The Pasch configuration (Encyclopaedia of Mathematics entry)

M.J. Grannell and T.S. Griggs
Department of Pure Mathematics
The Open University
Walton Hall
Milton Keynes MK7 6AA
UNITED KINGDOM

March 2001

Pasch configuration - The Pasch configuration or *quadrilateral* is a collection of four triples isomorphic to $\{a,b,c\}$, $\{a,y,z\}$, $\{x,b,z\}$, and $\{x,y,c\}$. They have been studied extensively in the context of Steiner triple systems.

A *Steiner triple system* of order v, $STS(v)$, is an ordered pair (V,B) where V is a set of cardinality v, called *elements* or *points*, and B is a collection of *triples*, also called *lines* or *blocks*, which collectively have the property that every pair of distinct elements of V occur in precisely one triple. $STS(v)$ exist if and only if $v \equiv 1$ or 3 (mod 6), [10]. To within isomorphism, the Steiner triple systems of orders 7 and 9 are unique but for all greater orders, the structure is not unique. A (p,l)-*configuration* in a Steiner triple system is a collection of l lines whose union contains precisely p points. A configuration whose number of occurrences in an $STS(v)$ depends only upon the order v and not on the structure of the $STS(v)$ is called *constant* and otherwise *variable*. There are two configurations with l=2 and five with l=3, all of which are constant. There are 16 configurations with l=4 of which the Pasch configuration is the unique (6,4)-configuration and the one containing the least number of points. Five of the 4-line configurations are constant but the Pasch configuration is variable. It was shown in [5] that the number of occurrences of all the other variable 4-line configurations can be expressed in terms of the order v, and the number c of Pasch configurations in the $STS(v)$.

The above gives motivation to the problem of constructing $STS(v)$ containing no Pasch configurations, so-called *anti-Pasch* or *quadrilateral free* Steiner triple systems. A solution for $v \equiv 3$ (mod 6) was first given by Brouwer ([1], see also [9]) and it was a long-standing conjecture that anti-Pasch $STS(v)$ also exist for all $v \equiv 1$ (mod 6), $v \neq 7$ or 13. This was settled in the affirmative in two papers, [11] and [8], published in 2000. The proof resolves the first case of a conjecture by Erdös, [3], that for every $m \geq 4$ there is an integer v_m so that for every $v \geq v_m$, $v \equiv 1$ or 3 (mod 6), there is an $STS(v)$ avoiding $(l+2,l)$-configurations for $4 \leq l \leq m$. Anti-Pasch $STS(v)$ have application to erasure-correcting codes, [2]. The theoretical maximum number of Pasch configurations in an $STS(v)$ is $v(v-1)(v-3)/24$ but this is achieved only in the point-line designs obtained from the projective spaces $PG(n,2)$, [12].

The Pasch configuration is an example of a trade. A pair of distinct collections of blocks (T_1,T_2) is said to be *mutually t-balanced* if each t-element subset of the base set V is contained in precisely the same number of blocks of T_1 as of T_2. Each collection T_1, T_2 is then referred to as a *trade*. The Pasch configuration is the smallest trade that can occur in a Steiner triple system. If T_1 is the collection $\{a, b, c\}$, $\{a, y, z\}$, $\{x, b, z\}$ and $\{x, y, c\}$ then, by replacing each triple with its complement, a collection T_2, $\{x, y, z\}$, $\{x, b, c\}$, $\{a, y, c\}$ and $\{a, b, z\}$ is obtained which contains precisely the same pairs as T_1. This transformation is known as a *Pasch switch* and when applied to a Steiner triple system yields another, usually non-isomorphic, Steiner triple system. There are 80 non-isomorphic $STS(15)$s of which precisely one is anti-Pasch. It was shown in [4] that all of the remaining 79 systems can be obtained from one another by successive Pasch switches. Other relevant papers in this area are [6] and [7].

The number of Pasch configurations and their distribution within a Steiner triple system is an invariant and provides a simple and useful test to help in determining whether two systems are isomorphic.

References

[1] BROUWER, A.E.: 'Steiner triple systems without forbidden subconfigurations', *Mathematisch Centrum Amsterdam* **ZW104/77** (1977), 8pp.

[2] COLBOURN, C.J., DINITZ, J.H., AND STINSON, D.R.: 'Applications of combinatorial designs to communications, cryptography and networking', *London Math. Soc. Lecture Notes* **267** (1999), 37-100.

[3] ERDÖS, P.: 'Problems and results in combinatorial analysis', *Creation in Mathematics* **9** (1976), 25.

[4] GIBBONS, P.B.: 'Computing techniques for the construction and analysis of block designs' *University of Toronto Department of Computer Science Technical Report* **92** (1976).

[5] GRANNELL, M.J., GRIGGS, T.S., AND MENDELSOHN, E.: 'A small basis for four-line configurations in Steiner triple systems', *J. Combin Designs* **3** (1995), 51-59.

[6] GRANNELL, M.J., GRIGGS, T.S., AND MURPHY, J.P.: 'Twin Steiner triple systems', *Discrete Math.* **167/168** (1997), 341-352.

[7] GRANNELL, M.J., GRIGGS, T.S., AND MURPHY, J.P.: 'Switching cycles in Steiner triple systems', *Utilitas Math.* **56** (1999), 3-21.

[8] GRANNELL, M.J., GRIGGS, T.S., AND WHITEHEAD, C.A.: 'The resolution of the anti-Pasch conjecture', *J. Combin. Designs* (to appear) (2000).

[9] GRIGGS, T.S., MURPHY, J.P., AND PHELAN, J.S.: 'Anti-Pasch Steiner triple systems', *J. Combin. Inf. & Sys. Sci.* **15** (1990), 79-84.

[10] KIRKMAN, T.P.: 'On a problem in combinations', *Cambridge and Dublin Math. J.* **2** (1847), 191-204.

[11] LING, A.C.H., COLBOURN, C.J., GRANNELL, M.J., AND GRIGGS, T.S.: 'Construction techniques for anti-Pasch Steiner triple systems', *J. London Math. Soc.* (to appear) (2000).

[12] STINSON, D.R., AND WEI, Y.J.: 'Some results on quadrilaterals in Steiner triple systems', *Discrete Math.* **105** (1992), 207-219.