Five-Line Configurations in Steiner Triple Systems

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1 Introduction

In [1], three of the present authors initiated the study of generating sets and bases for configurations in Steiner triple systems. A *configuration* is simply any collection of blocks which can occur as part of a Steiner triple system, i.e. no pair is repeated, and if the configuration consists of n blocks it will be referred to as an *n-configuration.* For some configurations e.g. the *n-star*, n lines intersecting at a common point, the number of occurrences in a Steiner triple system of order v, STS(v), can be expressed as a rational polynomial in v. Since for any given admissible v, this number is the same irrespective of the structure of the STS(v), such configurations will be called *constant.* Other configurations are *variable.* It is easy and was shown in [1] that all one, two and three-configurations in STS(v) are constant. For n-configurations, $n \geq 4$, this is not the case and the question arises of finding a set of m-configurations, $m \leq n$, such that the number of occurrences of each member of the set determines the number of occurrences of all possible n-configurations. Such a set is called a *generating set* and when it is minimal, a *basis.* There are 16 pairwise non-isomorphic 4-configurations and in [1], formulae for the number of occurrences of each of them was given. These all take the form of a rational polynomial in v plus or minus an integral multiple of the number of occurrences of the Pasch configuration. The latter, which is any configuration isomorphic to $\{A, B, C\}, \{A, Y, Z\}, \{X, B, Z\}, \{X, Y, C\}$, is known to be variable. Thus any constant configuration, and we could choose just a single block, together with any variable 4-configuration, in particular the Pasch configuration, form a basis for 4-configurations.

In a follow-up paper [2], the authors prove a general result concerning the generating set for n-configurations in STS(v). Define the *valency* of a point in a configuration as being the number of blocks in which the point occurs. The main theorem in [2] is then that a generating set for n-configurations in a Steiner triple system consists of any constant configuration together with all m-configurations, $m \leq n$ in which every point has valency at least 2. This result is then used to determine bases for 5-configurations and 6-configurations. For the former a basis consists of any constant configuration, the Pasch configuration and the *mitre configuration.* The latter is any configuration isomorphic to $\{A, B, C\}, \{A, D, E\}, \{A, F, G\}, \{B, D, F\}, \{C, E, G\}$. The proof of the theorem rests on recursive formulae relating the number of occurrences of particular n- and $(n-1)$-configurations. In theory therefore the whole process of determining a formula for the number of occurrences of each configuration may be systematized, but in practice this would be a major and impracticable undertaking. Apart from identifying all the configurations, much other information is required to be gleaned which is much simpler to be done by inspection. For 5-configurations, of which there are 56, it is in fact very straightforward to proceed by the methods as described in [1]. In this paper we present these formulae, simply for reference for future researchers who will need them. We give no details of the calculations, the methods employed are as in [1], but they have been obtained by the Toronto authors and Preston authors independently. Some of the results, in particular the more complex formulae, are obtained only after very lengthy calculations of the formulae for other configurations, a situation which also arises with the recursive formulae from [2].

Finally, the study of configurations in Steiner triple systems raises some interesting fundamental questions. One of these, the characterization of the constant configurations, is discussed in [2]. Based on the results below, in the last section of this paper we highlight another of these questions. In addition we discuss how this also relates to the identification of the constant configurations. It is a further aim of this paper to present these ideas.

2 Results

For each of the 56 5-configurations we first list the blocks of the configuration. These are ordered by ascending order of the number of points in each. Alongside, for information, the five integers separated by dashes give the 4-configurations which are obtained when each of the five blocks is removed in turn from the 5-configuration. The numbering of the 4-configurations is as in [1]. On the second line is given the formula, where v is the order of the Steiner triple system, m is the number of mitre configurations and p is the number of Pasch configurations. Configuration #1 is the mitre configuration and #2 is the *mia configuration*, obtained by adjoining an extra block through two non-adjacent points of a Pasch configuration. The formulae may alternatively be given in terms of the numbers of mitre and mia configurations simply by dividing the coefficient of the number of Pasch configurations by 3. For simplicity we write $n(v) = v(v-1)(v-3)$ which frequently occurs as a factor. All polynomials are irreducible over $\mathbb{Q}$, the set of rationals.
[The points of the configurations here are $0, 1, \ldots, 9, a, b, c, d, e$.]

