Cyclic bi-embeddings of Steiner triple systems on 31 points

G.K. Bennett, M.J. Grannell, T.S. Griggs
Department of Pure Mathematics
The Open University
Walton Hall
Milton Keynes
MK7 6AA
UNITED KINGDOM

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Corresponding author: G.K. Bennett, 57 Princess Way, Euxton, Chorley, Lancashire PR7 6PL, UK.

email:
G.K. Bennett: gkb53@hotmail.com
M.J. Grannell: m.j.grannell@open.ac.uk
T.S. Griggs: t.s.griggs@open.ac.uk

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Abstract. We investigate cyclic bi-embeddings in an orientable surface of Steiner triple systems on 31 points. Up to isomorphism, we show that there are precisely 2408 such embeddings. The relationship of these to solutions of Heffter's first difference problem is discussed and a procedure described which, under certain conditions, transforms one bi-embedding to another.

1 Introduction

In 1967 Ringel and Youngs completed the proof of the Heawood Map Colour Theorem. An account can be found in [6]. In particular they proved that the complete graph K_n can be embedded in an orientable surface of genus $\lceil (n-3)(n-4)/12 \rceil$. In the cases where $n \equiv 0,3,4$ or 7 (mod 12) the embeddings are triangulations and the faces form a Mendelsohn triple system. When $n \equiv 3$ or 7 (mod 12) there is potential for the Mendelsohn triple system to form two Steiner triple systems. In the literature, there is a wealth of material on graph embeddings but results on the embedding of designs are much more sparse. This paper is concerned with the latter. Throughout we only deal with orientable embeddings.

A Mendelsohn triple system of order n, MTS(n), is a pair (V,$\mathcal{B}$) where V is a set of points of cardinality n and $\mathcal{B}$ is a set of cyclically ordered triples of elements of V which collectively have the property that each ordered pair of elements of V is contained in precisely one triple. (A triple $\langle a, b, c \rangle$ "contains" the pairs $\langle a, b \rangle, \langle b, c \rangle, \langle c, a \rangle$.) Such systems exist for $n \equiv 0$ or 1 (mod 3), $n \neq 6$. A Steiner triple system of order n, STS(n), is a pair (V,$\mathcal{B}$) where V is a set of points of cardinality n and $\mathcal{B}$ is a set of triples of elements of V which collectively have the property that each unordered pair of elements of V is contained in precisely one triple. Such systems exist for $n \equiv 1$ or 3 (mod 6).

If the graph K_n is embedded in an orientable surface and every triple of a Steiner triple system is a face of this embedding then that system is also regarded as being embedded in the surface with these faces being coloured, say, black. If the remaining faces (white) also form a Steiner triple system we then have a face two-colourable bi-embedding of the two Steiner triple systems.

A particular question is whether, for $n \equiv 3$ or 7 (mod 12), all STS(n)s can be embedded. This seems to be a difficult question to answer. When $n \equiv 3$ (mod 12), the first non-trivial case is n=15. There are 80 non-isomorphic STS(15)s and it is currently known that three of these can be bi-embedded [1]. In this paper we focus on $n \equiv 7$ (mod 12) and, in particular, cyclic bi-embeddings of cyclic systems. By this we mean an embedding which has a cyclic automorphism of order n which necessarily extends to the two STS(n)s. In the case $n \equiv 3$ (mod 12), a cyclic STS(n) contains a unique short orbit. Consequently there can be no cyclic bi-embeddings of such a system. The STS(7) can be cyclically bi-embedded in a torus and details of this and the cyclic bi-embeddings of the STS(19)s can be found in [4]. The case examined in this paper is the cyclic bi-embedding of the 80 non-isomorphic cyclic STS(31)s. These are given in [3] and we follow the numbering therein.

