 0'\}$ and $\{\frac{k}{2}, \frac{k}{2}'\}$ but again not the edge $\{1, 1'\}$. If we alter $\mathcal{D}$ by transposing 1 and 0, and by transposing $1'$ and $0'$ in all cycles, we obtain a new decomposition. In this new decomposition in the cases $k \equiv 1, 3 \pmod 4$ the cycle that contains the edge $\{k, k'\}$ also contains the edge $\{1, 1'\}$ but not the edge $\{0, 0'\}$, and in the cases $k \equiv 0, 2 \pmod 4$ the cycle that contains the edge $\{\frac{k}{2}, \frac{k}{2}'\}$ also contains the edge $\{1, 1'\}$ but not $\{0, 0'\}$. In each case the second decomposition is different from the original in the sense required. This completes the proof of Theorem 3.2. $\square$

Finally, and as noted earlier, the results of Theorems 3.1 and 3.2 are easily combined to prove our main result, Theorem 3.3.

References

[1] B. R. Alspach and H. Gavlas, *Cycle decompositions of K_n and $K_n - I$*, J. Combin. Theory Ser. B, **81** (2001), 77–99.

[2] L. Babai, *On the minimum order of graphs with given group*, Canad. Math. Bull. **17** (1974), 467–470.

[3] L. Babai and A. J. Goodman, *Subdirectly reducible groups and edge-minimal graphs with given automorphism group*, J. London Math. Soc. **47** (1993), 417–432.

[4] M. Buratti *Rotational k-cycle systems of order $v < 3k$; another proof of the existence of odd cycle systems*, J. Combin. Des. **11** (2003), 433–441.

[5] M. Buratti and A. Del Fra, *Existence of cyclic k-cycle systems of the complete graph*, Discrete Math. **261** (2003), 113–125.

[6] I. J. Dejter, P. I. Rivera-Vega and A. Rosa, *Invariants for 2-factorizations and cycle systems*, J. Comb. Math. Comb. Comput. **16** (1987), 129–152.

[7] R. Frucht, *Herstellung von Graphen mit Vorgegebener abstrakter Gruppe*, Compositio Math. **6** (1939), 239–250.

[8] R. Frucht, *Graphs of degree three with a given abstract group*, Canadian J. Math. **1** (1949), 365–378.

[9] D. G. Hoffman, C. C. Lindner and C. A. Rodger, *On the construction of odd cycle systems*, J. Graph Theory **13** (1989), 417–426.

[10] G. J. Lovegrove, *Combinatorial designs and their automorphism groups*, Ph.D. thesis, The Open University, 2008.

[11] E. Mendelsohn, *On groups of automorphisms of Steiner triple and quadruple systems*, Congr. Numer. **13** (1975), 255–264.

[12] E. Mendelsohn, *On groups of automorphisms of Steiner triple and quadruple systems*, J. Combin. Theory Ser. A, **25** (1978), 97–104.

[13] M. Šajna, *Cycle decompositions III: complete graphs and fixed length cycles*, J. Combin. Des. **10** (2002) 27–78.