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COUNTABLY INFINITE STEINER TRIPLE SYSTEMS

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1 Introduction

A Steiner system is an ordered pair (V, B) where V is a set and B is a collection of k-element subsets of V, usually referred to as blocks, which have the property that every t-element subset of V is contained in precisely one block. In order to exclude trivial cases it is often assumed that $1 < t < k$.

The vast literature on Steiner systems which now exists (for a detailed bibliography see [2] or [3]) almost all relates to the case when the set V is finite; there appears to be very little published work on infinite Steiner systems. In this paper we confine our attention to the case where $t = 2$ and $k = 3$, so-called Steiner triple systems. There is a well established body of knowledge concerning such systems when the set V is finite. Our aim is to prove various results in the case where V has cardinality $\aleph_0$, We call such a system a countably infinite Steiner triple system, henceforth abbreviated to CIST.

2 Existence and Enumeration

In this section we give explicit constructions which answer the existence and enumeration question for CISTs. Our main result is that the number of non-isomorphic countably infinite Steiner triple systems has the cardinality of the continuum. The proof is in three stages. First we prove the existence of CISTs by giving a simple direct construction. Secondly we examine the quadrilateral structure of the constructed systems. Finally we complete the enumeration to establish the main result.

STAGE 1 - Existence

We take as our representation of a countably infinite set, $\bar{Q} = Q \cup \{\infty, -\infty\}$ where Q is the set of rationals. The construction of a CIST comprises three types of block which we consider separately. The second type of block depends on the selection of a certain function f and we shall denote the resulting CIST by CIST(f).

Block Type A. These consist of all blocks $\{x, y, z\}$ where $x + y + z = 0$ and x, y, z are unequal with $x, y, z \in Q$. Clearly all pairs of distinct rationals except those of the form $\{q, -2q\}$ where $q \in Q \setminus \{0\}$ will be contained in precisely one block of this type.

Block Type B. Let $f : \{r \in Q : \frac{1}{2} \leq |r| < 1\} \to \{-1, 1\}$. The blocks of type B are then taken to be all the blocks of the form

$$\{(-2)^s r,\ (-2)^{s+1} r,\ (-1)^s f(r)\infty\} \text{ where } s \text{ is an integer.}$$

(Here, and subsequently, we take $(-1)\infty$ to be $-\infty$.) Note that for any $q \in Q \setminus \{0\}$ there exists a unique $r \in Q$ such that $\frac{1}{2} \leq |r| < 1$ and integer s such that $q = (-2)^s r$. Blocks of type B exactly cover all pairs of the form $\{q, -2q\}, \{q, \infty\}$ and $\{q, -\infty\}$ where $q \in Q \setminus \{0\}$.

Block Type C. To complete the CIST(f) define a final block $\{0, \infty, -\infty\}$.

STAGE 2 - Quadrilateral structure

DEFINITION. A quadrilateral consists of four blocks of a Steiner triple system whose union has cardinality six. It is clear that a quadrilateral must have the following configuration. $\{x, y, z\}, \{a, b, z\}, \{a, y, c\}, \{x, b, c\}$.

DEFINITION. The quadrilateral graph of a Steiner triple system is the graph with vertex set the set of quadrilaterals and edge set the joins of vertices (i.e. quadrilaterals) which have at least one block in common.

NOTE. Both the number of quadrilaterals and the quadrilateral graph of a Steiner triple system are invariants. Therefore two Steiner triple systems with non-isomorphic quadrilateral graphs must themselves be non-isomorphic.

We firstly establish the three following lemmas.

LEMMA 1. No quadrilateral of a CIST(f) as constructed above contains the block $\{0, \infty, -\infty\}$.

Proof: Suppose a quadrilateral contained $\{0, \infty, -\infty\}$. Then since each element occurs in precisely two blocks, the element 0 must appear in one more block and from our construction this must be of the form $\{0, x, -x\}$ where $x \in Q \setminus \{0\}$. Similarly ∞ must occur in one more block and from our construction this has the form $\{\infty, y, -2y\}$ for $y \in Q \setminus \{0\}$. From the three blocks $\{0, \infty, -\infty\}, \{0, x, -x\}, \{\infty, y, -2y\}$ we must have a maximum of six distinct elements. Hence we may assume that either $y = x$ or $-2y = x$. Therefore the third block has the form $\{\infty, x, -2x\}$ or $\{\infty, -x/2, x\}$.

Finally the structure of a quadrilateral requires that $-\infty$ and $-x$ occur again in one block. Thus the fourth block must have one of the forms $\{-\infty, -x, 2x\}$ or $\{-\infty, -x, x/2\}$ from our construction. None of the four possibilities for the third and fourth blocks leads to a quadrilateral since in each case the cardinality of the union of the four blocks is greater than six.

LEMMA 2. No quadrilateral of a CIST(f) contains four blocks of type A.
Proof: Suppose some quadrilateral did and it was

$$\{x, y, z\}, \{a, b, z\}, \{a, y.c\}, \{x, b, c\}$$

where x, y, z, a, b, c are distinct elements of Q. By the construction, $x+y+z = a+b+z = a+y+c = x+b+c = 0$. From these equations we obtain $x+y = a+b$ and $x - y = a - b$, and so $x = a, y = b, z = c$. But this is a contradiction.

LEMMA 3. No quadrilateral of a CIST(f) contains four blocks of type B.
Proof: Suppose some quadrilateral did. Then it would have one of the two following forms:

(i) $\{\infty, x, -2x\}, \{\infty, y, -2y\}, \{-\infty, x, y\}, \{-\infty, -2x, -2y\}$

(ii) $\{\infty, x, -2x\}, \{\infty, y, -2y\}, \{-\infty, x, -2y\}, \{-\infty, -2x, y\}$

where $x, y, -2x, -2y$ are distinct elements of $Q \setminus \{0\}$.