2 Summary of Results

Each of the cyclic STS(31)s comprises five cyclic orbits. In a cyclic bi-embedding the blocks of each orbit are oriented consistently. Without loss of generality, one orbit may be oriented arbitrarily and there are then two possibilities for

each of the other orbits. This gives 16 potential orderings for consideration for each of the 80 systems, a total of 1280 possibilities.
An embedding of a graph (or design) in an orientable surface may be described by means of a rotation scheme. Given a vertex x of the graph (or point of the design) the rotation scheme at x comprises the cyclically ordered list of other vertices (points) which are adjacent to x taken in the order in which they appear around x in the embedding. The rotation scheme for the embedding is the set of all the vertices together with their rotations taken with a consistent orientation, i.e. all clockwise or all anti-clockwise. If the rotation scheme is cyclic then we can denote the vertices by $0, 1, ..., n-1$ in such a way that the rotation about x is obtained by adding x (modulo n) to the rotation about 0.
The 1280 orientations produce a total of 5536 cyclic bi-embeddings. These were found by the same computer program which was used in [4]. As a check on the correctness of the program it was used to verify the results produced by hand for n=19. The 5536 cyclic bi-embeddings can be reduced to 2408 non-isomorphic embeddings. The key to this is the observation that if system $\#i$ is embedded with system $\#j$ and ϕ is a multiplier automorphism of $\#i$ but not of $\#j$ then isomorphic embeddings of $\#i$ with $\#j$ and of $\#i$ with $\phi(\#j)$ will be obtained. Amongst the 80 cyclic STS(31)s there are 12 systems with a multiplier automorphism of order 3 and one system with a multiplier automorphism of order 5 (#80). For each of the 1280 orientations the program was used to determine all possible bi-embeddings. Hence a further check on the correctness of the program is the symmetry of the matrix N given in the Appendix whose entries give the numbers of non-isomorphic cyclic bi-embeddings of system $\#i$ with system $\#j$.
A brief summary of the computational results is as follows:

(a) Of the 80 cyclic STS(31)s, 76 can be cyclically bi-embedded (although this does not mean that the remaining four cannot be bi-embedded non-cyclically). This contrasts with the four cyclic STS(19)s, all of which may be cyclically bi-embedded [4].

(b) There are 64 non-isomorphic cyclic bi-embeddings of a system with itself, involving 44 distinct systems.

(c) System #80 ($PG(4, 2)$) does cyclically bi-embed, but not with itself. This is unlike the STS(7) ($PG(2, 2)$). The projective STS(15) ($PG(3, 2)$), as noted above, cannot have a cyclic bi-embedding; it does however have a non-cyclic bi-embedding with itself. This suggests that there is still hope for a non-cyclic bi-embedding of $PG(4, 2)$ with itself.

The distribution of the 5536 bi-embeddings of the cyclic STS(31)s is given in the matrix M (see Appendix). Each entry in the body of the matrix M gives the number of bi-embeddings of system $\#i$ with system $\#j$ where i and j are the system numbers which appear at the heads of the associated row and column. The numbering of the STS(31)s corresponds to that given in [3] but the systems are grouped into the classes discussed below. Because of this grouping the rows and columns of M are <u>not</u> numbered sequentially.The distribution of the 2408 non-isomorphic bi-embeddings is given, in similar format, in the matrix N. The matrices have a noticeable "block" structure which we show to be connected with the solution of Heffter's first difference problem. It is this which led us to group the STS(31)s into eight classes and explains the ordering of the 80 systems in the matrices (see Section 3 for details). The rotation schemes for the 2408 bi-embeddings are available from the authors in the format:

