Pairwise balanced designs on 4s+1 points with longest block of cardinality 2s

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This is a preprint of an article published in Utilitas Mathematica, 58, 2000, p97-107,

Abstract

The quantity $g^{(k)}(v)$ was introduced in [3] as the minimum number of blocks necessary in a pairwise balanced design on v elements, subject to the condition that the longest block have cardinality k. When $k \geq (v-1)/2$, except for the case where $v \equiv 1 \pmod 4$ and $k = (v-1)/2$, it is known that $g^{(k)}(v) = 1 + (v-k)(3k-v+1)/2$. The designs which achieve this bound contain, apart from the long block, only pairs and triples, all of which intersect the long block. This paper investigates the exceptional case where $v \equiv 1 (\text{mod } 4)$ and $k = (v-1)/2$. We prove that $PBD(v)$s with $g^{(k)}(v)$ blocks contain, apart from the long block, only pairs, triples, and quadruples, all of which intersect the long block. We also give a comprehensive description for the structure of the $PBD(v)$s.

1 Introduction

Let V be a set of cardinality v. The quantity $g^{(k)}(v)$ was introduced in [3] as the minimum number of blocks necessary in a pairwise balanced design, (with $\lambda = 1$), on v elements ($PBD(v)$), subject to the condition that the longest block have cardinality k. In [4], Woodall showed that, in effect, $g^{(k)}(v) \geq 1 + (v-k)(3k-v+1)/2$. It was later shown (Theorem 3.3 of [3]), that if $v \equiv 1 \pmod 4$ this bound is achieved for $k > (v-1)/2$ and otherwise for $k \geq (v-1)/2$. For $v > 7$ the longest block is uniquely defined. Moreover (Theorem 3.4 of [3]), the only designs which achieve this bound given above contain, apart from the long block, only pairs and triples all of which intersect the long block.

The case where $v = 4s+1, k = 2s, s \geq 2$, was considered in [2]. There (Theorem 4.1 of [2]), it was proved that $g^{(2s)}(4s+1) = 2s^2 + s + 1 + \lceil s/2 \rceil$. The designs which were constructed to achieve this bound contain, apart from the long block which is unique if $s > 2$, only pairs, triples, and quadruples. Also, all of these blocks intersect the long block. In this paper, we investigate this case further. In particular we show that all designs which achieve the bound have a similar structure and describe them in more detail. We remark that all $PBD(v)$s discussed in this paper have $\lambda = 1$.

First, we introduce some terminology and develop three fundamental equations from which our results will be deduced. Let $b_i, i \geq 2$, be the number of blocks of length i, apart from the long block. For all $i \geq 2$, let $b_{0,i}$ be the number of these blocks which are disjoint from the long block and let $b_{1,i}$ be the number which intersect the long block. Let $B_0 = \sum_{i\geq 2}(i-1)b_{0,i}$. Denote $g^{(k)}(v)$ by g.

The total number of blocks is

$$1 + \sum_{i\geq 2} b_i = g \tag{1.1}$$

The total number of pairs of elements, one of which is in the long block and the other is not, is given by $\sum_{i\geq 2}(i-1)b_{1,i} = k(v-k)$. Hence

$$\sum_{i\geq 2}(i-1)b_i = \sum_{i\geq 2}(i-1)b_{1,i} + \sum_{i\geq 2}(i-1)b_{0,i} = k(v-k) + B_0 \tag{1.2}$$

The total number of pairs of elements not in the long block is

$$\sum_{i\geq 2}\binom{i}{2}b_i = \binom{v}{2} - \binom{k}{2} \tag{1.3}$$

We consider the two cases where $s = 2t$ and $s = 2t+1$ separately.

