The content of an equiangular hexagon with integral sides

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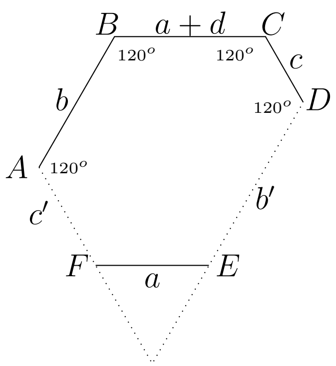
1976

In a recent issue of the Mathematical Gazette [1], A. C. Heath posed the question of tessellating a convex hexagon with interior angles of 120^o and integral sides, using equilateral triangles of unit side. The number of such triangles used is defined to be the content of the hexagon. In this article we show that it is possible to construct a hexagon with content $x \geq 6$ if and only if x is not one of the numbers $7, 8, 9, 11, 12, 15, 17, 20, 21, 23, 29, 36, 39, 41, 44, 84$.

Proposition 1. An equiangular hexagon with integral sides may be characterised by four integers a, b, c, d, where $a, b, c \geq 1$ and $d \geq 0$. The numbers a, b, c represent three alternate sides and $(a + d)$ represents the side opposite a (without loss of generality we may assume $d \geq 0$). The content of the hexagon is given by

$$\phi(a, b, c, d) = 2(ab + bc + ca) + 2d(a + b + c) + d^2.$$

Proof. Construct the configuration $ABCD$ and the dotted lines as shown in the diagram.



Place a line segment (EF) of length a in the unique position parallel to BC where its ends rest on the dotted lines. Put $b' = DE$, $c' = FA$.

Construction of equilateral triangles on AB, CD, EF shows that $b' = b + d$ and $c' = c + d$, so that b' and c' are both positive integers. By considering the circumscribed equilateral triangle and then removing the three smaller equilateral triangles it can be seen that the content of the hexagon is given by

$$\begin{aligned}\phi(a, b, c, d) &= (b + (a + d) + c)^2 - a^2 - b^2 - c^2 \\ &= 2(ab + bc + ca) + 2d(a + b + c) + d^2.\end{aligned}$$

In our subsequent discussion of $\phi(a, b, c, d)$ we shall assume that a, b, c are positive integers and that d is a non negative integer.

Proposition 2. For $x \equiv 0, 7, 8, 12, 15 \pmod{16}$ there exist values of a, b, c, d such that $\phi(a, b, c, d) = x$, with the possible exceptions of the cases $x = 7, 8, 12, 15, 23, 28, 31, 39, 44, 47$.

Proof. $\phi(a, 2, 2, 0) = 8a + 8$; apart from $x = 8$ this generates all numbers of the form $x \equiv 0, 8 \pmod{16}$.

$\phi(a, 4, 3, 1) = 16a + 39$; apart from $x = 7, 23, 39$ this generates all numbers of the form $x \equiv 7 \pmod{16}$.

$\phi(a, 4, 2, 2) = 16a + 44$; apart from $x = 12, 28, 44$ this generates all numbers of the form $x \equiv 12 \pmod{16}$.

$\phi(a, 4, 1, 3) = 16a + 47$; apart from $x = 15, 31, 47$ this generates all numbers of the form $x \equiv 15 \pmod{16}$.

It is possible to eliminate other residue classes in this fashion (for example see Proposition 5). However, we have not found a suitable integer all of whose residue classes can be so eliminated. The following proposition provides an alternative approach. We assume x is an arbitrary positive integer.

Proposition 3. A necessary and sufficient condition for the equation $\phi(a, b, c, d) = x$ to have a solution is that there exist positive integers $\alpha, \beta, \gamma, \epsilon$ such that

(i) $x = \epsilon^2 - \alpha^2 - \beta^2 - \gamma^2$,

(ii) $2(\alpha\beta + \beta\gamma + \gamma\alpha) \leq x$.

