Bi-embeddings of the projective space $PG(3,2)$

G. K. Bennett, M. J. Grannell and T. S. Griggs,
Department of Pure Mathematics,
The Open University,
Walton Hall, Milton Keynes, MK7 6AA,
United Kingdom.

This is a preprint of an article published in the Journal of Statistical Planning and Inference, 86, 2000, p321-329,

Running head:
Bi-embeddings of $PG(3,2)$

Corresponding author:
M. J. Grannell, 17 Royal Avenue, Leyland, Lancashire, PR25 1BQ, England.
Tel: 01772-433430 email: m.j.grannell@open.ac.uk

AMS classifications:
05B07, 05C10.

Abstract

We prove that, up to isomorphism, there is a unique embedding of the point-line design $PG(3,2)$ with itself in an orientable surface. In addition we prove that there are just three pairwise non-isomorphic embeddings of $PG(3,2)$ with itself in a non-orientable surface.

1 Background

The starting point for this paper is the classic result of Ringel and Youngs [6] that for all $n \equiv 0$, 3, 4 or 7 (mod 12), there exists a triangulation of the complete graph K_n in an orientable surface of appropriate genus. In such an embedding, the triangular faces form a *Mendelsohn triple system* of order n, MTS(n). This comprises a point set V of cardinality n and a collection, $\mathcal{B}$, of cyclically ordered triples of points having the property that every ordered pair of points is contained in precisely one triple. In the case where $n - 1$ is even, i.e. when $n \equiv 3$ or 7 (mod 12), the possibility exists that the faces of the embedding may be coloured alternately black or white in such a way that faces sharing a common edge have different colours. An embedding of this latter form is said to be *face 2-colourable* and the sets of black triangles and white triangles respectively form *Steiner triple systems* of order n, STS(n). Briefly, an STS(n) comprises a point set V of cardinality n and a collection, $\mathcal{B}$, of unordered triples of points having the property that every unordered pair of points is contained in precisely one triple. Face 2-colourable triangular embeddings are also called *bi-embeddings*. The embeddings described in Ringel's book [6] are bi-embeddings when $n \equiv 3$ (mod 12) but not when $n \equiv 7$ (mod 12). Bi-embeddings in the latter case can be found in a later paper by Youngs [8], a fact which, until recently, seems to have been overlooked.

Traditionally in this area of work, the designs obtained have been viewed as a by-product of constructions whose primary purpose is to provide embeddings of complete graphs. Little attention has been paid to the structural properties of the resulting designs. Recently, two of the present authors together with Jozef Širáň of the Slovak Technical University, Bratislava, wrote two papers [3, 4] in which designs were moved to centre stage and attention focused primarily on this aspect of the work. The efficacy of this approach was demonstrated by the fact that, having identified the particular Steiner triple systems obtained in Ringel's $n \equiv 3$ (mod 12) bi-embeddings, it was possible to use this information as the starting point of a simpler proof of the existence of these bi-embeddings. The same flavour also runs through [1] where it is proved that for all $n \equiv 7$ or 19 (mod 36), the number of non-isomorphic bi-embeddings of the complete graph K_n in an orientable surface is at least $2^{n^2/54-O(n)}$, a considerable improvement on what was previously known. The authors refer to this as "Topological Design Theory" rather than the traditional name "Topological Graph Theory" in order to emphasize the design-theoretical aspects of the work.

Three fundamental questions in Topological Design Theory are the following:

(1) Can every Mendelsohn triple system of order $n \equiv 0$, 3, 4 or 7 (mod 12) be embedded in an orientable surface of appropriate genus?

(2) Can every Steiner triple system of order $n \equiv 3$ or 7 (mod 12) be embedded in an orientable surface of appropriate genus?

(3) (An extension of (2)) Can every pair of Steiner triple systems of order $n \equiv 3$ of 7 (mod 12) be bi-embedded in an orientable surface of appropriate genus?

[Note that by this last question we mean that we seek a bi-embedding in which the black and the white systems are isomorphic to the given pair of STS(n)s. Achieving specific realisations of the two systems is clearly not generally possible, for example they may have a common block.]