2 The case $v = 8t+1, k = 4t$

The three fundamental equations become

$$\sum_{i\geq 2} b_i = 8t^2 + 3t \tag{2.1}$$

$$\sum_{i\geq 2}(i-1)b_i = 16t^2 + 4t + B_0 \tag{2.2}$$

$$\sum_{i\geq 2}\binom{i}{2}b_i = 24t^2 + 6t \tag{2.3}$$

By multiplying equations (2.1), (2.2), and (2.3) by 3, -3, and 1, respectively, and adding, we obtain

$$\sum_{i\geq 2}\frac{(i-3)(i-4)}{2}b_i = 3t - 3B_0 \tag{2.4}$$

Hence, we conclude

$$b_2 \leq 3t \tag{2.5}$$

Similarly, by mutiplying equations (2.1), (2.2), and (2.3) by 1, -2, and 1, respectively, and adding, we obtain

$$\sum_{i\geq 2}\frac{(i-2)(i-3)}{2}b_i = t - 2B_0 \tag{2.6}$$

Hence, we conclude

$$\sum_{j\geq 2} b_{2j} \leq t \tag{2.7}$$

Combination of inequalities (2.5) and (2.7) gives $\sum_{j\geq 1} b_{2j} \leq 4t$.

There are $4t+1$ elements not in the long block and hence at least one block intersecting each element of the long block is of even cardinality. Thus $\sum_{j\geq 1} b_{2j} \geq 4t$ and so, by comparison with the previous analysis, we obtain $\sum_{j\geq 1} b_{2j} = 4t$. Therefore $b_2 = 3t$ and, from equation (2.4), $B_0 = 0$ and

$b_i = 0, i \geq 5$. It now follows from equations (2.1) and (2.2) that $b_3 = 8t^2 - t$ and $b_4 = t$. A design having this structure was constructed, for all $t \geq 1$, in [2]. We state our results formally.

Theorem 2.1 *Let $v = 8t + 1$ and $k = 4t, t \geq 1$.*
All $PBD(v)$s in which the longest block has cardinality k and which contain only the minimum number of blocks, $g^{(k)}(v) = 8t^2+3t+1$, have the following structure:

1. *all blocks, apart from the long block of cardinality k, are pairs, triples, or quadruples and intersect the long block;*
2. *there are $3t$ pairs, $8t^2 - t$ triples, and t quadruples;*
3. *all elements in the long block are contained in either one pair and $2t$ triples (type I elements) or one quadruple and $2t - 1$ triples (type II elements);*
4. *there are $3t$ type I elements and t type II elements.*

It follows that, in order to construct the required design, it is necessary and sufficient to take the complete graph $K_{4t+1}, t \geq 1$, and decompose it into $4t$ 'factors', one for each element of the long block. The 'factors' corresponding to the $3t$ type I elements consist of $2t$ pairs (edges) and one singleton and those corresponding to the t type II elements consist of $2t-1$ pairs and one triple (triangle).

The method described in [2] is as follows. Let the vertex set of the complete graph K_{4t+1} be $\{0, 1, \ldots, 4t - 1, \infty\}$.
The 'factors' are

1. $\{\{\infty, 2i, 2i + 1\}, \{a, b\} : a + b \equiv 4i + 1(mod\ 4t), a, b \neq 2i\}$,
   $i = 0, 1, \ldots, t - 1$, (type A 'factors');
2. $\{\infty, \{a, b\} : a + b \equiv 4i + 3(mod\ 4t)\}$,
   $i = 0, 1, \ldots, t - 1$, (type B 'factors');
3. $\{i, \{i + 2t, \infty\}, \{a, b\} : a + b \equiv 2i(mod\ 4t), a \neq b\}$,
   $i = 0, 1, \ldots, 2t - 1$, (type C 'factors').

Type B and C 'factors' are assigned to type I elements and type A 'factors' are assigned to type II elements. When $t = 1$, this gives the following solution for $v = 9$ and $k = 4$. We list the blocks explicitly: $XYZW$ (the long block), $X\infty01$, $X23$, $Y\infty$, $Y03$, $Y12$, $Z0$, $Z2\infty$, $Z13$, $W1$, $W3\infty$, $W02$. It is easily seen that, up to isomorphism, this solution is unique. When $t \geq 2$, the solution produced above is <u>not</u> the unique solution.