If the conditions are satisfied then $\phi(\alpha, \beta, \gamma, \sigma) = x$, where $\sigma = \epsilon - (\alpha + \beta + \gamma)$. This value of σ is necessarily non-negative.

Proof.

(a) *Necessity.* If $\phi(\alpha, \beta, \gamma, \sigma) = x$, then

$$\sigma^2 + 2\sigma(\alpha + \beta + \gamma) + 2(\alpha\beta + \beta\gamma + \gamma\alpha) - x = 0.$$

Solving for σ we obtain

$$\sigma = -(\alpha + \beta + \gamma) + \sqrt{\alpha^2 + \beta^2 + \gamma^2 + x}.$$

Hence

(i) $\alpha^2 + \beta^2 + \gamma^2 + x$ is a perfect square, ϵ^2, say.

(ii) Since $\sigma \geq 0$, $\alpha^2 + \beta^2 + \gamma^2 + x \geq (\alpha + \beta + \gamma)^2$. Therefore $2(\alpha\beta + \beta\gamma + \gamma\alpha) \leq x$.

(b) *Sufficiency.* Suppose the conditions are satisfied.

Put $\sigma = \epsilon - (\alpha + \beta + \gamma)$. Then $\phi(\alpha, \beta, \gamma, \sigma) = x$. Moreover

$$\begin{aligned}\epsilon^2 &= x + \alpha^2 + \beta^2 + \gamma^2 \\ &\geq 2(\alpha\beta + \beta\gamma + \gamma\alpha) + \alpha^2 + \beta^2 + \gamma^2 \\ &= (\alpha + \beta + \gamma)^2\end{aligned}$$

so that $\sigma \geq 0$.

Proposition 4. If y is any positive integer not of the form $4^e(8f + 7)$ $(e, f = 0, 1, 2, \ldots)$ then

(i) y is a perfect square, or

(ii) y can be expressed as the sum of the squares of two positive integers, or

(iii) y can be expressed as the sum of the squares of three positive integers.

Proof. This result may be found in standard textbooks, For example, see [2].

We now establish the major part of our result.

Proposition 5. If $x \not\equiv 4 \pmod{16}$ then the equation $\phi(a, b, c, d) = x$ has a solution for $x \geq 317$. If $x \equiv 4 \pmod{16}$ then the equation has a solution for $x \geq 1268$.

Proof. Proposition 2 has already dealt with the cases $x \equiv 0, 7, 8, 12, 15$ (mod 16). This leaves

(a) $x \equiv 1, 2, 3, 5, 6 \pmod 8$, and

(b) $x \equiv 4 \pmod{16}$.

For both (a) and (b) we make use of Propositions 3 and 4. Firstly for the given x we define

$$g_0 = \lfloor \sqrt{x} \rfloor + 1,$$

where $\lfloor \ \rfloor$ denotes the integer part. We note that

$$\sqrt{x} < g_0 \leq \sqrt{x} + 1.$$

For $i = 0, 1, 2, \ldots, 7$, we define $g_i = g_0 + i$ and observe that

$$\begin{aligned} g_i^2 - x &\leq (\sqrt{x} + 1 + i)^2 - x \\ &= 2(1+i)\sqrt{x} + (1+i)^2. \end{aligned} \tag{5.1}$$

(a) $x \equiv 1, 2, 3, 5, 6 \pmod 8$.
For $i = 0, 1, 2, 3$ there are three distinct residues (mod 8) taken by g_i^2, namely $0, 1, 4$. For each possible value of x choose an appropriate i so that the values of g_i^2 and $g_i^2 - x$ are as tabulated below.

value of x	value of g_i^2	value of $g_i^2 - x$ (mod 8)
1	4	3
2	0	6
3	1	6
5	0	3
6	1	3