In case (i) the existence of the third block gives $x = -2y$ or $y = -2x$, both of which are contradictions. In case (ii) the third block gives $x = 4y$ and the fourth block gives $y = 4x$ and these together form a contradiction.

DISCUSSION. Lemmas 1, 2 and 3 establish that any quadrilateral in a CIST(f) must contain two blocks of type A and two blocks of type B. The two type B blocks must have the form $\{\pm\infty, x, -2x\}$ and $\{\pm\infty, y, -2y\}$ where $x, y, -2x, -2y$ are distinct elements of $Q \setminus \{0\}$. There are just two possibilities for the two type A blocks:

(i) $\{x, -2y, z\}$ and $\{y, -2x, z\}$

(ii) $\{x, y, z\}$ and $\{-2x, -2y, z\}$

where $z \in Q$. In case (i) we have $x - 2y + z = 0$ and $y - 2x + z = 0$ giving $x = y$ which is a contradiction. In case (ii) we have $x + y + z = 0$ and $-2x - 2y + z = 0$ and so $z = 0$ and $y = -x$. Thus any quadrilateral in a CIST(f) must have the form

$$\{\pm\infty, x, -2x\}, \{\pm\infty, -x, 2x\}, \{0, x, -x\}, \{0, 2x, -2x\},$$

where $x \in Q \setminus \{0\}$.

Further it is clear that if such a CIST(f) contains a quadrilateral with the above blocks it also will contain by our construction the two quadrilaterals with blocks

$$\{\mp\infty, \frac{x}{2}, -x\}, \{\mp\infty, -\frac{x}{2}, x\}, \{0, \frac{x}{2}, -\frac{x}{2}\}, \{0, x, -x\} \quad \text{and}$$

$$\{\mp\infty, 2x, -4x\}, \{\mp\infty, -2x, 4x\}, \{0, 2x, -2x\}, \{0, 4x, -4x\}$$

and it is evident that these are the only quadrilaterals in the CIST having a block in common with the original quadrilateral. In addition each of these two quadrilaterals will have a block in common with another quadrilateral namely

$$\{\pm\infty, \frac{x}{4}, -\frac{x}{2}\}, \{\pm\infty, -\frac{x}{4}, \frac{x}{2}\}, \{0, \frac{x}{4}, -\frac{x}{4}\}, \{0, \frac{x}{2}, -\frac{x}{2}\} \quad \text{and}$$

$$\{\pm\infty, 4x, -8x\}, \{\pm\infty, -4x, 8x\}, \{0, 4x, -4x\}, \{0, 8x, -8x\}$$

respectively.

By continued application of this argument it is clear that what is obtained is a doubly infinite chain in the quadrilateral graph.

The existence of quadrilaterals in a CIST(f) depends on the choice of f. By choosing f such that $f(r) \neq f(-r)$ for all r satisfying $\frac{1}{2} \leq r < 1$ no quadrilaterals will appear in the resulting CIST(f). Steiner triple systems without quadrilaterals are known as anti-Pasch systems. In the case of finite Steiner triple systems the spectrum of v for which anti-Pasch Steiner triple systems exist appears not to be completely determined. In [1] Doyen gives a construction for anti-Pasch Steiner triple systems when $v \equiv 3 \pmod 6$ and is relatively prime to 7. The discussion above enables us to state the following theorem.

THEOREM 1. There exists a countably infinite Steiner triple system having no quadrilaterals i.e. an anti-Pasch CIST.

By choosing f such that $f(r) = f(-r)$ for precisely one r satisfying $\frac{1}{2} \leq r < 1$ a single doubly infinite chain of quadrilaterals will be present in the quadrilateral graph of the resulting CIST(f). If f is chosen arbitrarily then there will be a doubly infinite chain of quadrilaterals for each r satisfying $\frac{1}{2} \leq r < 1$ for which $f(r) = f(-r)$; moreover these chains will be mutually disjoint.

STAGE 3 - Enumeration

We are now in a position to prove the main result of this section.

THEOREM 2. The number of countably infinite Steiner triple systems has the cardinality of the continuum.

Proof: The method of the proof is to start with a CIST(f) having exactly one doubly infinite chain in its quadrilateral graph. From this system we obtain $2^{\aleph_0}$ CISTs each of which is obtained by minor modifications of the original CIST(f). The modifications break the chain in the quadrilateral graph and this fracturing can be controlled to give $2^{\aleph_0}$ distinct patterns. The corresponding CISTs are therefore non-isomorphic. Initially, then, choose any f for which there is precisely one $r \in Q$ satisfying $\frac{1}{2} \leq r < 1$ and $f(r) = f(-r)$. For example

$$f(r) = \begin{cases} \;\;1 & \text{if } \frac{1}{2} \leq r < 1, \text{ or if } r = -\frac{1}{2}, \\ -1 & \text{if } -1 < r < -\frac{1}{2}. \end{cases}$$

The quadrilateral graph of such a CIST(f) comprises a single doubly infinite chain.

If a CIST contains a quadrilateral

$$\{\pm\infty, x, -2x\}, \{\pm\infty, -x, 2x\}, \{0, x, -x\}, \{0, 2x, -2x\},$$

then we define the opposite quadrilateral to be

$$\{0, -x, 2x\}, \{0, x, -2x\}, \{\pm\infty, 2x, -2x\}, \{\pm\infty, x, -x\},$$

that is the quadrilateral obtained from the original by set complementation. It is obvious that replacing a quadrilateral in a Steiner triple system by its opposite still gives a Steiner triple system.

