This is a preprint of an article accepted for publication in "Computers in Mathematical Research", N. M. Stephens and M. P.Thorne (eds.), published by the Institute of Mathematics and its Applications, 1988, 135–148.

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SOME APPLICATIONS OF COMPUTERS IN DESIGN THEORY

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1986

1 Introduction

The aim of this paper is to give a non-technical introduction to some uses of computers in Combinatorial Design Theory. In the space available it will only be possible to touch briefly on a very few topics but it is hoped that this will be sufficient to give a flavour of some of the important ideas in this area. For a detailed account of Design Theory the reader is referred to either of the two recent books by Hughes and Piper [18] or by Beth, Jungnickel and Lenz [3], the latter being particularly comprehensive. Certain chapters of the books by Anderson [1] and Street and Wallis [30] also contain expositions of some of this work. In addition, the Annals of Discrete Mathematics, volume 26, "Algorithms in Combinatorial Design Theory" contains a number of papers detailing the use of computers in helping to solve various Combinatorial problems. Within this latter volume the lengthy survey paper by M. J. Colbourn [6] would be well worth consulting by the reader who wishes to know more of this subject.

A t-design or simply a design, usually denoted by $t-(v, k, \lambda)$ or $\mathrm{S}_\lambda(t, k, v)$ consists of a v-element base set V together with a family of k-element subsets of V, called blocks which collectively have the property that every t-element subset of V occurs in precisely λ blocks. Blocks may be repeated, but the greatest interest centres on designs where no block is repeated; such designs are called simple. Designs in which t=2 are often referred to as balanced incomplete block designs or BIBDs for short. The case where $\lambda = 1$ is also of special interest; such designs are known as Steiner systems and are denoted by $\mathrm{S}(t, k, v)$. A comprehensive bibliography of published papers on Steiner systems is maintained by Doyen and Rosa [10], [11]. To exclude trivial cases in the above it is usually assumed that $1 < t < k < v$ and that not every k-element subset is a block.

Two easy examples of designs are the following:

(i) $2-(11,5,2)$
$V = Z_{11} = \{0,1,2,\ldots,10\}$
The blocks are $\{0,1,2,3,4\}, \{0,1,5,6,7\}, \{0,2,5,8,9\}$,
$\{0,3,6,8,10\}, \{0,4,7,9,10\}, \{1,2,7,8,10\}, \{1,3,5,9,10\}$,
$\{1,4,6,8,9\}, \{2,3,6,7,9\}, \{2,4,5,6,10\}, \{3,4,5,7,8\}$.

(ii) S$(3,4,8)$
$V = Z_8 = \{0,1,2,\ldots,7\}$
The blocks are $\{0,1,2,7\}, \{0,3,4,7\}, \{0,5,6,7\}, \{1,3,5,7\}$,
$\{1,4,6,7\}, \{2,3,6,7\}, \{2,4,5,7\}$ together with seven further blocks which are the complements of the above blocks.

Clearly, the first question to consider is the construction of such designs and one of the roles of the computer is in this context. There are many purely mathematical construction methods which give infinite families of $t-(v,k,\lambda)$ designs but many other interesting and important designs have only been constructed with the aid of a computer, (see for example Kramer, Leavitt and Magliveras [21] and Mills [27]). A rough idea of the computational scale associated with this may be obtained from considering the problem of constructing a simple $t-(v,k,\lambda)$ design from the collection of all $\binom{v}{k}$ k-element subsets of V. The correct number of blocks required as given in theorem 2.2 below is $b = \lambda\binom{v}{t}/\binom{k}{t}$ and these mav be selected in $\binom{\binom{v}{k}}{b}$ distinct ways. Even for relatively modest values of the parameters this is a large number and as each such collection would then need to be checked for the property that it should contain each t-element subset of V precisely λ times, such a frontal assault is computationally intractable, even with a computer. Obviously more refined methods are required which reduce the computations involved to a scale which are feasible for the computer to handle and one such important method is dealt with in more detail below in Section 3.

Before introducing some of the basic theory of block designs we note that the definition of a t-design is capable of both generalisation and modification. For example, it is possible to allow blocks of different but specified sizes and to introduce ideas of ordering into the blocks. Such designs have also been extensively studied. Much of the theory we develop in this paper and many of the ideas are also applicable, albeit with appropriate modification, to these wider classes of designs. However, for simplicity and ease of presentation our attention here will be confined to t-designs as defined above.