Now a number equal to 3 or 6 (mod 8) cannot be a perfect square or the sum of two squares since for any integers z_1, z_2, $z_1^2 \equiv 0, 1, 4 \pmod 8$ and $z_1^2 + z_2^2 \equiv 0, 1, 2, 4, 5 \pmod 8$. Thus if we can show that in each of the above cases $g_i^2 - x$ satisfies the conditions of Proposition 4 it will follow that there exist positive integers α, β, γ such that

$$x = g_i^2 - \alpha^2 - \beta^2 - \gamma^2.$$

Suppose that $g_i^2 - x \equiv 3 \pmod 8$. Then there exist $h = 0, 1, 2, \ldots$ such that $g_i^2 - x = 8h + 3$. But clearly $8h + 3 \neq 4^e(8f + 7)$ for any choice of e, f, h.
The case $g_i^2 - x \equiv 6 \pmod 8$ is dealt with similarly.

(b) $x \equiv 4 \pmod{16}$.
Choose $i = 0, 1, 2$ or 3 such that g_i is divisible by 4. Then $g_i^2 - x \equiv 12 \pmod{16}$. A number equal to 12 (mod 16) cannot be a perfect square or the sum of two squares, since for any integers z_1, z_2, $z_1^2 \equiv 0, 1, 4, 9 \pmod{16}$ and $z_1^2 + z_2^2 \equiv 0, 1, 2, 4, 5, 6, 9, 10, 13 \pmod{16}$. Take j to be the non-negative integer for which $g_i^2 - x = 16j + 12 = 4(4j + 3)$.

(i) Suppose j is even: $j = 2k$. Then $g_i^2 - x = 4(8k+3)$, and clearly $4(8k+3) \neq 4^e(8f+7)$ for any choice of e, f, k.

(ii) Suppose j is odd: $j = 2k-1$. Then

$$\begin{aligned} g_{i+4}^2 - x &= g_i^2 - x + 8g_i + 16 \\ &= 4(4j+3) + 8g_i + 16 \\ &= 4(8k-1) + 32\ell + 16 \quad \text{(where } 4\ell = g_i\text{)} \\ &= 4(8(k+\ell)+3). \end{aligned}$$

Hence $g_{i+4}^2 - x \equiv 12 \pmod{16}$ and $g_{i+4}^2 - x$ is not of the form $4^e(8f+7)$.

It follows that in either of the cases (i) or (ii) it is possible to find a value of $i = 0, 1, 2, \ldots, 7$ for which there are positive integers α, β, γ such that $x = g_i^2 - \alpha^2 - \beta^2 - \gamma^2$.

To complete the proof of Proposition 5 it remains to show that condition (ii) of Proposition 3 - i.e. $2(\alpha\beta + \beta\gamma + \gamma\alpha) \leq x$ - is satisfied for all suitably large values of x. From equation (5.1) we have in all cases

$$\alpha^2 + \beta^2 + \gamma^2 \leq 2(1+i)\sqrt{x} + (1+i)^2,$$

where $i \leq 3$ for $x \equiv 1, 2, 3, 5, 6 \pmod 8$ and $i \leq 7$ for $x \equiv 4 \pmod{16}$. For any numbers α, β we have $(\alpha - \beta)^2 \geq 0$ and so $2\alpha\beta \leq \alpha^2 + \beta^2$. Adding two similar inequalities we obtain $2(\alpha\beta + \beta\gamma + \gamma\alpha) \leq 2(\alpha^2 + \beta^2 + \gamma^2)$. It follows that

$$2(\alpha\beta + \beta\gamma + \gamma\alpha) \leq 4(1+i)\sqrt{x} + 2(1+i)^2.$$

Thus $2(\alpha\beta + \beta\gamma + \gamma\alpha) \leq x$ provided

$$x \geq 4(1+i)\sqrt{x} + 2(1+i)^2,$$

which is equivalent to

$$x \geq (1+i)^2(10 + 4\sqrt{6}).$$

For the cases $x \equiv 1, 2, 3, 5, 6 \pmod 8$ this gives $x \geq 317$, while for $x \equiv 4 \pmod{16}$ it gives $x \geq 1268$.