Returning now to the CIST(f) constructed above, we investigate the effect on the quadrilateral graph of replacing a quadrilateral of the CIST by its opposite. Firstly note that the vertex corresponding to the quadrilateral, its two adjacent vertices and all edges incident with any of these vertices are removed from the graph. It remains to ascertain what new structure is introduced. Observe that all new blocks are either of the form $\{0, x, -2x\}$ which we refer to as type D, or $\{\pm\infty, x, -x\}$ type E, where $x \in Q \setminus \{0\}$.

Any new vertex that is formed must contain a block of one of these types. Recalling that each element occurs in precisely two blocks of a quadrilateral there are six possibilities to consider. The quadrilateral must contain at least one of the following.

(i) two blocks of type D,

(ii) one block of type D and one of type A,

(iii) one block of type D and the block of type C,

(iv) two blocks of type E,

(v) one block of type E and one of type B,

(vi) one block of type E and the block of type C.

We investigate each of these in turn and reduce the possibilities to four different types (I), (II), (III) and (IV) given below.

(i) the quadrilateral contains blocks $\{0, x, -2x\}$ and $\{0, y, -2y\}$, both of type D. Then the remaining two blocks are of one of the forms:

  (a) $\{x, y, z\}$ and $\{-2x, -2y, z\}$ where $z \in Q \setminus \{0\}$. These are both of type A giving $x + y + z = -2x - 2y + z = 0$. Hence $z = 0$ and we have a contradiction.

  (b) $\{x, y, \pm\infty\}$ and $\{-2x, -2y, \pm\infty\}$. If the first of these was of type B then either $y = -2x$ or $x = -2y$ giving a repeated pair in each case. So we are left with $\{x, y, \pm\infty\}$ being of type E giving $y = -x$ and we have an opposite quadrilateral (Type I).

  (c) $\{x, -2y, z\}$ and $\{-2x, y, z\}$ where $z \in Q \setminus \{0\}$. Again these are both of type A so $x - 2y + z = y - 2x + z = 0$. Hence $x = y$ and we have a contradiction.

  (d) $\{x, -2y, \pm\infty\}$ and $\{-2x, y, \pm\infty\}$. If the first of these was of type B then since $x \neq y$, we must have $x = 4y$ and no such second block exists. If the first block was of type E then $x = 2y$ and again no such second block exists.

(ii) the quadrilateral contains blocks $\{0, x, -2x\}$ of type D and $\{0, y. - y\}$ of type A. Again the remaining two blocks are of one of the forms:

(a) $\{x, y, z\}$ and $\{-2x, -y, z\}$ where $z \in Q \setminus \{0\}$. These are both of type A giving $x + y + z = -2x - y + z = 0$. Hence $y = -3x/2$ and $z = x/2$ yielding the quadrilateral of Type II

$$\{0, x, -2x\}, \{0, -3x/2, 3x/2\}, \{x, -3x/2, x/2\}, \{-2x, 3x/2, x/2\}.$$

(b) $\{x, y, \pm\infty\}$ and $\{-2x, -y, \pm\infty\}$. If the first block was of type B then either $y = -2x$ giving a repeated pair or $y = -x/2$ and no such second block exists. If the first block was of type E then $y = -x$ giving a repeated pair.

(c) $\{x, -y, z\}$ and $\{-2x, y, z\}$ where $z \in Q \setminus \{0\}$. This becomes the same as (a) by replacing y by $-y$ and yields the same quadrilateral (Type II).

(d) $\{x, -y, \pm\infty\}$ and $\{-2x, y, \pm\infty\}$. Again this becomes the same as (b) by replacing y by $-y$ and may be eliminated in the same manner.

(iii) the quadrilateral contains blocks $\{0, x, -2x\}$ of type D and $\{0, \infty, -\infty\}$ of type C. The remaining two blocks are of the form $\{x, y, \pm\infty\}$ and $\{-2x, y, \mp\infty\}$ where $y \in Q \setminus \{0\}$, and with obvious notation as to selection of the "sign of the ∞s". If the first block was of type B then either $y = -2x$ which is clearly impossible or $y = -x/2$ and no such second block exists. If the first block was of type E then $y = -x$ and again no such second block exists.

(iv) the quadrilateral contains blocks $\{\pm\infty, x, -x\}$ and $\{\pm\infty, y, -y\}$ both of type E. The remaining two blocks are of one of the forms:

(a) $\{0, x, y\}$ and $\{0, -x, -y\}$. If the first block was of type A then $y = -x$ which is impossible. If the first block was of type D then either $y = -2x$ or $x = -2y$ and we have an opposite quadrilateral (Type I).

(b) $\{x, y, z\}$ and $\{-x, -y, z\}$ where $z \in Q \setminus \{0\}$. These blocks are of type A. Hence $x + y + z = -x - y + z = 0$ giving $z = 0$ and we have a contradiction.

(c) $\{x, y, \mp\infty\}$ and $\{-x, -y, \mp\infty\}$. If the first block was of type B then either $y = -2x$ or $x = -2y$ yielding without loss of generality the quadrilateral of Type IV

$$\{\pm\infty, x, -x\}, \{\pm\infty, 2x, -2x\}, \{\mp\infty, x, -2x\}, \{\mp\infty, -x, 2x\}.$$

If the first block was of type E then $y = -x$ which is impossible.

Now we must go on to consider the last two blocks of the forms $\{x, -y, \ \}$ and $\{-x, y, \ \}$ where again the missing element may be either $0, z \in Q \setminus \{0\}$ or $\mp\infty$. These are treated exactly similarly to (a), (b) or (c) above respectively where now x is replaced by $-x$. Again the same two quadrilaterals are obtained.