An answer to question (1) was given by Ducrocq and Sterboul [2]. Let $M = (V, \mathcal{B})$ be a Mendelsohn triple system of order n, where V is the point set and $\mathcal{B}$ is the collection of cyclically ordered triples. For each $x \in V$, the *neighbourhood digraph*, N_x is defined as follows. The set of vertices is $V \setminus \{x\}$. The ordered pair (a, b) is a directed edge if and only if $\langle x, a, b\rangle \in \mathcal{B}$. If, for each $x \in V$, the neighbourhood digraph N_x consists of a single directed cycle then the Mendelsohn triple system is said to be *neighbourhood uniform Hamiltonian.* Ducrocq and Sterboul's result is the following.

Theorem 1 *(Ducrocq and Sterboul [2])*
For all $n \equiv 0, 3, 4$ or 7 (mod 12), a Mendelsohn triple system, M, of order n can be embedded in an orientable surface if and only if M is neighbourhood uniform Hamiltonian.

Proof The cyclically ordered triples form the triangular faces of the embedding and these are glued together along common edges. This forms the required orientable surface which, because the Mendelsohn triple system is neighbourhood uniform Hamiltonian, contains no singular points and is therefore not a pseudosurface. □

Answers, even partial ones, to questions (2) and (3) seem much more difficult to obtain. The bi-embedding results of Ringel and Youngs guarantee only that for all $n \equiv 3$ or 7 (mod 12) there exists a single pair of STS(n)s which can be bi-embedded. In [4] it was suggested that the answer to question (2) may be affirmative. Admittedly the evidence is slim. But, for $n \equiv 7$ (mod 12), the toroidal bi-embedding of the complete graph K_7 is well known and embeddings of the four non-isomorphic cyclic STS(19)s are given in [4]. A study by the present authors, currently in progress, shows that there are **cyclic** embeddings of at least seventy-six of the eighty non-isomorphic cyclic STS(31)s. If the answer to question (2) really is affirmative then this would entail the existence of $n^{O(n^2)}$ non-isomorphic face 2-colourable triangular embeddings of K_n since there are $n^{n^2/6-o(n^2)}$ non-isomorphic STS(n)s [7].

Question (3) seems even more problematical. An approach which might yield more insight is to study possible bi-embeddings of pairs of STS(15)s. It is known that there exist precisely eighty non-isomorphic STS(15)s and these, together with some of their properties, are listed in the survey paper [5]. An invariant of a Steiner triple system, which aids in the distinguishing of non-isomorphic systems, is the number and distribution of *quadrilaterals* or *Pasch configurations*. A Pasch configuration in a Steiner triple system is a set of four triples whose union has cardinality six. Such a configuration is isomorphic to $\{\{a,b,c\},\{a,y,z\},\{x,b,z\},\{x,y,c\}\}$. The number of Pasch configurations which can occur in an STS(n) varies from a minimum of zero, in so-called *anti-Pasch* systems, to the maximum of $n(n-1)(n-3)/24$ which is only achieved in *projective spaces*. The projective space $PG(m,2)$ has point set $(GF(2))^{m+1} \setminus \{0\}$ and lines (or triples) $\{\{a,b,c\} : a+b+c=0\}$; this forms an STS($2^{m+1}-1$). The eighty STS(15)s number amongst their ranks two systems whose quadrilateral content is at the two extremes. In the listing of [5], system #1 is the projective space $PG(3,2)$ having 105 quadrilaterals, and system #80 is anti-Pasch. The bi-embedding described in Ringel's book [6] for $n = 15$ is of the anti-Pasch system with itself. Until now this was the only known embedding of any of the STS(15)s. In this paper we exhibit a bi-embedding of the projective space $PG(3,2)$ with itself. This is of particular interest for a number of reasons:

(1) It is only the second known triangulation of the complete graph K_{15} in an orientable surface, and only the second STS(15) which it is known can be embedded.

(2) In terms of quadrilateral content, the $PG(3,2)$ bi-embedding is at the opposite extreme to the Ringel bi-embedding.

(3) The toroidal embedding of the complete graph K_7 forms a bi-embedding of the projective space $PG(2, 2)$ with itself. We now have a bi-embedding of $PG(3, 2)$ with itself. This raises the general question of whether there is a bi-embedding of $PG(m, 2)$ with itself for all $m \geq 2$, and how such bi-embeddings may be constructed.

(4) The bi-embedding of $PG(3, 2)$ with itself described in this paper is unique up to isomorphism. The bi-embedding of $PG(2, 2)$ is also unique. This naturally raises the question that, if a bi-embedding of $PG(m, 2)$ can be constructed for $m \geq 4$, is it unique?