In principle, Proposition 5 provides the solution to the problem, at any rate for $x \geq 1268$. Quite a large number of individual cases remain however. To simplify their consideration we establish a further result on residue classes.

Proposition 6.

(i) For $x \not\equiv 1, 4, 6, 9 \pmod{10}$ there exist values of a, b, c, d such that $\phi(a, b, c, d) = x$, with the possible exceptions of the cases $x = 2, 3, 5, 7, 8, 10, 12, 13, 15, 17, 20, 23$.

(ii) For $x \not\equiv 0, 7 \pmod{14}$ there exist values a, b, c, d such that $\phi(a, b, c, d) = x$, with the possible exceptions of the cases $x = 1, 2, \ldots, 25, 27, 29, 30, 31, 32, 33, 36, 39, 41, 44, 47$.

Proof.

(i)

$$\begin{array}{rclcrcl} \phi(a,4,1,0) &=& 10a+8 & & \phi(a,3,2,0) &=& 10a+12 \\ \phi(a,3,1,1) &=& 10a+15 & & \phi(a,2,2,1) &=& 10a+17 \\ \phi(a,2,1,2) &=& 10a+20 & & \phi(a,1,1,3) &=& 10a+23 \end{array}$$

(ii)

$$\begin{array}{rclcrcl} \phi(a,6,1,0) &=& 14a+12 & & \phi(a,5,2,0) &=& 14a+20 \\ \phi(a,4,3,0) &=& 14a+24 & & \phi(a,5,1,1) &=& 14a+23 \\ \phi(a,4,2,1) &=& 14a+29 & & \phi(a,3,3,1) &=& 14a+31 \\ \phi(a,4,1,2) &=& 14a+32 & & \phi(a,3,2,2) &=& 14a+36 \\ \phi(a,3,1,3) &=& 14a+39 & & \phi(a,2,2,3) &=& 14a+41 \\ \phi(a,2,1,4) &=& 14a+44 & & \phi(a,1,1,5) &=& 14a+47 \end{array}$$

It is now possible to provide the complete solution to the original problem.

Theorem 1. $\phi(a, b, c, d)$ takes all positive integral values of x apart from $x = 1, 2, 3, 4, 5, 7, 8, 9, 11, 12, 15, 17, 20, 21, 23, 29, 36, 39, 41, 44, 84$.

Proof. Elimination of those numbers covered by Propositions 2, 5, and 6 leaves the following numbers still to be considered.

$1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 17, 19, 20, 21, 23, 29, 31, 36, 39, 41, 44, 49, 84, 91, 126, 154, 161, 189, 196, 231, 259, 266, 294, 301, 644, 756, 1204$.

The tabulation below deals with those which can be obtained from ϕ.

x	6	10	13	14	19	31	49	91	126	154	161	189
a	1	1	1	3	2	4	7	14	15	38	5	7
b	1	2	1	1	1	1	1	1	3	1	3	3
c	1	1	1	1	1	1	1	1	1	1	1	3
d	0	0	1	0	1	1	1	1	0	0	5	3

x	196	231	259	266	294	301	644	756	1204
a	32	12	42	20	23	49	32	40	99
b	2	4	1	3	3	1	3	6	4
c	1	3	1	1	3	1	2	3	2
d	0	1	1	2	0	1	4	0	0

The remaining numbers cannot be obtained from ϕ. To see this for a particular number x we use Proposition 3. It is only necessary to list all the possible values of α, β, γ which satisfy $2(\alpha\beta+\beta\gamma+\gamma\alpha) \leq x$ and to check that for each possibility, the number $(\alpha^2 + \beta^2 + \gamma^2 + x)$ is not a perfect square. We leave the details for the reader to complete.

References.

1. A. C. Heath, Problem 7, The Mathematical Gazette, Vol.60. Number 411 (March 1976).

2. L. J. Mordell, Diophantine equations. Academic Press (1969) pages 175–178.

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