A NEW LOOK AT AN OLD CONSTRUCTION FOR STEINER TRIPLE SYSTEMS

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1 Introduction

In this note we re-examine a construction for Steiner triple systems which although to our knowledge appears at least three times [4], [5], [6] in the published literature seems to be virtually unknown. In the authors' opinions this is a pity because it is direct, extremely simple and valid for all admissible orders of a Steiner triple system. The construction is presented in the papers by Schreiber [5] and Quiring [4] where it is used to prove the existence of a large set of mutually disjoint Steiner triple systems under a certain number theoretic condition (for definitions and precise statements of results see the relevant sections below). The same is done in the paper by R. M. Wilson [6] but in addition it is also shown that a single system can be constructed for all admissible orders. Indeed Wilson also comments on the elegance of the method. However it is not just our aim to popularise this work by presenting yet another version. The method can be extended very easily to construct a pair of mutually disjoint or mutually almost disjoint Steiner triple systems for some admissible orders and in the cases where a large set of mutually disjoint systems can be constructed these have the structure of being anti-Pasch or quadrilateral free, thus extending the known spectrum of such systems. In particular these are the first known quadrilateral free Steiner triple systems whose order $v \equiv 1 \pmod 6$.

2 Single systems

Recall that a Steiner triple system, denoted by STS(v), is a collection of subsets of cardinality three, called blocks, of a base set V of cardinality v having the property that every pair of elements of V occurs in precisely one block. The necessary and sufficient condition for the existence of an STS(v), [3] is $v \equiv 1$ or 3 (mod 6) and to exclude trivial systems we will also assume that $v \geq 7$. Let $n = v - 2$ and consider an Abelian group G of order n, whose operation we write additively for convenience and whose identity is denoted by 0. Note that neither 2 nor 3 divides n. Now list all unordered triples $\langle a, b, c\rangle$ where $a, b, c \in G$ and $a + b + c = 0$. These fall into three different types or possibilities.

(a) $\langle a, b, c\rangle$ where $a \neq b \neq c \neq a$. Except in the case dealt with under possibility (b), the set $\{a, b, c\}$ is included as a block of the system; we refer to these blocks as of type A.

(b) $\langle a, a, -2a\rangle$ where $a \in G,\ \ a \neq 0$. This collection of triples can be partitioned into orbits under the mapping $z \to -2z,\ \ z \in G$. If the number of triples in each orbit is even (note that the total number of such triples is even) then replace the repeated element in each triple by one of two further elements A or B in such a way that triples which have an element in common i.e. one of which can be mapped to the other by $z \to -2z$, receive different elements A or B. All sets $\{A, a, -2a\}$ and $\{B, a, -2a\}$ where $a \in G,\ \ a \neq 0$ are included as blocks of the system; we refer to these as of type B. If the number of triples in an orbit is odd then the above will not be possible but in that case such odd order orbits will occur in pairs; if one is generated by the triple $\langle a, a, -2a\rangle$ there will be another generated by $\langle -a, -a, 2a\rangle$. Include sets $\{0, a, -2a\}$ and $\{0, -a, 2a\}$ as blocks of the system; call these type A'. In addition delete blocks $\{0, a, -a\}$ and $\{0, 2a, -2a\}$ from the system replacing these with $\{A, a, -a\}$ and $\{B, 2a, -2a\}$; type B'. Other triples in the pair of orbits may then be dealt with as previously providing we begin with replacing both $\langle 2a, 2a, -4a\rangle$ and $\langle -2a, -2a, 4a\rangle$ by $\{A, 2a, -4a\}$ and $\{A, -2a, 4a\}$ respectively.

(c) a single triple $\langle 0, 0, 0\rangle$ which becomes the set $\{0, A, B\}$; type C.

It is very easy to verify that all of the above blocks constitute a Steiner triple system and we leave this as an exercise for the reader.

3 Two disjoint or almost disjoint systems

Two Steiner triple systems are said to be disjoint if they have no block in common and almost disjoint if they have precisely one block in common. Firstly construct a single system $S(0)$ as indicated in Section 2. Now choose $a \in G,\ a \neq 0$ and apply the mapping $z \to z + a,\ z \in G,\ A \to A, B \to B$ to the system to construct a second system $S(a)$. Assume that all of the orbits under possibility (b) in Section 2 are of even length i.e. that no sets of type A' or B' occur. It is then easy to verify that in this case the only block which systems $S(0)$ and $S(a)$ may have in common is $\{C, -a, 2a\}$ which in $S(a)$ is the map of the block $\{C, -2a, a\}$ in $S(0)$. Hence the two systems are either disjoint or almost disjoint according as the orbit of the triple $\langle a, a, -2a\rangle$ under $z \to -2z$ is singly or doubly even respectively. If the mapping $z \to z + a$, $z \in G,\ A \to B, B \to A$ is also applied to $S(0)$ to form a third system $S'(a)$ it is then clear that the systems $S(0)$ and $S'(a)$ are either almost disjoint or disjoint in the respective cases.