2 Basic Results

That t-designs can not exist for all values of the parameters is clear by attempting to construct an $\mathrm{S}(1,2,7)$ or $2-(5,3,2)$ for example. Below we give some of the basic theory of designs and in particular theorem 2.3 gives an important necessary condition on the parameters. Proofs are only given of the elementary results and even these are sketches.

Theorem 2.1
If there exists a $t-(v,k,\lambda)$ design then there also exists a $(t-1)-(v-1,k-1,\lambda)$ design.
Proof: Choose any element of the base set V, delete it from all blocks containing it and then also delete all blocks which do not contain the element. What remains is a $(t-1)-(v-1,k-1,\lambda)$ design. This design is said to be derived from the $t-(v,k,\lambda)$ design.

Theorem 2.2
If there exists a $t-(v,k,\lambda)$ design then $\binom{k}{t}$ divides $\lambda\binom{v}{t}$.
Proof: Each block contains precisely $\binom{k}{t}$ distinct t-element subsets and there are $\binom{v}{t}$ t-element subsets in all. It follows that the number of blocks b in a $t-(v,k,\lambda)$ design is given by $b=\lambda\binom{v}{t}/\binom{k}{t}$.

Repeated application of theorem 2.1 together with theorem 2.2 gives a simple necessary condition for the existence of a $t-(v,k,\lambda)$ design.

Theorem 2.3
A necessary condition for the existence of a $t-(v,k,\lambda)$ design is that $\binom{k-i}{t-i}$ divides $\lambda\binom{v-i}{t-i}$ for $i=0,1,2,\ldots,v-1$.

Remark: For given t,k,λ the values of v satisfying this condition are called “admissible”. Although in certain cases e.g. $t=2$ and $k=3$ the admissibility condition on v is also known to be sufficient (Hanani [15]) in general this is not so. Euler’s 36 officers problem, equivalent to the construction of a Steiner system $\mathrm{S}(2,6,36)$ was shown to be impossible by Tarry [31] and other possible designs which are known not to exist include a simple $3-(11,5,2)$, (Dehon [8]) and $4-(15,5,1)$, (Mendelsohn and Hung [26]). The latter paper makes extensive use of the computer and illustrates very well another role for the computer in determining non-existence results for designs having certain theoretically admissible parameters and/or a certain structure. Mendelsohn and Hung’s basic methodology was firstly to enumerate all non-isomorphic $3-(14,4,1)$ designs and then to show that none of these can be extended to form a design with the given parameters. As is evident from consulting the paper, such a technique makes huge computational demands which can only be provided by use of a computer. Probably therefore the most fundamental question concerns the determination for each triple (t, k, λ) of those values of v for which a $t-(v, k, \lambda)$ design exists. It has been conjectured that the condition given in theorem 2.3 is "almost" sufficient in the sense that for given t, k, λ there exists v_0 such that for any $v > v_0$ satisfying the admissibility condition there does exist a $t-(v, k, \lambda)$ design. A further necessary condition is known in the special case when $t = 2$ and $v = 1 + k(k-1)/\lambda$. This result, known as the Bruck-Ryser-Chowla theorem, is stated below.

Theorem 2.4
A necessary condition for the existence of a $2-(v, k, \lambda)$ design where $v = 1 + k(k-1)/\lambda$ is that

(a) $k - \lambda$ is a perfect square if v is even, or

(b) $x^2 = (k-\lambda)y^2 + (-1)^{(v-1)/2}\lambda z^2$ has a solution in integers x, y, z with $(x, y, z) \neq (0, 0, 0)$, if v is odd.

Two other important results are given below.

Theorem 2.5 (R.M. Wilson [34])
Given t, k, v there is an integer λ_0 such that for any $\lambda \geq \lambda_0$, for which v is admissible there is a $t-(v, k, \lambda)$ design.

Remark: The proof is non-constructive and the designs are not necessarily simple.

Theorem 2.6 (Teirlinck [32])
If $v \equiv t \pmod{((t+1)!)^{2t+1}}$ and $v \geq t+1$ then there is a simple $t-(v, t+1, ((t+1)!)^{2t+1})$ design.

Remark: The proof is recursive and Teirlinck proves rather more than we state here.

The following tabulation for $\lambda = 1$ (Steiner systems) gives those values (t, k) for which the state of knowledge regarding the admissibility condition is complete (or almost complete).