bi-embedding number	*rotation at 0*	*the two STS(31)s embedded therein*

3 Heffter's First Difference Problem

In 1897, Heffter posed the following difference problem [5]: can the integers 1,2,...,$3k$ be partitioned into k triples $\{a, b, c\}$ so that for each triple, $a+b\pm c \equiv 0$ (mod $6k$+1)? Heffter observed that a solution to this problem produces cyclic Steiner triple systems of order $6k$+1. Each difference triple $\{a, b, c\}$ gives rise to a cyclic orbit of the system generated by either $\{0, a, a+b\}$ or $\{0, b, a+b\}$. In [2] it was found that for the case where k=5 there are 64 solutions to Heffter's difference problem (HDP) producing $64\times 2^5 = 2048$ distinct cyclic STS(31)s which lie in 80 isomorphism classes. Two solutions to HDP are said to be *multiplier equivalent* if one set of triples may be obtained from the other by first multiplying by a constant factor (modulo $6k$+1) and then reducing any residue x in the range $3k+1 \leq x \leq 6k$ to $(6k+1-x)$. Clearly, two solutions to HDP which are multiplier equivalent will produce isomorphic sets of Steiner triple systems. The 64 solutions to HDP for $k = 5$ partition into eight "Heffter classes" under multiplier equivalence. A representative solution for each of these eight classes is listed as A to H below.

A:	{1,2,3}	{4,7,11}	{5,12,14}	{6,9,15}	{8,10,13}
B:	{1,2,3}	{4,8,12}	{5,9,14}	{6,10,15}	{7,11,13}
C:	{1,3,4}	{2,8,10}	{5,12,14}	{6,9,15}	{7,11,13}
D:	{1,2,3}	{4,7,11}	{5,10,15}	{6,12,13}	{8,9,14}
E:	{1,3,4}	{2,10,12}	{5,11,15}	{6,7,13}	{8,9,14}
F:	{1,5,6}	{2,7,9}	{3,13,15}	{4,10,14}	{8,11,12}
G:	{1,5,6}	{2,10,12}	{3,13,15}	{4,7,11}	{8,9,14}
H:	{1,11,12}	{2,7,9}	{3,5,8}	{4,13,14}	{6,10,15}

Each of these eight Heffter classes produces 2^5 distinct STS(31)s, some of which will be isomorphic. In fact, three of the classes produce 16 systems, three produce 8 systems and two produce 4 systems. The distribution of the 80 cyclic STS(31)s into the eight Heffter classes is given in the table below using the standard numbering of the 80 systems given in [3].

A:	5	6	7	8	9	10	11	12	33	34	35	36	37	38	39	40
B:	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30
C:	41	42	43	44	45	46	47	48	55	56	57	58	59	60	61	62
D:	1	2	3	4	13	14	31	32								
E:	49	50	51	52	53	54	63	64								
F:	65	66	67	68	69	70	71	76								
G:	72	73	74	75												
H:	77	78	79	80												

These 80 numbers, in the order which they appear above, form the headers for the rows and columns of the matrices M and N given in the Appendix.
The significance of Heffter's first difference problem for the results is now explained.
Consider the oriented cyclic orbit containing an oriented triple $\langle 0, \alpha, \alpha+\beta \rangle$. This orbit will also contain the oriented triples $\langle 0, \beta, -\alpha \rangle$ and $\langle 0, -(\alpha+\beta), -\beta \rangle$. If this orbit forms part of a cyclic embedding then there are just two possibilities for the rotation about zero:

(a) $\quad (\alpha+\beta)\ \ldots\ \beta\ (-\alpha)\ \ldots\ (-(\alpha+\beta))\ (-\beta)\ \ldots\ \alpha,$ $\quad$ or

(b) $\quad (\alpha+\beta)\ \ldots\ (-(\alpha+\beta))\ (-\beta)\ \ldots\ \beta\ (-\alpha)\ \ldots\ \alpha.$