(v) the quadrilateral contains blocks $\{\pm\infty, x, -x\}$ of type E and $\{\pm\infty, y, -2y\}$ of type B. The remaining two blocks are of one of the forms:

(a) $\{0, x, y\}$ and $\{0, -x, -2y\}$. If the first block was of type D then either $y = -2x$ and no such second block exists or $y = -x/2$ and we have a repeated pair. If the first block was of type A then $y = -x$ which is impossible.

(b) $\{x, y, z\}$ and $\{-x, -2y, z\}$ where $z \in Q \setminus \{0\}$. Both blocks are of type A. Hence $x + y + z = -x - 2y + z = 0$ giving $y = -2x/3$ and $z = -x/3$ yielding the quadrilateral of Type III $\{x, -2x/3, -x/3\}, \{-x, 4x/3, -x/3\}, \{\pm\infty, x, -x\}, \{\pm\infty, -2x/3, 4x/3\}$.

(c) $\{x, y \mp \infty\}$ and $\{-x, -2y, \mp\infty\}$. If the first block was of type B then either $y = -2x$ and no such second block exists or $y = -x/2$ and we have a repeated pair. If the first block was of type E then $y = -x$ and again we have a repeated pair.

Now we must go on to consider the last two blocks of the forms $\{-x, y, \ \}$ and $\{x, -2y, \ \}$ where again the missing element may be either $0, z \in Q \setminus \{0\}$ or $\mp\infty$. These are treated exactly similarly to (a), (b) or (c) above respectively where now x is replaced by $-x$. Again the only quadrilateral obtained is the one as in (b) above but with $-x$ replacing x, i.e. Type III

$$\{-x, 2x/3, x/3\}, \{x, -4x/3, x/3\}, \{\pm\infty, x, -x\}, \{\pm\infty, 2x/3, -4x/3\}.$$

(vi) the quadrilateral contains blocks $\{\pm\infty, x, -x\}$ of type E and $\{0, \infty, -\infty\}$ of type C. The remaining two blocks are of one of the forms:

(a) $\{x, \mp\infty, y\}$ and $\{-x, 0, y\}$ where $y \in Q \setminus \{0\}$. From the second of these blocks $y \neq -x$ so the first block is not of type E. It therefore must be of type B. Hence either $y = -2x$ or $y = -x/2$ and no such second block exists in either case.

(b) $\{-x, \mp\infty, y\}$ and $\{x, 0, y\}$ where $y \in Q \setminus \{0\}$. This is the same as (a) with x replaced by $-x$ and is eliminated in the same manner.

Summarising, four new types of quadrilateral may appear, of one of the following forms:

(I) $\{0, -x, 2x\}, \{0, x, -2x\}, \{\pm\infty, 2x, -2x\}, \{\pm\infty, x, -x\}$ which is an opposite quadrilateral.

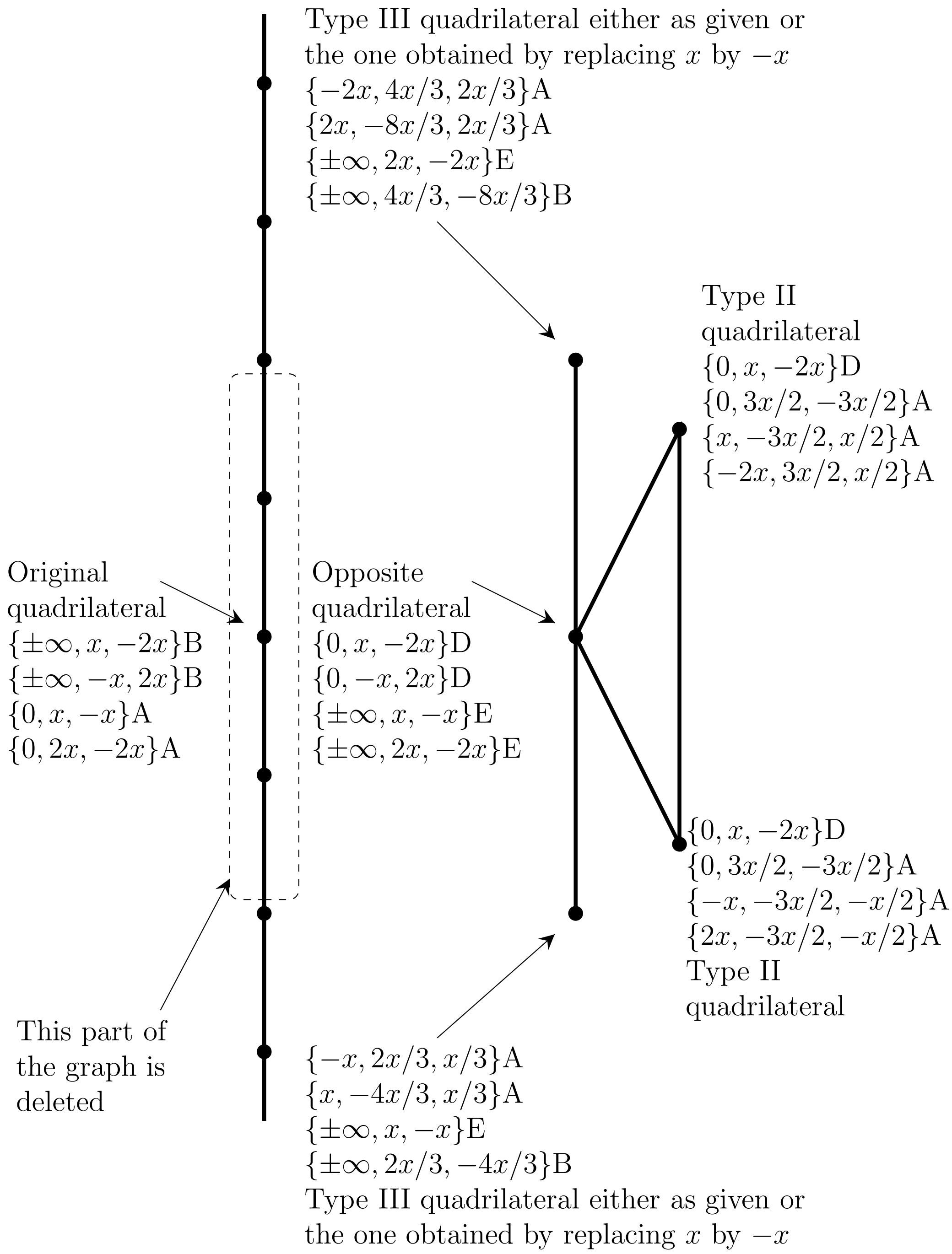
(II) $\{0, x, -2x\}, \{0, -3x/2, 3x/2\}, \{x, -3x/2, x/2\}, \{-2x, 3x/2, x/2\}$.