	System	Admissibility Condition	Reference
a)	$S(2,3,v)$	$v \equiv 1$ or $3 \pmod 6$ Sufficient	Kirkman [19]
b)	$S(3,4,v)$	$v \equiv 2$ or $4 \pmod 6$ Sufficient	Hanani [14]
c)	$S(2,4,v)$	$v \equiv 1$ or $4 \pmod{12}$ Sufficient	Hanani [15], [17]
d)	$S(2,5,v)$	$v \equiv 1$ or $5 \pmod{20}$ Sufficient	Hanani [16], [17]
e)	$S(2,6,v)$	$v \equiv 1$ or $6 \pmod{15}$ Sufficient for $v > 11151$ Not Sufficient for $v = 16, 21, 36$ (165 values to be settled)	Mills [28]

A variety of other families of Steiner systems are known for $t = 2$ and $t = 3$ but these do not cover all the admissible values of v. In the cases when $t > 3$, there are only 14 parameter sets for which Steiner systems are known. These are as follows:

a) $S(5,8,24)$, $S(4,7,23)$, $S(5,6,12)$, $S(4,5,11)$; Witt [36].

b) $S(5,6,24)$, $S(4,5,23)$, $S(5,6,48)$, $S(4,5,47)$, $S(5,6,84)$, $S(4,5,83)$, $S(5,7,28)$, $S(4,6,27)$; Denniston [9].

c) $S(5,6,72)$, $S(4,5,71)$; Mills [27].

No Steiner system with $t > 5$ is known; the construction of such a design would be of considerable interest.

For block designs with $\lambda > 1$ the state of knowledge is again very much incomplete. The admissibility conditions are known to be sufficient for $(t,k) = (2,3), (2,4)$ and $(2,5)$ apart from the non-existence of a $2-(15,5,2)$ design; Hanani [15], [16], [17]. Infinite classes of designs with $t > 2$ were known before Teirlinck's result described above but for $t > 5$ only a finite number of designs has been constructed and with the exception of the $6-(33,8,36)$ found by Magliveras and Leavitt [23] and the $6-(20,9,112)$ found by Kramer, Leavitt and Magliveras [21], none were simple.

3 Construction of Designs

One possible method of constructing designs is via the use of a suitable automorphism group.

Definition: Let V be a v-element set and ϕ a permutation of V. If D is a $t-(v,k,\lambda)$ design and ϕ maps the blocks of D onto the blocks of D then ϕ is said to be an automorphism of D. As is well known the set of all such automorphisms forms a group called the automorphism group of D and denoted by $\mathrm{Aut}(D)$.

If G is a subgroup of $\mathrm{Aut}(D)$ and B is a block of D then the set $\{\phi(B) : \phi \in G\}$ is a set of blocks of D and is called the orbit of B under G. Thus the entire design D may be constructed from the group G and a collection of 'starter blocks", this collection consisting of one representative block drawn from each orbit. For example, the two starter blocks $\{0,1,4\}$ and $\{0,2,7\}$ generate a Steiner system $\mathrm{S}(2,3,13)$ under the action of the cyclic group generated by the mapping $z \to z+1 \pmod{13}$, the orbits being $\{0,1,4\}$, $\{1,2,5\}$, ..., $\{12,0,3\}$ and $\{0,2,7\}$, $\{1,3,8\}$, ..., $\{12,1,6\}$.

When the design D is unknown, $\mathrm{Aut}(D)$ will also be unknown and so the construction of D is speculative. Firstly a suitable group G of potential automorphisms is selected. The collection of all k-element subsets of V is then partitioned into orbits under the action of G. These orbits may be regarded as the pieces of an over-complete jig-saw puzzle. It is required to select from this collection of orbits a subcollection which collectively forms a $t-(v,k,\lambda)$ design. For this method the choice of G is critical since if it is not a subgroup of $\mathrm{Aut}(D)$ it is impossible to construct D from G. In practice it is often found that elementary mathematical arguments can be used to eliminate certain possible automorphisms and this restricts the choice of G. Within this restriction in broad terms the larger the group G, the smaller will be the number of starter blocks required and this will in turn aid the construction of the design. However, the smaller the group G, the more likely that there will be a design D having G as a subgroup of $\mathrm{Aut}(D)$. For the pessimistic, Babai [2] has proved that almost all Steiner systems $\mathrm{S}(2,3,v)$ have only the identity permutation as an automorphism and one might reasonably conjecture that this applies to most classes of t-designs.