In Section 2 we describe the method used in constructing the bi-embedding of $PG(3, 2)$ with itself. In Section 3 we analyse the results. We particularly thank Paul Bonnington of the University of Auckland, New Zealand, who first suggested to us that determination of the existence of a $PG(3, 2)$ bi-embedding was computationally feasible. As a by-product of the method we also determine all face 2-colourable triangular embeddings of $PG(3, 2)$ with itself in a non-orientable surface. These embeddings are also given in Section 3.

2 Method

The aim is to produce face 2-colourable triangular embeddings of K_n in orientable or non-orientable surfaces, for the particular case $n = 15$, in which the black and the white systems are both isomorphic to $PG(3, 2)$. In such an embedding there will be 15 vertices, 105 edges and 70 triangular faces, and the genus of the surface may then be determined from Euler's formula. In the orientable case the surface is S_{11}, the sphere with 11 handles, and in the non-orientable case it is $\bar{S}_{22}$, the sphere with 22 crosscaps.

A triangular embedding of K_n may be described by means of a *rotation scheme*. This comprises a set of circularly ordered lists, one for each vertex of K_n. The list corresponding to the vertex x, the *rotation at x*, gives the remaining $n - 1$ vertices in the order in which they appear around x in the given embedding. If the embedding is in an orientable surface then a consistent orientation, say clockwise, may be selected for the entire rotation scheme. As an example, Table 1 gives a rotation scheme for an embedding of K_7 in a torus. In fact this embedding is unique up to isomorphism. The vertices of K_7 are taken to be the points of Z_7.

0:	1	3	2	6	4	5
1:	2	4	3	0	5	6
2:	3	5	4	1	6	0
3:	4	6	5	2	0	1
4:	5	0	6	3	1	2
5:	6	1	0	4	2	3
6:	0	2	1	5	3	4

Table 1. Rotation scheme for embedding K_7.

Given a triangular embedding of K_n, by considering each pair of adjacent triangular faces, $\langle i,j,k\rangle$ and $\langle i,k,l\rangle$, it is easy to see that the rotation scheme must satisfy the following:

Rule R. If the rotation at i contains $\ldots jkl \ldots$ then the rotation at k contains either $\ldots lij \ldots$ or $\ldots jil \ldots$.

The converse is also true (see for example [6], p76), namely a rotation scheme on n points (with the rotation at each point x containing all the $n-1$ points apart from x) which satisfies Rule R represents a triangular embedding of K_n in some surface. The surface may or may not be orientable. It will be orientable if it is possible to orient the rotations at the vertices consistently, i.e. to satisfy the following:

Rule R*. If the rotation at i contains $\ldots jkl \ldots$ then the rotation at k contains $\ldots lij \ldots$.

The necessity of Rule R* may be seen in a similar fashion to that of Rule R. A proof of its sufficiency is given in [6].

We take the vertices of K_{15} to be the elements of Z_{15}. Without loss of generality, the rotation at 0 can be taken as:

$$0: \quad 1\ \ 2\ \ 3\ \ 4\ \ 5\ \ 6\ \ 7\ \ 8\ \ 9\ \ 10\ \ 11\ \ 12\ \ 13\ \ 14$$

Since a face 2-colourable embedding is sought, it can be assumed that, for $i = 1, 2, \ldots, 7$, the triangles $\langle 0, 2i-1, 2i\rangle$ are coloured black and the triangles $\langle 0, 2i, 2i+1\rangle$ (with "15" replaced by "1") are coloured white. We concentrate initially on the white system, the argument regarding the black system being identical.

Given a realisation of $PG(3,2)$ there are precisely $15\times(14\times12\times10\times\ldots\times2)$ ways of mapping the blocks through a single point onto the seven white triples $\{0,2i,2i+1\}$. However, $PG(3,2)$ has an automorphism group of order 20160 [5]. Consequently the number of differently labelled copies of $PG(3,2)$ containing the seven specified triples is $15.2^7.7!/20160 = 480$. We now show how these may be obtained without examining all $15.2^7.7!$ possibilities. The key property of $PG(3,2)$ used is that every intersecting pair of blocks lies in two quadrilaterals and these in turn lie in a unique STS(7).