To see this, observe that if the element ∞ is removed, then the type A and type B 'factors' form a one-factorization of the complete bipartite graph $K_{2t,2t}$ on the sets $\{0, 2, \ldots, 4t-2\}$ and $\{1, 3, \ldots, 4t-1\}$. For $t \geq 2$, this may be replaced by a non-isomorphic one-factorization. A one-factorization of $K_{2t,2t}$ is equivalent to a Latin square of side $2t$ and a simple argument shows that this has a partial transversal of length t. The equivalent object in the one-factorization is a set of t edges incident with $2t$ of the $4t$ points and lying in t distinct one-factors. Without loss of generality, these can be taken as $\{0, 1\}, \{2, 3\}, \ldots, \{2t-2, 2t-1\}$. Now assign the element ∞ to each of these pairs, thus forming triples, and assign ∞ as a singleton to each of the remaining one-factors. It is easily shown that non-isomorphic one-factorizations of $K_{2t,2t}$ produce non-isomorphic solutions to the problem.

3 The case $v = 8t + 5, k = 4t + 2$

The three fundamental equations become

$$\sum_{i \geq 2} b_i = 8t^2 + 11t + 4 \tag{3.1}$$

$$\sum_{i \geq 2} (i-1) b_i = 16t^2 + 20t + 6 + B_0 \tag{3.2}$$

$$\sum_{i \geq 2} \binom{i}{2} b_i = 24t^2 + 30t + 9 \tag{3.3}$$

By multiplying equations (3.1), (3.2), and (3.3) by 4, -5, and 2, respectively, and adding, we obtain

$$\sum_{i \geq 2} (i-3)^2 b_i = 4t + 4 - 5B_0 \tag{3.4}$$

There are $4t+3$ elements not in the long block and hence at least one block intersecting each element of the long block is of even cardinality. Thus $\sum_{j \geq 1} b_{2j} \geq 4t + 2$ and, by comparison with equation (3.4), it follows that $B_0 = 0$. Further comparison then establishes that $b_i = 0, i \geq 5$. Equations (3.1), (3.2), and (3.3) can now be solved for b_2, b_3, and b_4, and we obtain $b_2 = 3t + 3$, $b_3 = 8t^2 + 7t$, and $b_4 = t + 1$. Thus, this case is slightly more complex than the case discussed in the previous section. The total number of pairs and quadruples, $b_2 + b_4 = 4t + 4$, is greater than the cardinality of the long block. All of these blocks intersect the long block and every element of the long block must be contained in an odd number of them. It follows that there is a distinguished element of the long block which is contained in three of these blocks. There are four possibilities:

(A) three pairs,
(B) two pairs and one quadruple,
(C) one pair and two quadruples,
(D) three quadruples.
The remaining elements of the long block are each contained in precisely one block of even cardinality.

The solution described in [2] is the first of these. We describe it in similar terminology to the previous case. Let the vertex set of the complete graph K_{4t+3} be $\{0, 1, \ldots, 4t+1, \infty\}$.
The 'factors' are

1. $\{\{\infty, 2i, 2i+1\}, \{a, b\} : a+b \equiv 4i+1 (mod\ 4t+2), a, b \neq 2i\}$,
   $i = 0, 1, \ldots, t$;
2. $\{\infty, \{a, b\} : a+b \equiv 4i+3 (mod\ 4t+2)\}, i = 0, 1, \ldots, t-1$;
3. $\{0, 2t+1, \infty, \{a, b\} : a+b \equiv 0 (mod\ 4t+2), a \neq b\}$,
   the 'factor' assigned to the distinguished element;
4. $\{i, \{i+2t+1, \infty\}, \{a, b\} : a+b \equiv 2i (mod\ 4t+2), a \neq b\}$,
   $i = 1, 2, \ldots, 2t$.

The above 'factors' can be modified to produce solutions corresponding to possibility (B).

1. $\{\infty, \{a, b\} : a+b \equiv 4i+1 (mod\ 4t+2)\}, i = 0, 1, \ldots, t$;
2. $\{\{\infty, 2i, 2i+3\}, \{a, b\} : a+b \equiv 4i+3 (mod\ 4t+2), a, b \neq 2i\}$,
   $i = 0, 1, \ldots, t-1$;
3. $\{0, 2t+1, \{\infty, 2t, 2t+2\}, \{a, b\} : a+b \equiv 0 (mod\ 4t+2), a, b \neq 2t, a \neq b\}$,
   the 'factor' assigned to the distinguished element;
4. $\{2t+2, \{1, \infty\}, \{a, b\} : a+b \equiv 2 (mod\ 4t+2), a \neq b\}$;
5. $\{i, \{i+2t+1, \infty\}, \{a, b\} : a+b \equiv 2i (mod\ 4t+2), a \neq b\}$,
   $i = 2, 3, \ldots, 2t$.