Groups which have been found useful in practice are the following:

a) Cyclic groups which may be represented as

$$\langle z \to z+n,\ n = 0,1,2,\ldots,v-1\rangle$$

on the set of residue classes modulo v

b) Affine groups which have the form

$$\langle z \to az+b,\ b = 0,1,2,\ldots,v-1,\ a \text{ and } v \text{ coprime } \rangle$$

on the same residue classes.

c) Projective special linear groups PSL$(2, v-1)$ where $v-1$ is prime

$$\langle z \to \frac{az+b}{cz+d} \text{ where } a, b, c, d \in \{0, 1, 2, \ldots, v-2\} \text{ and } ad - bc = 1\rangle$$

defined on the residue classes modulo $v-1$ augmented by ∞.

(d) The exceptional Mathieu groups M_{22}, M_{23} and M_{24} in their permutation representations on 22, 23 and 24 elements respectively.

The transitivity properties of the above groups greatly reduces the amount of checking needed to be done in combining orbits to form a design.

Constructions of designs using suitable automorphism groups substantially reduce the number of blocks which need to be considered. For example S$(5, 6, 24)$ has 7084 blocks but Denniston [9] constructed such a system from only 3 starter blocks using the group PSL$(2, 23)$; indeed all the designs produced in [9] were obtained by hand calculation. However, in the case of designs containing larger numbers of blocks the computer is a valuable tool. Even in [9] the author provides a program for checking the results.

A computer-based construction would have four aspects.

a) Determination of all k-element set orbits under the group G.

b) Determination of all t-element set orbits under G.

c) For each k-element set orbit, identification of the t-element set orbits which the k-element set orbit covers.

d) Selection if possible of a subcollection of k-element set orbits which collectively cover each t-element set orbit precisely λ times.

To illustrate the situation we choose an example due to Kramer, Magliveras and Mesner [20]. Using M_{24} represented as a permutation group acting on the set $\{\infty, 0, 1, 2, \ldots, 22\}$ the authors show for example that there are five 12-element set orbits and three 8-element set orbits. They identify the location of 8-element set orbits within 12-element set orbits by the 3×5 matrix $A = (a_{i,j})$ where $a_{i,j}$ is the number of occurrences of the i^{th} 8-element set orbit in the j^{th} 12-element set orbit. In fact

$$A = \begin{pmatrix} 140 & 1680 & 0 & 0 & 0 \\ 35 & 945 & 630 & 210 & 0 \\ 22 & 844 & 696 & 256 & 2 \end{pmatrix}$$

To find an $8-(24, 12, \lambda)$ design it is necessary and sufficient to find a vector $\mathbf{x} = (x_1, x_2, x_3, x_4, x_5)^T$ of non-negative integers such that $A\mathbf{x} = (\lambda, \lambda, \lambda, \lambda, \lambda)^T$.

4 Enumeration and Isomorphism Testing

For both of these related problems the computer is a useful aid. Given a quadruple (t, v, k, λ) we may wish to enumerate all (generally non-isomorphic) $t-(v, k, \lambda)$ designs or to enumerate all such designs with a particular structure or collection of automorphisms. (We say that two designs are isomorphic if there is a one-one mapping between the two base sets which takes the blocks of one system onto the blocks of the other).

Generally speaking, enumeration of all non-isomorphic designs is only feasible, even with the aid of a computer, for relatively small values of v. For example, in the case of small Steiner triple systems $\mathrm{S}(2, 3, v)$ the situation is as follows.

System parameters	Number of non-isomorphic systems
$\mathrm{S}(2, 3, 7)$	1
$\mathrm{S}(2, 3, 9)$	1
$\mathrm{S}(2, 3, 13)$	2
$\mathrm{S}(2, 3, 15)$	80 [7], [33], [12]
$\mathrm{S}(2, 3, 19)$	$> 2 \times 10^6$ [29]
$\mathrm{S}(2, 3, 21)$	$> 2 \times 10^6$ [35]
$\mathrm{S}(2, 3, 25)$	$> 10^{14}$ [35]

The huge increase in the number of non-isomorphic Steiner triple systems between $v = 15$ and $v = 19$ is an example of a phenomenon known as the combinatorial explosion. The same feature occurs in the enumeration of other combinatorial objects. A convenient listing of the 80 $\mathrm{S}(2, 3, 15)$s is given in [22]. Other enumeration results for general block designs are given in [13] and for BIBD's in [25]. Results for cyclic Steiner 2-designs i.e. Steiner systems $\mathrm{S}(2, k, v)$ which admit an automorphism of order v are contained in [4] and an encyclopaedic reference for Steiner triple systems is the paper by Mathon, Phelps and Rosa [24].

