This is a preprint of an article published in Congressus Numerantium, 86, 1992, 19-25.
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EQUIVALENCE CLASSES OF STEINER TRIPLE SYSTEMS

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1 Preliminaries

Let \( V \) be a set of cardinality \( v \equiv 1 \) or \( 3 \) (mod 6). Such a \( v \) will be called admissible. Denote by \( R(V) \) the collection of all realisations of a Steiner triple system, \( STS(v) \), on \( V \). Recall that a quadrilateral (also known as a Pasch configuration) in an \( STS(v) \) is a collection of four blocks whose union has cardinality six. It follows immediately that the structure of a quadrilateral must be \( \{a, b, c\}, \{a, y, z\}, \{x, b, z\}, \{x, y, c\} \). It also follows that, for any Steiner triple system \( S_1 \in R(V) \) which contains a quadrilateral, we can obtain a different system \( S_2 \in R(V) \) by replacing any quadrilateral in \( S_1 \) by the “opposite” quadrilateral. The “opposite” quadrilateral is obtained by replacing each block of the quadrilateral by its complement with respect to the six points involved. We call this replacement operation, quadrilateral switching and write \( S_1 \leftrightarrow S_2 \).

Now choose any two Steiner triple systems \( S, T \in R(V) \). We say that \( T \) is quadrilaterally obtainable from \( S \) and write \( S \q T \) if and only if the system \( T \) can be realized from the system \( S \) by a sequence of \( n \geq 0 \) quadrilateral switches:

\[
S = S_0 \leftrightarrow S_1 \leftrightarrow S_2 \leftrightarrow \ldots \leftrightarrow S_{n-1} \leftrightarrow S_n = T.
\]

Clearly \( q \) is an equivalence relation and it is natural to ask about the equivalence classes thus induced. Denote the number of equivalence classes by \( Q(v) \). For all \( v \equiv 3 \) (mod 6) there exist quadrilateral-free (also know as anti-Pasch) \( STS(v) \) ([1], [6]) and it is conjectured that for all \( v \equiv 1 \) (mod 6) with \( v \geq 19 \) similar quadrilateral-free \( STS(v) \) exist. Each realization of such a system on a set \( V \) will, by itself, constitute an equivalence class and so will contribute 1 to \( Q(v) \). Apart from this observation little is known about \( Q(v) \) for most values of \( v \). For small values of \( v \) we have, trivially, \( Q(1) = 0 \) and \( Q(3) = 1 \). It is easy to verify that \( Q(7) = 1 \). Finally, because all realizations of an \( STS(9) \) are isomorphic and anti-Pasch it follows that \( Q(9) = 9!/|Aut\ STS(9)| = 9!/432 = 840 \).

A related and, probably, more important problem is the following. If \( S, T \in R(V) \) we say that \( T \) is isomorphically quadrilaterally obtainable from \( S \) and write \( S \i T \) if and only if \( S \q T' \) for some realization \( T' \) which is isomorphic to \( T \). Again, \( i \) is an equivalence relation and we denote the number of equivalence classes by \( I(v) \). In this case it is trivial that \( I(1) = 0 \) and \( I(3) = I(7) = I(9) = I(13) = 1 \). The number of pairwise non-isomorphic \( STS(15) \)s is 80 ([2], [3], [7], [10]), of which just one is anti-Pasch. It was shown by Gibbons [4], that under the relation \( i \), the remaining 79 form a single equivalence class and thus \( I(15) = 2 \). This leads immediately to the following question. If the equivalence classes which contain just a single anti-Pasch system are ignored, because they are clearly “special” in the sense that they contain no quadrilaterals to switch, do all the remaining \( STS(v) \) for a given \( v \) belong to just one equivalence class? In this short note we show that the answer is in the negative at least for the values \( v = 19, 21, 25, 27 \) and 31. We would conjecture even on this slim evidence that the answer is also negative for all admissible values of \( v \geq 19 \).
2 Twin Steiner triple systems

We define a set of twin Steiner triple systems to be two $STS(v)$s each of which contains precisely one quadrilateral which when switched produces the other system. Below are listed, in compact notation, sets of twin $STS(v)$s for $v = 19, 21, 25, 27$ and $31$. For the nine pairs of twin $STS(19)$s given, the base set $V$ is $\{a, b, c, \ldots, r, s\}$. The blocks are represented by a string of symbols $S_1, S_2, \ldots, S_{57}$, one symbol for each block, where each $S_i \in \{A, B, C, \ldots, R, S\}$. The string is constructed as follows. Using the usual lexicographical order, we order elements within each block and then we order the blocks. Thus, if the $i^{th}$ block is the triple $\{x_i, y_i, z_i\}$ then $x_i < y_i < z_i$. Moreover, every pair $\{x, y\}$ with $x < y$ and either

(i) $x < x_i$, or

(ii) $x = x_i$ and $y < y_i$

will appear in an earlier block. We take the symbol $S_i$ to be the upper-case letter corresponding to $z_i$. For each set of twins, only one system is listed, the four blocks which form the quadrilateral being underlined. To form the twin system, symbols which are marked with an asterisk are replaced as indicated.