Consider the two triples $\{0,2,3\}$ and $\{0,4,5\}$. If these are to lie in a copy of $PG(3,2)$ then there exist points a and b such that $\{2,4,a\},\{3,5,a\}$, $\{2,5,b\},\{3,4,b\}$ and $\{0,a,b\}$ are white triples. There are **ten** possible choices for a and, once a is selected, b is determined by the triple $\{0,a,b\}$. Having selected a (and b), choose any triple $\{0,c,d\}$ where $c,d \notin \{0,2,3,4,5,a,b\}$ and consider the intersecting pair $\{0,2,3\}$ and $\{0,c,d\}$. As before there are points e and f such that $\{2,c,e\},\{3,d,e\},\{2,d,f\},\{3,c,f\}$ and $\{0,e,f\}$ are white triples. There are **six** remaining choices for e and, once e is selected, f is determined by the triple $\{0,e,f\}$, Having selected e (and f), choose one of the remaining triples containing 0, say $\{0,g,h\}$ and consider the intersecting pair $\{0,2,3\}$ and $\{0,g,h\}$. We now find that there are points k and l such that $\{2,g,k\},\{3,h,k\},\{2,h,l\},\{3,g,l\}$ and $\{0,k,l\}$ are white triples. There are just **two** choices for k and, once k is selected, then l is determined by the triple $\{0,k,l\}$.

The triples listed above may be chosen in $10\times6\times2 = 120$ ways and they comprise all nineteen white triples through the points 0, 2 and 3. Having chosen these triples, consider any two points which do not appear together in one of these nineteen triples, say x and y. Suppose that the white triple containing these two points is $\{x,y,z\}$. Since x and y appear in a total of six white triples containing 0, 2 and 3, and these triples cover six distinct points, in addition to $0,2,3,x$ and y, we see that z cannot be any one of these eleven points. Thus there are just **four** possible choices for z making a total of $120\times4 = 480$ ways of choosing the foregoing twenty triples. The white system is now fully determined; to see this we show how the remaining fifteen triples can be deduced.

Consider the intersecting triples $\{0,x,u\}$ and $\{0,y,v\}$, where $x,y,u,v \notin \{0,2,3\}$. There is a point w such that $\{x,y,z\},\{u,v,z\},\{x,v,w\},\{y,u,w\}$ and $\{0,z,w\}$ are white triples. Note that $z,w \notin \{0,2,3\}$. Consequently the triples $\{u,v,z\},\{x,v,w\}$ and $\{y,u,w\}$ form three additional triples, making a total of twenty-three. Similarly, by considering the intersecting pairs of triples

$\{2,x,u'\},\{2,y,v'\}$ and $\{3,x,u''\},\{3,y,v''\}$, we generate a further six white triples $\{u',v',z\},\{x,v',w'\},\{y,u',w'\},\{u'',v'',z\},\{x,v'',w''\}$ and $\{y,u'',w''\}$, making a total of twenty-nine triples. The fifteen points $0,2,3,x,y,z,u,v,w,u',v',w',u'',v''$ and w'' are distinct. A quadrilateral containing the intersecting triples $\{0,x,u\}$ and $\{2,x,u'\}$ guarantees that $\{3,u,u'\}$ is a white triple. Likewise, $\{2,u,u''\}$ is a white triple. Hence, amongst the twenty-nine triples so far specified, u has occurred with all but the four points v',w',v'' and w''. A quadrilateral containing the intersecting triples $\{2,y,v'\}$ and $\{3,y,v''\}$ guarantees that $\{0,v',v''\}$ is a white triple. Since $\{x,v',w'\}$ is also a white triple, the pair $\{u,v'\}$ cannot appear in a white triple with either v'' or w'. Consequently $\{u,v',w''\}$ must be a white triple and so therefore also must $\{u,v'',w'\}$. Thus, by this argument, the two remaining white triples containing u are determined and a similar argument regarding u' and u'' gives four additional triples, making thirty-five in all.

There are also precisely 480 black systems containing the specified black triples through 0. By considering the rotation at 0 it can be seen that these may be derived from the 480 white systems by applying the permutation $(0)(14\ 13\ 12\ldots\ 1)$. Putting a black and a white system together, the assumed rotation at 0 together with the lists of black and white triples determines a potential rotation scheme. As a consequence, there are 480×480 potential embeddings of $PG(3,2)$ with itself. Each of these was examined to check firstly that the potential rotation at each vertex indeed comprises a single 14-cycle and, in such cases, secondly that the whole scheme satisfies Rule R. The rotation schemes so identified were then further tested against Rule R* to determine those which are orientable.