One method of enumeration is to find all realisations of a certain design on a given base set and then to determine the isomorphism classes. It is to this problem of isomorphism testing that we will devote the remainder of this paper.

Basically, a proof that two designs with the same parameters are non-isomorphic involves the identification of some property which remains invariant under permutation of the base set and which differentiates between the two designs. Ideally the invariant should be quick to compute but effective in discriminating between non-isomorphic designs. An example of a generally poor invariant is the automorphism group of the design; poor because it is difficult to compute and ineffective anyway in that many non-isomorphic designs with the same parameters can have the same automorphism group.

In the case of Steiner triple systems $\mathrm{S}(2,3,v)$ an invariant which has proved both easy to compute and effective in discriminating is the quadrilateral structure. A quadrilateral of an $\mathrm{S}(2,3,v)$ is a collection of four blocks whose union has cardinality six. We can always express a quadrilateral in the following format:

$$\{a,b,c\},\{a,y,z\},\{x,b,z\},\{x,y,c\}$$

At the most basic level we may simply count the number of quadrilaterals present in a given Steiner triple system. For Steiner triple systems containing a relatively small number of blocks it is of course possible to do this counting by hand but it is extremely tedious and very easy to make errors. Such a task is clearly ideally suited to the computer. However when the system under consideration is larger or it is necessary to consider possible isomorphisms of a large number of designs, as did Stinson and Ferch [29] in their construction of 2,000,000 Steiner triple systems $\mathrm{S}(2,3,19)$ the computer is the only way to perform the calculations. Plainly two systems which have different numbers of quadrilaterals cannot be isomorphic. However, it is quite possible for non-isomorphic systems to have the same number of quadrilaterals. As a refinement, we may count the number of times each element appears in a quadrilateral of the given system thereby obtaining a frequency table for the elements of the base set. Again, two systems with different frequency tables cannot be isomorphic, although the converse may not be true. If the frequency tables are the same then this method may have the added bonus of giving information about any possible isomorphism. For example, if a is in the base set of the system S, b is in the base set of the system T and a,b are the only two elements with frequency f then any possible isomorphism from S to T must map a to b.

Another invariant applicable to Steiner triple systems and indeed to any Steiner system of the form $\mathrm{S}(t,t+1,v)$ is the train structure. For a system $\mathrm{S}(2,3,v)$ we may construct its train digraph as follows. Choose any set $\{a,b,c\}$ not necessarily a block of the system. Then there will be blocks of the system containing the pairs $\{a,b\}$, $\{a,c\}$ and $\{b,c\}$, say $\{a,b,z\}$, $\{a,y,c\}$ and $\{x,b,c\}$. Thus the set $\{a,b,c\}$ generates another set $\{x,y,z\}$ and this process may be continued until we reach either a block of the system which gives rise to itself or a set which has been considered before. Proceeding in this fashion we construct a digraph T with vertices which are the collection of all 3-element subsets and directed edges from $\{a,b,c\}$ to $\{x,y,z\}$ if $\{a,b,c\}$ generates $\{x,y,z\}$ as described above. In practice these digraphs are very effective for discriminating between non-isomorphic systems even though there do exist non-isomorphic systems with the same train structure [5]. A major disadvantage is that the train digraph for an $S(t, t+1, v)$ will contain $\binom{v}{t+1}$ vertices.

Invariants similar to quadrilateral structure and train structure may be identified for other classes of block designs. There is no known invariant which is both easy to compute and will always distinguish non-isomorphic designs; generally speaking the simpler the computation, the less discriminating the invariant. The computation of invariants may be analysed for time and space complexity. Using the normal terminology both quadrilateral structure and train structure have time complexity $O(v^3)$ but, in general, there is no known polynominal time algorithm for deciding whether or not two block designs are isomorphic. Indeed, the evidence currently available [6] suggests that a general polynominal-time algorithm does not exist. If this is indeed the case the best that we can hope for are polynominal-time invariants with good discrimination in specific cases. Although digital computers are now an essential tool it therefore appears that there are intrinsic limitations to what can be achieved even with their aid.

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