In case (b) consider the alternative rotation about zero given by

(c) $\quad (\alpha+\beta)\ \ldots\ (-(\alpha+\beta))\ (-\alpha)\ \ldots\ \alpha\ (-\beta)\ \ldots\ \beta,$

where the three sections of the rotation, namely $(\alpha+\beta)\ \ldots\ (-(\alpha+\beta))$, $(-\alpha)\ \ldots\ \alpha$, and $(-\beta)\ \ldots\ \beta$ are exactly as in (b). If (b) generates a cyclic bi-embedding, then so does (c). The cyclic orbits of the two Steiner triple systems of (c) will be identical to those of (b) except that the oriented cyclic orbit containing $\langle 0, \alpha, \alpha+\beta \rangle$ is replaced by the oriented cyclic orbit containing $\langle 0, \beta, \alpha+\beta \rangle$. Both these orbits come from a common Heffter difference triple. As a consequence, we see that if $\langle 0, \alpha, \alpha+\beta \rangle$ is an oriented triple of system $\#i$ bi-embedded with a system $\#j$ in which the rotation about zero has the structure given in (b), then we obtain a bi-embedding of a system $\#i'$ with system $\#j$. The systems $\#i$ and $\#i'$ will lie in the same Heffter class and differ only in the orbit corresponding to the Heffter difference triple $\{\alpha, \beta, \alpha+\beta\}$. The system $\#j$ is common to both biembeddings.
The above observations provide an explanation for the "block" structure of the matrices M and N. The operation described in moving from (b) to (c) is an example of a combinatorial *trade* where one substructure is replaced by an equivalent substructure; in this case the oriented orbit containing $\langle 0, \alpha, \alpha+\beta \rangle$ is replaced by that containing $\langle 0, \beta, \alpha+\beta \rangle$. Finally we note that the four STS(31)s which are not cyclically bi-embedded are #72 - #75 and comprise Heffter class G in the table given above.

4 Conclusion

The results of this paper give rise to the following questions:

1. The Heffter class G does not give rise to any cyclic bi-embeddings. Is it possible to identify structural features of this class which clearly preclude such bi-embeddings?

2. (An extension of (1)). There are several pairs of Heffter classes which do not give rise to corresponding bi-embeddings. Again, is it possible to identify structural features of these classes which explain this phenomenon?

3. The matrices M and N have a clear block structure related to the Heffter classes. Within a block it is noticeable that most entries are non-zero and that in many cases it is possible to move from one non-zero entry to another using a trade of the form described above. A worthwhile future investigation would be the extent to which it is possible to obtain all the non-zero entries in such a block from any one entry by performing a sequence of such trades.

Less important but still of interest are:

4. The systems #72 - #75, which comprise Heffter class G, do not cyclically bi-embed with any other system. However, this does not imply that these systems will not embed in some other way and it would be of interest to find bi-embeddings of these systems.

5. $PG(4,2)$ does not cyclically bi-embed with itself but the possibilty still exists for this system to have a non-cyclic bi-embedding with itself. The basis for this suggestion is twofold:

   (a) $PG(2,2)$ does cyclically bi-embed with itself and

   (b) $PG(3,2)$ (which *cannot* have a cyclic bi-embedding) does have a non-cyclic bi-embedding with itself.

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Appendix

The matrices M and N described in the text are given below.

For the sake of clarity and to reveal the block structure, “0” entries have been suppressed and replaced by blanks.