4 Large sets of mutually disjoint systems

A collection of pairwise mutually disjoint Steiner triple systems is called a large set if every triple occurs as a block of one of the systems. Elementary counting determines that this consists of $v-2$ mutually disjoint Steiner triple systems. Using the same arguments as in Section 3 it is an easy generalisation of the results given there that if the orbits under possibility (b) in Section 2 are all singly even, the Steiner triple systems $S(a)$ as a runs through all elements of the group G form a large set of mutually disjoint Steiner triple systems. Again the reader will be able to supply the details of the proof for himself or alternatively refer to [4], [5], [6]. The condition that the orbits under possibility (b) are all singly even is better given number theoretically. For every prime divisor p of $n = v - 2$, the order of $-2 \pmod p$ must be singly even.

Alternatively this is equivalent to the order of $+2 \pmod p$ being odd or in fact that the order of $+2 \pmod n$ is odd. It follows that $+2$ must be a quadratic residue and hence that p must be of the form $8s+1$ or $8s+7$. In the latter case -2 is a quadratic non-residue and hence the order of -2 is even. In addition this must be singly even since the order divides $p-1 = 2(4s+3)$. To summarize, $v - 2$ must be of the form of a product of any primes of the form $8s + 7$ and certain primes of the form $8s + 1$. Up to 100, these latter include the primes 73 and 89 but not 17, 41 or 97.

5 Quadrilateral free Steiner triple systems

A quadrilateral consists of four blocks of a Steiner triple system whose union has cardinality six. It is clear that a quadrilateral must have the following configuration: $\{x,y,z\},\{a,b,z\},\{a,y,c\},\{x,b,c\}$. Steiner triple systems having no quadrilaterals are known as anti-Pasch systems or we prefer the term quadrilateral free. Bose [1] gave the following construction of an STS(v) where $v \equiv 3 \pmod 6$. Let H be an Abelian group of odd order $2s+1$ and let S be the Cartesian product $H \times \{0,1,2\}$. Then the following subsets of S,

$$\begin{array}{l} \{(x,0),(x,1),(x,2)\} \text{ for all } x \in H, \\ \{(x,0),(y,0),(z,1)\} \\ \{(x,1),(y,1),(z,2)\} \\ \{(x,2),(y,2),(z,0)\} \text{ where } xy = z^2,\ x \neq y \end{array}$$

form the blocks of an STS($6s+3$).

In [2], Doyen observed that if the order of the group H is not divisible by 7 then the system so formed is quadrilateral free and this is the only construction known to the authors of such systems.

We now prove that using the method described in this paper, the condition for which the Steiner triple systems so constructed are quadrilateral free is precisely the same as the condition to construct a large set of mutually disjoint systems i.e. the orbits under possibility (b) in Section 2 are all singly even.

Consider the system $S(0)$. Firstly observe that no quadrilateral may comprise four blocks of type A; $\{x,y,z\},\{a,b,z\},\{a,y,c\},\{x,b,c\}$ since the conditions $x+y+z = a+b+z = a+y+c = x+b+c = 0$ imply $x=a, y=b, z=c$. Now assume that the quadrilateral includes a block of type B; $\{A,a,-2a\}$ and further assume that the block which intersects it at element A is also of type B; $\{A,b,-2b\}$. Then there are four possibilities for the remaining two blocks.

(i) $\{a,b,c\}$ and $\{-2a,-2b,c\}$ giving $a+b+c = -2a-2b+c = 0$. Thus $c=0$ and $b=-a$ giving a bona fide quadrilateral the only way of which to avoid is that the orbits under possibility (b) in Section 2 are all singly even since in that case alone if $\{A,a,-2a\}$ is a block then so is $\{B,-a,2a\}$ and not $\{A,-a,2a\}$. Hence, the condition is necessary and blocks of types A' and B' will not occur in a quadrilateral free system and can be ignored.

(ii) $\{a,b,B\}$ and $\{-2a,-2b,B\}$ so from the first block $b=-a/2$ giving a contradiction with the second block.

(iii) $\{a, -2b, c\}$ and $\{-2a, b, c\}$ giving $a - 2b + c = -2a + b + c = 0$. This leads to $a = b$, a contradiction.

(iv) $\{a, -2b, B\}$ and $\{-2a, b, B\}$ so from the first block $-2b = -a/2$ or $a = 4b$. Hence from the second block $-2a = \;\; 8b = -b/2$ or $15b = 0$. But the order of the group G is a product of primes of the form $8s + 7$ or $8s + 1$ so $b = 0$, a contradiction.

The only other possibility to consider is if the quadrilateral includes a block of type B; $\{A, a, -2a\}$ and the block of type C: $\{A, B, 0\}$. Then the remaining two blocks are either $\{0, a, -a\}$ and $\{B, -2a, -a\}$, or $\{0, -2a, 2a\}$ and $\{B, a, 2a\}$ the first case of which is possible only if $-2a = 2a$ and the second only if $5a = 0$ both leading to $a = 0$, again a contradiction. Hence the condition is also sufficient. Up to 100, the result establishes the existence of previously unknown quadrilateral free STS(v) for $v = 25, 49, 73$ and 91.

6 References

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