(III) $\{x, -2x/3, -x/3\}, \{-x, 4x/3, -x/3\}, \{\pm\infty, x, -x\}, \{\pm\infty, -2x/3, 4x/3\}$.

(IV) $\{\pm\infty, x, -x\}, \{\pm\infty, 2x, -2x\}, \{\mp\infty, x, -2x\}, \{\mp\infty, -x, 2x\}$.

However in possibility (IV) the only way in which blocks $\{\pm\infty, x, -x\}$ and $\{\pm\infty, 2x, -2x\}$ can appear in the CIST is either by replacing the quadrilateral $\{\pm\infty, x, -2x\}, \{\pm\infty, -x, 2x\}, \{0, x, -x\}, \{0, 2x, -2x\}$ by its opposite in which case neither of the blocks $\{\mp\infty, x, -2x\}$ and $\{\mp\infty, -x, 2x\}$ exist or by replacing both of the quadrilaterals which are adjacent to the above in the quadrilateral graph i.e. quadrilaterals corresponding to two vertices a distance two apart. Since in what follows we never replace quadrilaterals less than a distance three apart, quadrilaterals of type (IV) do not appear and can be disregarded.

So the overall effect on the quadrilateral graph of replacing a quadrilateral in the CIST can be illustrated in the following diagram. The infinite chain is broken and an offshoot configuration obtained.



Type III quadrilateral either as given or the one obtained by replacing x by $-x$
$\{-2x, 4x/3, 2x/3\}$A
$\{2x, -8x/3, 2x/3\}$A
$\{\pm\infty, 2x, -2x\}$E
$\{\pm\infty, 4x/3, -8x/3\}$B
Type II quadrilateral
$\{0, x, -2x\}$D
$\{0, 3x/2, -3x/2\}$A
$\{x, -3x/2, x/2\}$A
$\{-2x, 3x/2, x/2\}$A
Original quadrilateral
$\{\pm\infty, x, -2x\}$B
$\{\pm\infty, -x, 2x\}$B
$\{0, x, -x\}$A
$\{0, 2x, -2x\}$A
Opposite quadrilateral
$\{0, x, -2x\}$D
$\{0, -x, 2x\}$D
$\{\pm\infty, x, -x\}$E
$\{\pm\infty, 2x, -2x\}$E
$\{0, x, -2x\}$D
$\{0, 3x/2, -3x/2\}$A
$\{-x, -3x/2, -x/2\}$A
$\{2x, -3x/2, -x/2\}$A
Type II quadrilateral
This part of the graph is deleted
$\{-x, 2x/3, x/3\}$A
$\{x, -4x/3, x/3\}$A
$\{\pm\infty, x, -x\}$E
$\{\pm\infty, 2x/3, -4x/3\}$B
Type III quadrilateral either as given or the one obtained by replacing x by $-x$

In the CIST(f) which we are considering let r_0 be the unique rational in $[\frac{1}{2}, 1)$ such that $f(r_0) = f(-r_0)$. Then, in the above diagram, x is of the form $\pm(-2)^s r_0$ for some $s \in Z$ because only if x is of this form can both the blocks $\{\pm\infty, x, -2x\}$ and $\{\pm\infty, -x, 2x\}$ exist with the same choice of the signs of the ∞s. Since $2x/3$ and $4x/3$ are then not of the form $\pm(-2)^s r_0$ for any $s \in Z$ there are only two type III quadrilaterals to which each opposite quadrilateral is adjacent, as indicated. For the same reason and because $3x/2$ is also not of the form $\pm(-2)^s r_0$ for any $s \in Z$, none of the type II or type III quadrilaterals can be adjacent to any quadrilateral in the original doubly infinite chain. Again the fact that we never replace quadrilaterals corresponding to vertices in the original doubly infinite chain less than a distance three apart ensures that no new type of quadrilateral is adjacent to any other new type of quadrilateral other than as indicated on the diagram i.e. the effect of the replacement operation is local to each vertex representing the quadrilateral replaced.

We are now finally in a position to construct $2^{\aleph_0}$ pairwise non-isomorphic CISTs. Choose any subset S of Z^+. First replace the quadrilateral

$$\{\pm\infty, r_0, -2r_0\}, \{\pm\infty, -r_0, 2r_0\}, \{0, r_0, -r_0\}, \{0, 2r_0, -2r_0\},$$

in the original CIST(f) by the opposite quadrilateral

$$\{0, -r_0, 2r_0\}, \{0, r_0, -2r_0\}, \{\pm\infty, 2r_0, -2r_0\}, \{\pm\infty, r_0, -r_0\},$$

The quadrilateral graph of the CIST so obtained will consist of two singly infinite chains and the offshoot configuration.