	5	6	7	8	9	10	11	12	33	34	35	36	37	38	39	40	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	41	42	43	44	45	46	47	48	55	56	57	58	59	60	61	62	1	2	3	4	13	14	31	32	49	50	51	52	53	54	63	64	65	66	67	68	69	70	71	76	72	73	74	75	77	78	79	80
5		1	2	1	3	2	1	2	2	1	2	1	1	2	3																							2	1	1			1		1	1		1																																
6	1	2	1	2	2	1	2	1	1		3		2	3	2	1																	1	1				2			1					1	1	1																																
7	2	1	4	1	5	2	3	2	2	3	2	3	1	4	3	2																			1	1		2				1			1		1	1																																
8	1	2	1	2	2	1	2	1	1		3		2	3	2	1																					1	1						1	1	1	1	2																																
9	3	2	5	2	2	3	2	1	1	2	1	2	4	3	4	3																		1		1		1	1					1	2			1																																
10	2	1	2	1	3	4	5	2	2	3	4	3	3	2	1	2																	1	2		1		1		1	1		1	1	1		1	1																																
11	1	2	3	2	2	5	2	3	3	2	3	2	4	1	4	1																	1			1	1	2	1		1				1																																			
12	2	1	2	1	1	2	3			1	2	1	3	2	1	2																	1		1		2	1	1		1	1		1	1	1		1																																
33	2	1	2	1	1	2	3			1	2	1	3	2	1	2																		1		1		1	1	2	1	1	1		1	1	1																																	
34	1		3		2	3	2	1	1	2	1	2	2	1	2	1																	1			1		1		1	2	1				1																																		
35	2	3	2	3	1	4	3	2	2	1	4	1	5	2	3	2																		1	1	2		1		1	1				1																																			
36	1		3		2	3	2	1	1	2	1	2	2	1	2	1																	2	1	2	1	1		2	1	1	1	2	1	1	1	2	1																																
37	1	2	1	2	4	3	4	3	3	2	5	2	2	3	2	1																	1	1	1	1	1	1					1	2	1	1		1																																
38	2	3	4	3	3	2	1	2	2	1	2	1	3	4	5	2																			1	1	1	1	1	1	1	2				1	1	1																																
39	3	2	3	2	4	1	4	1	1	2	3	2	2	5	2	3																			1	1		2			1		1	1		1																																		
40		1	2	1	3	2	1	2	2	1	2	1	1	2	3																						1	1			1	1	1			2		1																																
15																	2	3	2	2	3	4	3	3	2	3	3	4	5	6	3	4	1	1	2	3	4	1	4	2	2	1	1	2		2	1	1																																
16																	3	4	2	2	2	3	1	3	2	1	3	2	2	3	2	1	2	1	2	2	1	1	2	1	1	3	2	2	2	5	1																																	
17																	2	2	4	3	1	3	3	4	6	3	4	3	3	2	2	3	2		1	2	4	1	4		1	2	1	2	1	1	1	1																																
18																	2	2	3	2	3	3	2	3	3	2	1		3	2	2	3	1	4	3	3	3	5	4	3	2	4	2	4	3	2	5	4																																
19																	3	2	1	3		3	2	2		1	3	2	3	2	3	2	4	3	3	3	3	3	3	4	3	2	3	2	2	2	2	2																																
20																	4	3	3	3	3	2	4	2	2	3	5	4	2	3	4	1	1	1	1	2	2	1	3	1	2	2	2	2	4	3	2	3																																
21																	3	1	3	2	2	4	2	1	2	3	2	1	3	2	2	3	3	2		2	1	1	1	2	2	1	1	2	3	3	2	2																																
22																	3	3	4	3	2	2	1		3	4	3	2	3	2	2	3	2		2	1	2	1	1	3	3	1	2	2	3	4		1																																
23																	2	2	6	3		2	2	3	2		4	4	3	1	2	4	2	3	2	2		2	1	2	4	2	2	2	1	2	3	2																																
24																	3	1	3	2	1	3	3	4		2	2	2	3	1	2	4	2	4		1		3	1	3	2	2	1	1	2	1	1																																	
25																	3	3	4	1	3	5	2	3	4	2	2	2	2	2	1	1	3	1		3	1	2	1	1	5	2	1		1	3	1	3																																
26																	4	2	3		2	4	1	2	4	2	2	2	4		3	1	3	2		2	1	1	3		3	2	2	1	2		1	1																																
27																	5	2	3	3	3	2	3	3	3	3	2	4	2	4	2	4	3	2	3	3	2	2	1	2	1	3	2	2	2	1	1	2																																
28																	6	3	2	2	2	3	2	2	1	1	2		4	2	2	2	3	1	3			1	1	3	1	5	1	3	3	2	1																																	
29																	3	2	2	2	3	4	2	2	2	2	1	3	2	2	2	2	2	2	1	2	1	2	1	3	2	1	3		2		1	1																																
30																	4	1	3	3	2	1	3	3	4	4	1	1	4	2	2	2	2	1	1	1	3	1	1	2	2	3	2	1	1	1	1	1																																
41		1				1	1	1		1		2	1				1	2	2	1	4	1	3	2	2	2	3	3	3	3	2	2			1	1	2	2	3	1	1	1	2	2	1	1		2	2	2	1		1	2											5	4	1	6	4	3	2	3								
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43			1					1			1	2	1	1	1		2	2	1	3	3	1		2	2				3	3	1	1	1	1			1	1	2		1	1	2	2	1	1		2		2	1		1	4											6	3	2	1	1		1	2								
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45				1			1	2				1	1	1		1	4	1	4	3	3	2	1	2			1	1	2		1	3	2		1	1		2	1	1		2	3	1			1	1	2	2		1	2	1											2	3		4	4	1	1	1								
46	2	2	2	1	1	1	2	1	1	1	1		1	1	2	1	1	1	1	5	3	1	1	1	2	3	2	1	2	1	2	1	2	2	1	3	2	4	1	1	1	1	2	2	1	1	2	2	4	3	1	1	5	3	1	2									3	4	1	2	2	1		3								
47	1				1		1	1	1			2		1			4	2	4	4	3	3	1	1	1	1	1	3	1	1	1	1	3	1	2	2	1	1	4	2	2	2	1	1	2	2	1	1	2	4		1	2	3											7	4	1	7	5	2	2									
48	1					1			2	1	1	1		1			2	1		3	4	1	2	3	2	3	1		2	3	3	2	1	1		2	1	1	2		1	1			1	3	2		1	2		2		2		1									5	4	1		2	1	2	1								
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56			1					1	1	1		1		2		1	1	3	2	4	2	2	1	1	2	2	2	2	3	5	1	3	1		1		2	1	2	1	1	2	3	2		1	2	1		1	1	1	1	3	1										7	3	2	3	2	1		2								
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61		1	1	1		1			1			2		1			1	1	1	5	2	2	2		3	1	1	1	1	1	1	1		1		1	1	2	1	2	1	2	1		2	3	2	1	1	2	1	1	1	3	2	1									6		1	3	2	1	1	2								
62	1	1	1	2	1	1		1				1	1	1		1	1		1	4	2	3	2	1	2		3	1	2		1	1	2	1	2	1	1	2	1			1	2	1	1	2	1	2	3	2	1	1	3	1		1									4	4	1	5	6	1	1	2								
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Table 1. The Matrix M