**Systems 1**
MIPOSKNRQEHLRSPQOSKQL*M*PRQJLORIPM*L*NSJNQRSMNO
OPRRPQOSSRSQS
Twin system: replace LMML by MLLM.

**Systems 2**
HOMFLNPSRQKJSMPORHMIPRNSPNQLSRORQKPLRQRJSQ*MN
SRQ*QOMPQPS
Twin system: replace QO by OQ.

**Systems 3**
IHNOGLPSRKRSNHPQSSLOJQPRPIMOQLN*KIRQ*SMQ*RQPOSNS
RKROSORS*N*QS
Twin system: replace NQQN by QNNQ.

**Systems 4**
FJENKSMPROSMRNKQPKPQHQSRLMIROQGQRSSONSP*M*QRJPQSO
RPQ*N*L*SRQSS
Twin system: replace PMNP by MPPN.
Using the same conventions, four sets of twin $STS(21)$s on the base set $V = \{a, b, c, \ldots, r, s, t, u\}$ are represented by strings of symbols $S_1, S_2, \ldots, S_{70}$ below.

**Systems 1**

$IJHSRMLTUQKSTNLOQRUGPIQ*NSTUQOKNUPTJIUNROUPMQTSoP$

$RTJR*TNSQURTTRMUSRULPUQSS$

Twin system: replace QR by RQ.

**Systems 2**

$KQNHUTIMPSSTJORMUPQMFQK*N*PTUQLHKSURPLSOUTNRPTQ$

$SQMUSTQU*Q*S*R*ORUNTRRPUTSUT$

Twin system: replace KNUOOR by NKOURO.
Finally, and again using the same conventions, we list a pair of twin $STS(25)$s on the base set $V = \{a, b, c, \ldots, x, y\}$, a pair of twin $STS(27)$s on the base set $V = \{\sigma, a, b, c, \ldots, x, y, z\}$ and a pair of twin $STS(31)$s on the base set $V = \{\gamma, \delta, \lambda, \pi, \sigma, a, b, c, \ldots, x, y, z\}$.

For each value of $v$, it has been verified that all the systems $STS(v)$ are pairwise non-isomorphic by computing the cycle structure of each pair of elements of each system.

The method of construction of these twin systems was, essentially, a hill-climbing algorithm. In [5], two of the present authors used a modification of the hill-climbing algorithm described by Stinson [8] in order to construct anti-Pasch $STS(19)$s. The modification was
to check at each stage of the construction that no quadrilaterals had been introduced. The final block of a system constructed in this way is, of course, completely determined. Completing the system with this final block can result in a number of different outcomes. One of these is that the system is anti-Pasch. However, quadrilaterals are often introduced by this final block. Sometimes only one quadrilateral is introduced. In these cases one can try switching this quadrilateral and checking to see if further quadrilaterals are introduced. We find that in a small proportion of these cases no further quadrilaterals are generated by the switch and the two systems therefore form a twin pair. It seems that twin $STS(v)$s are comparatively rare amongst the collection of all pairwise non-isomorphic $STS(v)$s. On the basis of our computations, twin $STS(19)$s seem to be much sparser than anti-Pasch $STS(19)$s. However, this does not preclude there being a considerable number of such twin systems. Indeed, we would conjecture that they exist for all admissible $v \geq 19$.

3 Open Problems

The main problem of obtaining a reasonable description of the equivalence classes induced on $R(V)$ by the relations $q$ and $i$ remains. With respect to the latter, we have shown that for the values $v = 19, 21, 25, 27$ and 31 the removal of all anti-Pasch $STS(v)$s from $R(V)$ still leaves behind more than one equivalence class. We would conjecture that this is the case for all admissible values of $v \geq 19$. However, it is still feasible that $i$ partitions $R(V)$ into a relatively small number (perhaps only one or, more likely, $O(v)$) of “large” equivalence classes accompanied by a number of “small” equivalence classes each containing only a few systems. Examples of the latter are the anti-Pasch and twin systems. It seems more than likely that other equivalence classes containing a very limited number of realizations can be found. We are actively pursuing a search for these.

With regard to the above discussion two questions come immediately to mind.

(1) For a given admissible $v$, do all non-isomorphic $STS(v)$s which contain “enough” quadrilaterals fall into just a single equivalence class under the relation $i$? The word “enough” is difficult to define. There may be, up to isomorphism, a unique $STS(v)$ which contains more quadrilaterals than all other pairwise non-isomorphic $STS(v)$s. In such a case, the answer to (1) is trivially positive by choosing that system. This is known to be the case for $v = 2^d - 1$, $d \in \mathbb{Z}^+ [9]$. However, by “enough” we really intend a relatively small number of quadrilaterals, perhaps $O(v)$.

(2) Which equivalence classes under the relation $i$ are not further partitioned by the relation $q$?
Clearly the relation $q$ gives a finer partition of $R(V)$ into equivalence classes than the relation $i$, and the equivalence classes under $i$ which consist of either a single anti-Pasch system or a pair of twin systems are further partitioned into many equivalence classes under $q$. However, the single equivalence class under $i$ for the case $v = 7$ is not sub-partitioned further by $q$. We tentatively conjecture that this may be the distinguishing
characteristic between “large” and “small” equivalence classes.

Unfortunately, we feel that it will be very difficult to answer the two questions above. However, the following three problems may be more tractable and the solutions would shed some light on the general situation.

(a) Determine the value of $Q(13)$.

(b) For $v = 19$, determine other equivalence class structures under the relation $i$.

(c) Prove that there exists a pair of twin $STS(v)$s for all admissible $v \geq 19$.

References