The procedure just described does indeed lead to embeddings of $PG(3,2)$ with itself, both orientable and non-orientable. Isomorphisms between these realizations may be determined in the manner given below. The same approach can also be used to determine the automorphism groups. It is convenient to permit mappings which reverse the colours as well as those which preserve them. This is a departure from the terminology of [1] where colour-reversing mappings were disallowed as potential isomorphisms and automorphisms. The difference between the two approaches amounts to at most a factor 2 in the orders of the automorphism groups.

Consider two rotation schemes, R_1 and R_2, defined on the points of Z_n and representing triangular embeddings of K_n. To determine those mappings (if any) $\phi: Z_n\to Z_n$ which take R_1 to R_2 we only need consider $2n(n-1)$ possibilities. For suppose two points x and y are fixed in R_1, then once their images $\phi(x)$ and $\phi(y)$ are chosen in R_2, the circularly ordered rotations at x in R_1 and at $\phi(x)$ in R_2 must correspond. Since y corresponds to $\phi(y)$, the images of the remaining points are determined up to a reversal of one of these rotations. In the case $n = 15$, given two distinct embeddings there are 420 possible mappings which might provide an isomorphism and, similarly, there are 420 possible mappings of an embedding which might provide an automorphism.

3 Results

From the 480×480 possibilities described above, 504 embeddings of $PG(3, 2)$ with itself were identified and these fall into just four isomorphism classes. A representative of each class is given in Table 2. Of these 504 embeddings, 42 satisfy Rule R* and therefore represent orientable bi-embeddings; all of these lie in isomorphism class #1. Classes #2, #3 and #4 contain respectively 210, 210 and 42 embeddings; these satisfy Rule R but cannot be oriented to satisfy Rule R*, and therefore represent non-orientable embeddings of $PG(3, 2)$ with itself. The embeddings from classes #1 and #4 each have (full) automorphism groups of order ten, in fact the dihedral group D_5. Those from classes #2 and #3 have (full) automorphism groups of order two. In each case the automorphisms of odd order preserve the colouration whilst those of even order reverse the colouration. Table 2 also gives generators for the automorphism groups of the four representative embeddings.

Up to isomorphism therefore, there is a *unique* orientable bi-embedding of $PG(3, 2)$ and just three non-orientable embeddings of $PG(3, 2)$ with itself.

Class #1 Representative														
0:	1	2	3	4	5	6	7	8	9	10	11	12	13	14
1:	2	0	14	3	7	11	8	12	4	13	5	10	9	6
2:	0	1	6	10	12	7	14	4	9	5	8	11	13	3
3:	4	0	2	13	10	7	1	14	6	12	11	5	9	8
4:	0	3	8	10	6	11	7	9	2	14	13	1	12	5
5:	6	0	4	12	14	7	10	1	13	8	2	9	3	11
6:	0	5	11	4	10	2	1	9	12	3	14	8	13	7
7:	8	0	6	13	9	4	11	1	3	10	5	14	2	12
8:	0	7	12	1	11	2	5	13	6	14	10	4	3	9
9:	10	0	8	3	5	2	4	7	13	11	14	12	6	1
10:	0	9	1	5	7	3	13	12	2	6	4	8	14	11
11:	12	0	10	14	9	13	2	8	1	7	4	6	5	3
12:	0	11	3	6	9	14	5	4	1	8	7	2	10	13
13:	14	0	12	10	3	2	11	9	7	6	8	5	1	4
14:	0	13	4	2	7	5	12	9	11	10	8	6	3	1
Automorphisms														
Order 2: (7)(0 3)(1 8)(2 4)(5 13)(6 10)(9 14)(11 12)														
Order 5: (0 2 11 14 5)(1 8 10 7 6)(3 13 9 12 4)														

Class #2 Representative														
0:	1	2	3	4	5	6	7	8	9	10	11	12	13	14
1:	2	0	14	3	9	11	5	7	13	4	10	12	6	8
2:	0	1	8	5	7	6	4	14	9	12	10	11	13	3
3:	4	0	2	13	10	6	5	9	1	14	8	12	11	7
4:	0	3	7	12	8	11	14	2	6	9	13	1	10	5
5:	6	0	4	10	14	11	1	7	2	8	13	12	9	3
6:	0	5	3	10	8	1	12	14	13	11	9	4	2	7
7:	8	0	6	2	5	1	13	10	9	14	12	4	3	11
8:	0	7	11	4	12	3	14	10	6	1	2	5	13	9
9:	10	0	8	13	4	6	11	1	3	5	12	2	14	7
10:	0	9	7	13	3	6	8	14	5	4	1	12	2	11
11:	12	0	10	2	13	6	9	1	5	14	4	8	7	3
12:	0	11	3	8	4	7	14	6	1	10	2	9	5	13
13:	14	0	12	5	8	9	4	1	7	10	3	2	11	6
14:	0	13	6	12	7	9	2	4	11	5	10	8	3	1
Automorphism														
Order 2: (10)(0 14)(1 13)(2 6)(3 12)(4 7)(5 9)(8 11)														