1) 012 034 135 236 456 — 14 – 14 – 14 – 15 – 15
$m :$ MITRE

2) 012 034 135 245 236 — 15 – 15 – 15 – 15 – 16
$3p :$ MIA

3) 012 034 135 236 147 — 11 – 11 – 14 – 15 – 15
$n(v)/2 - 12p$

4) 012 034 135 236 457 — 11 – 12 – 12 – 14 – 15
$n(v)/2 - 12p - 6m$

5) 012 034 135 245 067 — 11 – 11 – 12 – 12 – 16
$3(v-7)p$

6) 012 034 135 246 257 — 10 – 11 – 11 – 14 – 14
$n(v)/2 - 12p$

7) 012 034 135 246 567 $10-10-12-12-14$
$n(v)/4-6p-3m$
8) 012 034 135 067 168 $10-11-11-11-11$
$n(v)(v-8)/4+6p$
9) 012 034 135 067 268 $9-11-11-12-12$
$n(v)(v-10)/2+36p+6m$
10) 012 034 135 067 568 $8-10-10-11-12$
$n(v)(v-9)/2+24p$
11) 012 034 135 236 078 $8-11-11-12-15$
$n(v)(v-9)/4+12p$
12) 012 034 135 236 378 $7-11-11-11-15$
$n(v)(v-7)/12:$ CONSTANT
13) 012 034 135 236 478 $6-8-12-12-15$
$n(v)(v-11)/4+24p+6m$
14) 012 034 135 245 678 $6-6-6-6-16$
$(v-7)(v-12)p/6$
15) 012 034 135 246 078 $11-11-13-13-14$
$n(v)(v-9)/8-3(v-9)p+3m$
16) 012 034 135 246 178 $8-9-11-12-14$
$n(v)(v-9)/2-12(v-9)p$
17) 012 034 135 246 578 $6-9-9-12-14$
$n(v)(v-11)/4-6(v-11)p+6m$
18) 012 034 135 267 468 $9-9-10-12-12$
$n(v)(v-11)/2+48p+12m$
19) 012 034 156 357 468 $10-10-10-13-13$
$n(v)(v-10)/12+6p+2m$
20) 012 034 135 067 089 $7-8-8-11-11$
$n(v)(v-7)(v-9)/16:$ CONSTANT
21) 012 034 135 067 189 $8-8-9-11-11$
$n(v)(v-9)^2/8-12p$
22) 012 034 135 067 289 $5-8-11-12-13$
$n(v)(v^2-20v+103)/4+12(v-13)p-12m$
23) 012 034 135 067 589 $4-9-9-11-12$
$n(v)(v-9)(v-11)/8+6(v-11)p$
24) 012 034 135 067 689 $6-8-9-9-11$
$n(v)(v^2-20v+101)/4-60p-6m$

25) 012 034 135 236 789 $4-6-6-6-15$
$n(v)(v^2-22v+123)/36-12p-2m$
26) 012 034 135 246 789 $5-5-6-6-14$
$n(v)(v^2-22v+123)/24-(v^2-22v+123)p-3m$
27) 012 034 135 267 289 $3-8-8-12-12$
$n(v)(v-9)(v-11)/16+6(v-9)p$
28) 012 034 135 267 489 $5-5-9-12-12$
$n(v)(v^2-22v+125)/8+12(v-12)p-12m$
29) 012 034 135 267 689 $3-6-9-9-12$
$n(v)(v^2-22v+123)/4+12(v-16)p-18m$
30) 012 034 156 357 089 $9-9-10-10-11$
$n(v)(v-9)(v-10)/4+6(v-13)p$
31) 012 034 156 357 289 $5-10-10-11-14$
$n(v)(v^2-21v+114)/4+6(v-23)p-18m$
32) 012 034 156 378 579 $9-9-9-9-9$
$n(v)(v^2-21v+113)/10-36p-6m$
33) 012 034 056 078 09a $7-7-7-7-7$
$n(v)(v-5)(v-7)(v-9)/3840:$ 5-STAR, CONSTANT
34) 012 034 056 078 19a $4-7-8-8-8$
$n(v)(v-7)(v-9)(v-11)/96:$ CONSTANT
35) 012 034 056 178 19a $3-8-8-8-8$
$n(v)(v-9)^2(v-11)/128+3p$
36) 012 034 056 178 29a $2-8-8-13-13$
$n(v)(v-11)(v^2-20v+107)/64-6(v-11)p+3m$
37) 012 034 056 178 39a $5-5-8-8-9$
$n(v)(v-11)(v^2-20v+103)/16-6(v-19)p+6m$
38) 012 034 056 178 79a $3-4-8-9-9$
$n(v)(v-9)(v-11)^2/16-12(v-11)p$
39) 012 034 135 067 89a $4-5-5-6-11$
$n(v)(v-12)(v^2-20v+103)/24-6(v-17)p+6m$
40) 012 034 135 267 89a $2-5-5-6-12$
$n(v)(v-12)(v^2-22v+129)/24+2(v^2-28v+207)p+18m$
41) 012 034 135 678 69a $3-3-3-6-6$
$n(v)(v-13)(v^2-21v+114)/48-6(v-15)p+6m$
42) 012 034 156 278 39a $2-2-5-9-9$
$n(v)(v^3-33v^2+371v-1427)/16-12(2v-27)p+24m$

43) 012 034 156 357 89a $5-5-5-5-10$
$n(v)(v-12)(v^2-21v+116)/48+(v^2-25v+216)p/2+6m$