When $t = 1$, possibilities (C) and (D) can not occur. The reason in the latter case is trivial. There are seven elements not in the long block, but the assignment of three quadruples to the distinguished element requires nine elements. Before considering possibility (C), we state and prove an easy but very useful lemma which applies both to the cases in this section and in the previous section.

Lemma 3.1 *Let $v = 4s + 1$ and $k = 2s, s \geq 2$.*
An element x, not in the long block, occurs the same number of times in the pairs as in the quadruples.
Proof. Let x occur in α pairs, β triples, and γ quadruples. Since x must occur with every element of the long block, it follows that $\alpha + \beta + \gamma = 2s$. Since x must also occur with every other element not in the long block, it follows that $\beta + 2\gamma = 2s$. Hence $\alpha = \gamma$. $\square$

Now, returning to possibility (C) when $v = 13$ and $k = 6$, consider the two quadruples assigned to the distinguished element. These contain six elements not in the long block which must occur in pairs, each containing different elements of the long block. Hence, including the distinguished element, the long block must have cardinality at least seven.

Nevertheless, solutions to the problem corresponding to possibilities (C) and (D) do exist for all $t \geq 2$. First, we give the 'factors' corresponding to possibility (C).

1. $\{\{\infty, 2t + 2i, 2t + 2i + 5\}, \{a, b\} : a + b \equiv 4i + 3(mod\ 4t + 2),$
   $a, b \neq 2t + 2i\}, i = 0, 1, \ldots, t - 2;$
2. $\{\infty, \{a, b\} : a + b \equiv 4i + 1(mod\ 4t + 2)\}, i = 1, 2, \ldots, t - 2,$
   (hence there are no 'factors' of this type if $t = 2$);
3. $\{4t + 1, \{j + 1, \infty\}, \{a, b\} : a + b \equiv j(mod\ 4t + 2), a, b \neq j + 1\},$
   $j = 1, 4t - 3;$
4. $\{2t, \{2t + j, \infty\}, \{a, b\} : a + b \equiv 4t + j(mod\ 4t + 2), a, b \neq 2t + j\},$
   $j = -1, 1;$
5. $\{i + 2t + 1, \{i, \infty\}, \{a, b\} : a + b \equiv 2i(mod\ 4t + 2), a \neq b\},$
   $i = 0, 1, \ldots, 2t - 2, i \neq 2;$
6. $\{i, \{i + 2t + 1, \infty\}, \{a, b\} : a + b \equiv 2i(mod\ 4t + 2), a \neq b\}, i = 2, 2t - 1;$
7. $\{\infty, \{2t - 1, 2t, 2t + 1\}, \{2, 4t - 2, 4t + 1\},$
   $\{a, b\} : a + b \equiv 4t(mod\ 4t + 2), a, b \neq 2, 2t - 1, a \neq b\},$
   the 'factor' assigned to the distinguished element.

When $t \geq 3$, the above 'factors' can be modified to produce solutions corresponding to possibility (D).

1. $\{\infty, \{a, b\} : a + b \equiv 4i + 3(mod\ 4t + 2)\}, i = 0, 1, \ldots, t - 2;$
2. $\{\{\infty, 2t - 2 + 2i, 2t + 5 + 2i\}, \{a, b\} : a + b \equiv 4i + 1(mod\ 4t + 2),$
   $a, b \neq 2t - 2 + 2i\}, i = 1, 2, \ldots, t - 2;$

3. $\{4t+1, \{j+1, \infty\}, \{a,b\} : a+b \equiv j(mod\ 4t+2), a,b \neq j+1\}$, $j = 1, 4t-3$;

4. $\{2t, \{2t+j, \infty\}, \{a,b\} : a+b \equiv 4t+j(mod\ 4t+2), a,b \neq 2t+j\}$, $j = -1, 1$;