Then for each $n \in S$, compute $m = m(n) = \sum_{\substack{i \in S \\ i \leq n}} (i+3)$.

Replace each quadrilateral
$\{\pm\infty, 2^m r_0, -2^{m+1} r_0\}, \{\pm\infty, -2^m r_0, 2^{m+1} r_0\}, \{0, 2^m r_0, -2^m r_0\},$
$\{0, 2^{m+1} r_0, -2^{m+1} r_0\}$
by its opposite
$\{0, -2^m r_0, 2^{m+1} r_0\}, \{0, 2^m r_0, -2^{m+1} r_0\}, \{\pm\infty, 2^{m+1} r_0, -2^{m+1} r_0\},$
$\{\pm\infty, 2^m r_0, -2^m r_0\}$.

Each such replacement produces a further break in the infinite chain together with further (disconnected) offshoot configurations. The total effect of all such replacements is to produce a CIST whose quadrilateral graph contains two singly infinite chains, a finite chain of length n for each $n \in S$ and no further chains. Hence distinct subsets $S \subset Z^+$ give rise to CISTs having non-isomorphic quadrilateral graphs and therefore these CISTs are themselves non-isomorphic. Since there are $2^{\aleph_0}$ distinct subsets of Z^+ there are at least $2^{\aleph_0}$ pairwise non-isomorphic CISTs.

The fact that the cardinality of the number of CISTs can not be greater than $2^{\aleph_0}$ is guaranteed by the fact that the set $\bar{Q}$ and therefore also the collection of all 3-subsets of $\bar{Q}$ from which blocks of a CIST are chosen, are both of cardinality $\aleph_0$. Hence the result is proved; the number of countably infinite Steiner triple systems has the cardinality of the continuum.

3 Disjoint and almost disjoint CISTs

DEFINITION. Two Steiner systems (V, B_1) and (V, B_2) are said to be mutually disjoint (MD)/respectively mutually almost disjoint (MAD) if the number of blocks common to both B_1 and B_2 is zero/one. A collection of pairwise MD Steiner triple systems is said to be a large set if every 3-element subset of the base set V occurs as a block in some Steiner system in the collection. Similarly a large set of MAD Steiner triple systems is one in which the systems are pairwise MAD and every 3-element subset of V occurs in some Steiner system in the collection.

Our main result concerns MD systems. For the case where V is finite, Lu Jia Xi [8],[9] has proved in a sequence of by now classic papers, the existence of a large set of MD Steiner triple systems for all $v \equiv 1$ or $3 \pmod 6$ and $v > 7$ with the possible exception of $v = 141, 283, 501, 789, 1501$ and 2365. We prove the same result in the case where the set V has cardinality $\aleph_0$

THEOREM 3. There exists a large set of mutually disjoint countably infinite Steiner triple systems.

Proof: Consider an anti-Pasch CIST, $\mathcal{A}$ as constructed in Theorem 1. For each $u \in Q$, we generate an isomorphic copy of $\mathcal{A}$, denoted by $\mathcal{A}_u$ by applying to the points of the system the mapping

$$x \to x + u,\ x, u \in Q, \quad \infty \to \infty, \quad -\infty \to -\infty.$$

The family $\{\mathcal{A}_u : \ u \in Q\}$ provides a large set of MD CISTs. This can be seen as follows.

The blocks of type A are mapped to blocks of the form $\{x, y, z\}$ where $x+y+z = 3u$ and x, y, z are unequal with $x, y, z \in Q$: type A'. The block of type C is mapped to $\{u, \infty, -\infty\}$: type C'. The blocks of type B are mapped to $\{x+u, -2x+u, \pm\infty\}$ where $x \in Q \setminus \{0\}$: type B'.

Now let u run through the rationals, thus furnishing $\aleph_0$ CISTs. In this manner we certainly obtain all blocks of the form $\{x, y, z\}$ where x, y, z are unequal with $x, y, z \in Q$ and $x+y+z=r,\ r \in Q$, and also all blocks of the form $\{r, \infty, -\infty\},\ r \in Q$.

Additionally, choosing $u', v' \in Q$ such that $u' \neq v'$ and applying the mapping defined above for $u = u'$ and $v = v'$ the blocks of types A and C occuring in the two CISTs will be disjoint. Thus all blocks of types A' and C' occur in one and only one system.

There remains to consider blocks of type B' for varying values of u. Two questions arise:

(i) do all blocks of the forms $\{\alpha, \beta, \infty\}$ and $\{\alpha, \beta, -\infty\}$ where $\alpha, \beta \in Q$ and $\alpha \neq \beta$ occur among the systems and

(ii) do any such blocks occur in more than one system?

To answer these questions we firstly show that for the mappings of blocks of type B the pair $\{\alpha, \beta\}\ \ \alpha, \beta \in Q,\ \alpha \neq \beta$ occurs in precisely two systems in a block of type B'. This is easily seen, for take any pair $\{\alpha, \beta\}$ as above, then the mapping applied to a block of type B, say $\{x, -2x, \pm\infty\}$ produces $\{x+u, -2x+u, \pm\infty\}$. Two values of x and u will now produce the required pair in the block, that is $x = (\alpha-\beta)/3,\ u = (2\alpha+\beta)/3$ or $x = (\beta-\alpha)/3,\ u = (\alpha+2\beta)/3$.

Now from our construction of $\mathcal{A}$ if for any $q \in Q \setminus \{0\}$ there is a block of this system $\{q, -2q, \infty\}$ there is also a block of this system $\{-q, 2q, -\infty\}$.