	5	6	7	8	9	10	11	12	33	34	35	36	37	38	39	40	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	41	42	43	44	45	46	47	48	55	56	57	58	59	60	61	62	1	2	3	4	13	14	31	32	49	50	51	52	53	54	63	64	65	66	67	68	69	70	71	76	72	73	74	75	77	78	79	80
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32																																						1		2		1			1		1		1	1				4			4		1	2	1	2	1	2	2	1	1	1	3	2	2	2	3	2								
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67																																	1	1	2	3		1	1	1		2		1	1		1	1	3	3	3	3	3	3	3	3						2	2		2	2	4	1	1	1	4	1								
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69																																	4	3	1	3	4	2	5	2	4	2	2	3	4	5	2	6			2	2			2	2		3		3		1	1		3	3	1			2	1	2								
70																																	3			2	1	1	2	1	1	1	1		1		1	1			2	2			2	2	2	1	2	1	2	1	1	2	1	1	1	2	2	4	1	4								
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76																																	3	4	2	2	1	3		1	2	2	2	3	2	1	2	2			2	2			2	2	2	1	2	1	2	1	1	2	1	1	1	2	2	4	1	4								
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78																																																									11	10	11	10	2	3	3	6																
79																																																									12	7	6	7	1	4	4	3																
80																																																									3	2	1	2	2	1	1																	

Table 2. The Matrix N