5. $\{i+2t+1, \{i, \infty\}, \{a,b\} : a+b \equiv 2i(mod\ 4t+2), a \neq b\}$, $i = 0, 1, \ldots, 2t-2, i \neq 2, 4$, (note that this requires $t \geq 3$);

6. $\{i, \{i+2t+1, \infty\}, \{a,b\} : a+b \equiv 2i(mod\ 4t+2), a \neq b\}, i = 2, 4, 2t-1$;

7. $\{\{\infty, 4, 4t-4\}, \{2t-1, 2t, 2t+1\}, \{2, 4t-2, 4t+1\}$, $\{a,b\} : a+b \equiv 4t(mod\ 4t+2), a,b \neq 2, 4, 2t-1, a \neq b\}$, the 'factor' assigned to the distinguished element.

This leaves unresolved a solution for the case $t = 2$ that is $v = 21$ and $k = 10$, corresponding to possibility (D). But this can be achieved by using the following 'factors' of the complete graph K_{11} on the vertex set $\{0, 1, \ldots, 9, \infty\}$. Set brackets and delimiting commas are omitted for clarity.

1, 02, ∞3, 47, 58, 69;	2, 03, ∞1, 49, 57, 68;
3, 01, ∞2, 48, 59, 67;	4, 05, ∞6, 17, 28, 39;
5, 06, ∞4, 19, 27, 38;	6, 04, ∞5, 18, 29, 37;
7, 08, ∞9, 14, 25, 36;	8, 09, ∞7, 16, 24, 35;
9, 07, ∞8, 15, 26, 34;	123, 456, 789, ∞0.

The results in this section are stated formally in the following theorem.

Theorem 3.1 *Let $v = 8t+5$ and $k = 4t+2, t \geq 1$.*
All $PBD(v)$s in which the longest block has cardinality k and which contain only the minimum number of blocks, $g^{(k)}(v) = 8t^2 + 11t + 4$, have the following structure:

1. *all blocks, apart from the long block of cardinality k, are pairs, triples or quadruples, and intersect the long block;*

2. *there are $3t+3$ pairs, $8t^2+7t$ triples, and $t+1$ quadruples;*

3. *there is a distinguished element of the long block which is contained in x quadruples, $2t-x$ triples, and $3-x$ pairs for some $x : 0 \leq x \leq 3$;*

4. *all other elements in the long block are contained in either one pair and $2t+1$ triples (type I elements) or one quadruple and $2t$ triples (type II elements);*

5. *there are $3t+x$ type I elements and $t+1-x$ type II elements.*

Moreover there exists such a $PBD(v)$ for all $t \geq 1$ and $0 \leq x \leq 3$ except for $(t,x) =$ (1,2) or (1,3).

4 The case $v = 13, k = 6$

We have already observed that when $v = 9$ and $k = 4$, the $PBD(v)$ which achieves the minimum bound is unique up to isomorphism. The only other case in which it is feasible to enumerate solutions would appear to be when $v = 13$ and $k = 6$. This has already been done in [1] but is complex. With the information about the structure of the designs obtained in this paper, in particular Lemma 3.1, the following approach is more transparent.

Let $V = \{A, B, C, D, E, F, 1, 2, 3, 4, 5, 6, 7\}$, and let the long block be $ABCDEF$. Let the distinguished element be A. Only possibility (A) three pairs, and possibility (B) two pairs and one quadruple, occur in this case, and we first consider the former. It is to be understood that the arguments will often proceed 'without loss of generality'. Let the blocks containing the distinguished element be $A12$, $A34$, $A5$, $A6$, and $A7$. There are three type I elements and two type II elements. Let the latter be B and C. Since each of the elements 5, 6, and 7 appear in a pair, we see from Lemma 3.1 that each of these elements must appear in at least one of the two quadruples. Thus there are two alternatives for the blocks containing B and C. The first is that the blocks containing B are $B567$, $B13$, and $B24$. Then the blocks containing C are $C145$, $C26$, and $C37$; hence by Lemma 3.1, we also have $D1$, $E4$, and $F5$. It is then easy to verify that the design may be completed in two ways: (i) $D47$, $D25$, $D36$, $E16$, $E27$, $E35$, $F17$, $F46$, $F23$; or (ii) $D46$, $D27$, $D35$, $E17$, $E25$, $E36$, $F16$, $F47$, $F23$.