Thus since $x = \pm(\alpha-\beta)/3$ the type B' blocks containing the pair $\{\alpha, \beta\}$ will have oppositely signed infinities and so under the mappings one will be $\{\alpha, \beta, \infty\}$ and the other $\{\alpha, \beta, -\infty\}$. Hence all blocks of the form $\{\alpha, \beta, \infty\}$ and $\{\alpha, \beta, -\infty\}$ occur each in just one system. Thus all blocks of the form $\{x, y, z\}$ with x, y, z unequal and $x, y, z \in \bar{Q}$ are included among these systems and no block occurs in more than one system; hence we have produced a large set of mutually disjoint countably infinite Steiner triple systems as required.

Turning now to MAD systems it is immediately clear that if the construction described in Theorem 3 were applied to a CIST(f) with $f(r) = f(-r)$ for all $r \in Q$ satisfying $\frac{1}{2} \leq |r| < 1$ then a collection of $\aleph_0$, pairwise MAD CISTs would be obtained. Unfortunately it is equally clear that the collection is not a large set nor can be extended to such. However in [6], Lindner and Rosa give for the finite case $v \equiv 1$ or $3 \pmod 6$ an easy construction of a large set of mutually almost disjoint Steiner triple systems $\mathrm{S}(2,3,v)$ from a Steiner quadruple system $\mathrm{S}(3,4,v+1)$. The Lindner and Rosa method is also applicable to the infinite case and so we are able to state:

THEOREM 4. If there exists a countably infinite Steiner quadruple system (i.e. the parameters defined in the introductory section of this paper have the values $t = 3, k = 4$ and $|V| = \aleph_0$) then there exists a large set of mutually almost disjoint countably infinite Steiner triple systems.
Proof: Take as representation of a countably infinite set, $Q^* = \bar{Q} \cup \{\star\}$ and construct a countably infinite Steiner quadruple System with this as the base set V. For each $x \in Q$ consider the derived CIST obtained by deleting all blocks not containing x as well as x itself. Now rename the element $\star$ as x. As x runs through Q a large set of mutually almost disjoint countably infinite Steiner triple systems is obtained. For details see [6].

NOTE. In a forthcoming paper [4] we give a construction for a countably infinite Steiner quadruple system. The existence of such systems has also been proved by Köhler in [5].

Finally in this section we observe that the large set of mutually disjoint countably infinite Steiner triple systems consists of isomorphic copies of an anti-Pasch CIST. It would be interesting to construct a large set of MD CISTs in which each system was isomorphic to a non-anti-Pasch CIST; indeed we ask the more general question of whether there exists a large set of MD isomorphic copies of any CIST?

4 Pairs of CISTs with prescribed intersections

DEFINITION. For the case where V is finite and of cardinality $v \equiv 1$ or 3 (mod 6) denote by $I(v)$ the set of all integers m such that there exists a pair of Steiner triple systems each of order v having precisely m blocks in common.

In [7], Lindner and Rosa proved that $I(v) = \{n : n \in Z, 0 \leq n \leq b - 6\} \cup \{b - 4, b\}$ where $b = v(v - 1)/6$ is the number of blocks in each system. We consider the same problem for CISTs and present a complete solution in the two theorems below.

THEOREM 5. For any non-negative integer m or m countably infinite, there exists a pair of countably infinite Steiner triple systems having precisely m blocks in common.
Proof: Consider an anti-Pasch cist $\mathcal{A}$, as constructed in Theorem 1 and apply two mappings to the points of this CIST as follows.
Mapping 1: $x \to x + 1,\ x \in Q;\quad \infty \to \infty;\quad -\infty \to -\infty.$

Mapping 2: $x \to x-1,\ x \in Q;\quad \infty \to \infty;\quad -\infty \to -\infty.$
The two CISTs obtained denoted by $\mathcal{A}_1$, and $\mathcal{A}_{-1}$ respectively, are by the proof of Theorem 3, disjoint.

Our method of proof is to devise a mapping $F : \bar{Q} \to \bar{Q}$ such that the CIST $F(\mathcal{A}_1)$ has precisely m blocks in common with $\mathcal{A}_{-1}$. The form of F is an involution; we firstly require that $F(\pm\infty) = \pm\infty$ and $F(0) = 0$ and will determine a subset $S \subset Q^+$ such that $F(\pm x) = \mp x$ if $x \in S$, and $F(\pm x) = \pm x$ otherwise, with obvious notation as to the selection of the signs. Clearly any blocks common to $F(\mathcal{A}_1)$ and $\mathcal{A}_{-1}$ will originate from the same type of block A, B or C in $\mathcal{A}$. We will arrange that all common blocks originate from type A. To do this we consider the three types in turn.

(i) If $1 \notin S$ then the blocks originating from type C in both systems are distinct.

(ii) Consider typical blocks originating from type B. These are of the form $\{r+1, -2r+1, \pm\infty\}$ where $r \in Q \setminus \{0\}$ in $\mathcal{A}_1$ and $\{s-1, -2s-1, \pm\infty\}$ where $s \in Q \setminus \{0\}$ in $\mathcal{A}_{-1}$. For the former to be equal to the latter when the mapping F is applied one of the following three necessary conditions must hold. (They are not sufficient because the "signs of the ∞ s" may be different).

  (a) $\{-r-1, 2r-1\} = \{s-1, -2s-1\}$.
  Hence either $-r-1 = s-1$ and $2r-1 = -2s-1$ giving $r = -s$, but in this case from the construction of $\mathcal{A}$ the "signs of the ∞ s" are different. Alternatively $-r-1 = -2s-1$ and $2r-1 = s-1$ giving $r = 2s = 4r$ which is impossible.