44) 012 034 156 378 59a $3-5-5-9-9$
$n(v)(v^3-33v^2+367v-1383)/8-24(v-18)p+36m$

45) 012 034 056 078 9ab $4-4-4-4-7$
$n(v)(v-7)(v-9)(v-11)(v-13)/2304:$ CONSTANT

46) 012 034 056 178 9ab $2-4-5-5-8$
$n(v)(v^2-20v+103)(v^2-24v+147)/96+12(v-14)p-6m$

47) 012 034 056 789 7ab $3-3-3-4-4$
$n(v)(v-9)(v-11)^2(v-18)/384+3(v-11)p$

48) 012 034 135 678 9ab $2-2-2-6-6$
$n(v)(v^2-22v+129)(v^2-25v+162)/432$
$-2(v^2-28v+219)p/3-6m$

49) 012 034 156 278 9ab $1-5-5-5-13$
$n(v)(v^4-46v^3+808v^2-6442v+19743)/288$
$-2(v^2-37v+318)p/3-10m$

50) 012 034 156 378 9ab $2-2-5-5-9$
$n(v)(v^4-46v^3+804v^2-6330v+18987)/48$
$-2(v^2-40v+405)p-42m$

51) 012 034 156 789 7ab $2-3-3-5-5$
$n(v)(v^4-46v^3+800v^2-6242v+18495)/64$
$+6(5v-74)p-27m$

52) 012 034 056 789 abc $2-2-2-4-4$
$n(v)(v-13)(v^4-45v^3+767v^2-5847v+16812)/3456$
$-3(v-13)p+m$

53) 012 034 156 789 abc $1-2-2-5-5$
$n(v)(v^5-60v^4+1458v^3-17944v^2+111909v-283428)/576$
$+2(v^2-37v+345)p+30m$

54) 012 034 567 589 abc $2-2-2-2-3$
$n(v)(v^5-60v^4+1454v^3-17784v^2+109809v-274284)/768$
$+(v^2-67v+768)p/2+18m$

55) 012 034 567 89a bcd $1-1-2-2-2$
$n(v)(v^6-75v^5+2370v^4-40402v^3+391905v^2-2051235v$
$+4530492)/10368-(v^2-40v+381)p-15m$

56) 012 345 678 9ab cde $1-1-1-1-1$
$n(v)(v^7-91v^6+3588v^5-79510v^4+1069873v^3-8742231v^2$
$+40167162v-80101224)/933120+(v-16)(v-21)p/6+2m$

3 Conclusion

As mentioned in [2], the authors make the conjecture that an n-configuration is constant if and only if it can be obtained from an $(n-1)$-star by adjoining a further block. In general this can be done in precisely five ways. They prove that the condition is sufficient and state that they strongly believe that it is also necessary. We do too. Below we introduce a different but we think essentially equivalent conjecture on the characterization of the constant configurations. However before we do so we firstly need to discuss another interesting fundamental question concerning configurations in Steiner triple systems.

It would appear that the following result, analogous to the graph reconstruction conjecture, is true.

Conjecture 1: For all $n \geq 2$, the collection of n $(n-1)$-configurations obtained from an n-configuration by removing each of the n blocks in turn, uniquely determines the n-configuration apart from four exceptions.

The exceptions are when $n = 2$, the pair of parallel blocks and the pair of intersecting blocks, and when $n = 3$, the 3-star and the *triangle* $\{A, F, B\}, \{A, E, C\}, \{B, D, C\}$. These are just the usual "small" exceptions which often occur in Combinatorics. The results in the previous section prove that the conjecture is true when $n = 5$ and it is straightforward to verify it for $n < 5$. However we suspect that in general it could be quite difficult to prove.

Within this framework we can now make the following conjecture concerning the constant configurations.

Conjecture 2: For all $n \geq 5$, an n-configuration is constant if and only if each of the n $(n-1)$-configurations which can be obtained from it by removing a block in turn is also constant.

Observe that this result is however false for $n = 4$ though interestingly enough not for $n = 2$ or 3. It is worth looking at the situation in a little more detail. Let C_n be a particular n-configuration. Then consider the following three statements.

A: C_n is constant.

B: C_n can be obtained from an $(n-1)$-star by adjoining a further line.

C: All n $(n-1)$-configurations obtained from C_n by removing a block in turn are constant.

We believe that for all $n \geq 5$, the three statements are equivalent. However for $n = 2, 3, 4$ there will be sporadic exceptions. In what follows $n \geq 2$. The present state of knowledge seems to be as follows. It is proved in [2] that B $\Rightarrow$ A. It is also conjectured that A $\Rightarrow$ B but there will be a single exception the *3-partial parallel class* $\{A, B, C\}, \{D, E, F\}, \{G, H, I\}$. It is also easily seen that B $\Rightarrow$ C. Again if C $\Rightarrow$ B, the 3-partial parallel class will be an exception. We have made the conjecture above that A $\Rightarrow$ C for all $n \geq 5$ and it is trivially true for $n = 2, 3, 4$. However C $\Rightarrow$ A must have as exceptions the eleven variable 4-configurations. To our mind the intricacy of the situation may make progress on the above questions difficult.

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References

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