The other alternative is that the blocks containing B are $B156$, $B24$, and $B37$. There are now four cases for the blocks containing C which, for ease of checking by the reader, we list first: (iii) $C257$, $C13$, $C46$; (iv) $C257$, $C14$, $C36$; (v) $C457$, which then forces $C13$ and $C26$; (vi) $C147$, $C25$, $C36$. In case (iii), the design has a unique completion, $D1$, $D26$, $D35$, $D47$, $E2$, $E17$, $E36$, $E45$, $F5$, $F14$, $F23$, $F67$. In case (iv) the design can be completed in two ways: (a) $D1$, $D23$, $D45$, $D67$, $E2$, $E17$, $E35$, $E46$, $F5$, $F13$, $F26$, $F47$; or (b) $D1$, $D26$, $D35$, $D47$, $E2$, $E13$, $E45$, $E67$, $F5$, $F23$, $F17$, $F46$. However these two designs are isomorphic under the permutation $(BC)(DE)(12)(67)$. Cases (v) and (vi) both have unique completions. The former is (v) $D1$, $D27$, $D35$, $D46$, $E4$, $E17$, $E25$, $E36$, $F5$, $F14$, $F23$, $F67$; the latter is (vi) $D1$, $D23$, $D45$, $D67$, $E1$, $E27$, $E35$, $E46$, $F4$, $F13$, $F26$, $F57$.

Now consider possibility (B) and let the blocks containing the distinguished element be $A123$, $A45$, $A6$, and $A7$. There are four type I elements and one type II element, say B. Then, without loss of generality, and using Lemma 3.1, the blocks containing B are $B167$, $B24$, and $B35$. This in turn forces further blocks $C1$, $D1$, $E2$, and $F3$. Now consider the pairs 14 and 15. The first alternative is (vii) blocks $E14$ and $F15$, which then forces, without loss of generality, $E36$ and $E57$; the design has a unique completion

$C25$, $C37$, $C46$, $D27$, $D34$, $D56$, $F26$, $F47$. The second alternative is (viii & ix) blocks $E15$ and $F14$ which again forces, without loss of generality, $E36$ and $E47$. But there are now two ways of completing the design: (viii) $D25$, $D37$, $D46$, $C26$, $C34$, $C57$, $F27$, $F56$; or (ix) $D25$, $D37$, $D46$, $C27$, $C34$, $C56$, $F26$, $F57$.

For ease of reference, we list the nine solutions below in compact notation. By examining the structure of these, in particular the elements 5, 6, and 7, it is easily verified that the nine solutions are pairwise non-isomorphic.

POSSIBILITY (A).
The long block is $ABCDEF$. A is the distinguished element.
All designs contain the blocks $A12$, $A34$, $A5$, $A6$, $A7$.
The six designs are

B: 567 13 24	B: 567 13 24	B: 156 24 37
C: 145 26 37	C: 145 26 37	C: 257 13 46
D: 1 47 25 36	D: 1 46 27 35	D: 1 26 35 47
E: 4 16 27 35	E: 4 17 25 36	E: 2 17 36 45
F: 5 17 46 23	F: 5 16 47 23	F: 5 14 23 67

B: 156 24 37	B: 156 24 37	B: 156 24 37
C: 257 14 36	C: 457 13 26	C: 147 25 36
D: 1 23 45 67	D: 1 27 35 46	D: 1 23 45 67
E: 2 17 35 46	E: 4 17 25 36	E: 1 27 35 46
F: 5 13 26 47	F: 5 14 23 67	F: 4 13 26 57

POSSIBILITY (B).
The long block is $ABCDEF$. A is the distinguished element.
All designs contain the blocks $A123$, $A45$, $A6$, $A7$.
The three designs are

B: 167 24 35	B: 167 24 35	B: 167 24 35
C: 1 25 37 46	C: 1 26 34 57	C: 1 27 34 56
D: 1 27 34 56	D: 1 25 37 46	D: 1 25 37 46
E: 2 14 36 57	E: 2 15 36 47	E: 2 15 36 47
F: 3 15 26 47	F: 3 14 27 56	F: 3 14 26 57

This listing verifies the results given in [1].

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