  (b) $\{-r-1, -2r+1\} = \{s-1, -2s-1\}$.
  If $-r-1 = s-1$ and $-2r+1 = -2s-1$ it follows that $r = -s = \frac{1}{2}$ and as in (a) the "signs of the ∞ s" are different. Alternatively if $-r-1 = -2s-1$ and $-2r+1 = s-1$ then $r = 4/5$ and $s = 2/5$. Hence we must require that $r+1 = 9/5 \notin S$.

  (c) $\{r+1, 2r-1\} = \{s-1, -2s-1\}$.
  If $r+1 = s-1$ and $2r-1 = -2s-1$ then $r = -s = -1$ and the same applies as in (a) and (b). Alternatively $r+1 = -2s-1$ and $2r-1 = s-1$ gives $r = -2/5$ and $s = -4/5$; $-2r+1 = 9/5$ as in (b).

Hence, to summarise, if $9/5 \notin S$ then the blocks originating from type B in $F(\mathcal{A}_1)$ and $\mathcal{A}_{-1}$ are distinct.

iii) Any block in $\mathcal{A}_1$, originating from type A will be of the form $\{x, y, z\}$ where $x + y + z = 3$ and x, y, z are unequal with $x, y, z \in Q$. Without loss of generality, applying the mapping F to such a block leads to one of the four following possibilities which are examined in turn.

(a) $\{x, y, z\} \to \{x, y, z\}$. The block can not be in $\mathcal{A}_{-1}$.

(b) $\{x, y, z\} \to \{-x, y, z\}$. For the latter to be a block of $\mathcal{A}_{-1}$ then $-x + y + z = -3$ giving $x = 3$ and $z = -y$. Requiring $3 \notin S$ avoids this possibility.

(c) $\{x, y, z\} \to \{-x, -y, z\}$. Again for the latter to be a block of $\mathcal{A}_{-1}$ then $-x - y + z = -3$ giving $z = 0$ and $y = 3 - x$. So for any block of the form $\{0, x, 3 - x\}$ where $x \in Q \setminus \{0, \frac{3}{2}, 3\}$ in $\mathcal{A}_1$ the condition that both $|x| \in S$ and $|3 - x| \in S$ will result in a block common to both $F(\mathcal{A}_1)$ and $\mathcal{A}_{-1}$.

(d) $\{x, y, z\} \to \{-x, -y, -z\}$. The latter would of necessity be a block of $\mathcal{A}_{-1}$ but we avoid the occurrence of this by our choice of the set S below which ensures that no block $\{x, y, z\}$ contained in $\mathcal{A}_1$, maps to $\{-x, -y, -z\}$. To see this note that if $x + y + z = 3$ then not all of $|x|, |y|$ and $|z| \in S$.

To complete the proof let m be as in the statement of the theorem. Put

$$\epsilon_n = \sum_{i=1}^{n} 10^{-i} \quad \text{and let} \quad S = \{x : \ x = 3/2 \pm \epsilon_n \ \text{ for } n \in Z \text{ and } 1 \le n \le m\}.$$

Then from the conditions derived in (i), (ii) and (iii) above it is clear that $F(\mathcal{A}_1)$ will have precisely m blocks in common with $\mathcal{A}_{-1}$ and the theorem is proved.

Superficially the above theorem might appear slightly surprising in that it is not analogous to the result in the finite case proved by Lindner and Rosa [7]. However observe that in the finite case the number of common blocks between two Steiner systems automatically determines the number of blocks not in common. This is not the case for CISTs and in the construction of Theorem 5 of two countably infinite Steiner triple systems having any prescribed number of common blocks, the number of blocks not in common has cardinality $\aleph_0$. To complete the solution therefore we state and prove the theorem below.

THEOREM 6. For any $m \in \{0, 4\} \cup \{n : \ n \in Z, n \ge 6\}$ and no other values there exists a pair of countably infinite Steiner triple systems differing

in precisely m blocks, i.e. there are precisely m blocks in the first system but not in the second and vice-versa.
Proof: Consider any CIST constructed on the set $\bar{Q}$ as indicated. Let V be a finite set of cardinality $v \equiv 1$ or $3 \pmod 6$ and further consider any Steiner triple system $\mathrm{S}(2,3,v)$ constructed on V. Then another CIST can be constructed on the Cartesian product $V \times \bar{Q}$ in the usual way, i.e. the blocks of the new CIST are

(i) $\{(a,x),(b,y),(c,z)\}$

(ii) $\{(a,x),(a,y),(a,z)\}$ for all $a \in V$

(iii) $\{(a,x),(b,x),(c,x)\}$ for all $x \in \bar{Q}$

where $\{a,b,c\}$ and $\{x,y,z\}$ are blocks of the $\mathrm{S}(2,3,v)$ and original CIST respectively.

Trivially all blocks of the form $\{(a,0),(b,0),(c,0)\}$ form a finite Steiner triple subsystem of order v of the CIST. Now let any m defined as in the statement of the theorem be given. Choose any v such that $v(v-1)/6 \geq m$ and construct a CIST on the set $V \times \bar{Q}$ as indicated. Now construct a second CIST in the same way but replacing m blocks of the form $\{(a,0),(b,0),(c,0)\}$ using the m blocks of a second $\mathrm{S}(2,3,v)$ which has precisely these blocks not in common with the first $\mathrm{S}(2,3,v)$.

It remains to show that there does not exist a pair of CISTs differing in 1, 2, 3 or 5 blocks. Suppose otherwise. Consider one of the CISTs, then the blocks not in common with the other CIST form a finite partial Steiner triple system which by Treash's theorem [10] can be embedded in a (complete) finite Steiner triple system. Replacing the non-common blocks of the first CIST with those of the second CIST in the finite Steiner triple system would then give a contradiction to Lindner and Rosa's result concerning the sets $I(v)$. Hence the theorem is proved.

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