(Continued on the next page)

Class #3 Representative

0:	1	2	3	4	5	6	7	8	9	10	11	12	13	14
1:	2	0	14	3	5	8	6	7	9	11	12	10	13	4
2:	0	1	4	14	5	7	10	11	6	8	12	9	13	3
3:	4	0	2	13	8	9	12	6	10	7	11	5	1	14
4:	0	3	14	2	1	13	7	9	10	8	11	6	12	5
5:	6	0	4	12	7	2	14	9	11	3	1	8	10	13
6:	0	5	13	9	14	10	3	12	4	11	2	8	1	7
7:	8	0	6	1	9	4	13	11	3	10	2	5	12	14
8:	0	7	14	11	4	10	5	1	6	2	12	13	3	9
9:	10	0	8	3	12	2	13	6	14	5	11	1	7	4
10:	0	9	4	8	5	13	1	12	14	6	3	7	2	11
11:	12	0	10	2	6	4	8	14	13	7	3	5	9	1
12:	0	11	1	10	14	7	5	4	6	3	9	2	8	13
13:	14	0	12	8	3	2	9	6	5	10	1	4	7	11
14:	0	13	11	8	7	12	10	6	9	5	2	4	3	1

Automorphism

Order 2: (11)(0 13)(1 8)(2 3)(4 9)(5 6)(7 10)(12 14)

Class #4 Representative

0:	1	2	3	4	5	6	7	8	9	10	11	12	13	14
1:	2	0	14	3	11	9	7	6	8	5	13	4	12	10
2:	0	1	10	11	14	4	8	6	12	9	5	7	13	3
3:	4	0	2	13	6	12	7	10	14	1	11	5	8	9
4:	0	3	9	7	11	6	10	8	2	14	13	1	12	5
5:	6	0	4	12	14	9	2	7	10	13	1	8	3	11
6:	0	5	11	4	10	14	9	13	3	12	2	8	1	7
7:	8	0	6	1	9	4	11	13	2	5	10	3	12	14
8:	0	7	14	11	13	12	10	4	2	6	1	5	3	9
9:	10	0	8	3	4	7	1	11	12	2	5	14	6	13
10:	0	9	13	5	7	3	14	6	4	8	12	1	2	11
11:	12	0	10	2	14	8	13	7	4	6	5	3	1	9
12:	0	11	9	2	6	3	7	14	5	4	1	10	8	13
13:	14	0	12	8	11	7	2	3	6	9	10	5	1	4
14:	0	13	4	2	11	8	7	12	5	9	6	10	3	1

Automorphisms

Order 2: (0)(6)(13)(1 11)(2 10)(3 9)(4 8)(5 7)(12 14)

Order 5: (0 1 4 8 11)(2 13 10 14 12)(3 5 6 7 9)

Table 2. Isomorphism class representatives and their automorphisms.

References

[1] C. P .Bonnington, M. J. Grannell, T. S. Griggs and J. Širáň, *Exponential families of non-isomorphic triangulations of complete graphs*, submitted.

[2] P. M. Ducrocq and F. Sterboul, *Les G-systemes triples*, Ann. Discrete Math. **9** (1980), 141-145.

[3] M. J. Grannell, T. S. Griggs and J. Širáň, *Face 2-colourable triangular embeddings of complete graphs*, J. Combin. Theory (B) **74** (1998), 8-19.

[4] M. J. Grannell, T. S. Griggs and J. Širáň, *Surface embeddings of Steiner triple systems*, J. Combin. Designs **6** (1998), 325-336.

[5] R.A. Mathon, K.T. Phelps and A. Rosa, *Small Steiner triple systems and their properties*, Ars Combinatoria **15** (1983), 3-110.

[6] G. Ringel, *Map Color Theorem*, Springer, 1974.

[7] R. M. Wilson, *Nonisomorphic Steiner triple systems*, Math. Z. **135** (1974), 303-313.

[8] J. W. T. Youngs, *The mystery of the Heawood conjecture*, in: "Graph Theory and its Applications" (B. Harris, Ed.), Academic Press, 1970, 17-50.