Biembeddings of Latin squares and Hamiltonian decompositions

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Abstract

Face 2-colourable triangulations of complete tripartite graphs $K_{n,n,n}$ correspond to biembeddings of Latin squares. Up to isomorphism, we give all such embeddings for $n = 3, 4, 5$ and 6, and we summarize the corresponding results for $n = 7$. Closely related to these are Hamiltonian decompositions of complete bipartite directed graphs $K^*_{n,n}$, and we also give computational results for these in the cases $n = 3, 4, 5$ and 6.

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Biembeddings of Latin squares

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1 Introduction

A number of recent papers [2, 5, 6] have dealt with biembeddings of pairs of Steiner triple systems (STSs) in both orientable and nonorientable surfaces. Such a biembedding corresponds to a face 2-colourable triangulation of a complete graph K_n. The vertices of the graph form the points of the Steiner triple systems and the triangular faces in each of the two colour classes respectively form the triples of each system. We here recall that an STS(n) may be formally defined as an ordered pair $(V, \mathcal{B})$, where V is an n-element set (the *points*) and $\mathcal{B}$ is a set of 3-element subsets of V (the *triples*), such that every 2-element subset of V appears in precisely one triple. Such systems are known to exist if and only if $n \equiv 1$ or 3 (mod 6). We say that two STS(n)s are *biembedded* in a surface if there is a face 2-colourable triangulation of K_n in which the face sets forming the two colour classes give copies of the two systems. We will take the colour classes of face 2-colourable embeddings to be black and white.

One may consider embeddings which involve other types of combinatorial design. Embeddings of complete tripartite graphs are discussed in [7, 11] and form a useful tool in recursive constructions for biembeddings of Steiner triple systems. A face 2-colourable triangulation of the complete tripartite graph $K_{n,n,n}$ may be considered as a biembedding of a pair of transversal designs TD$(3, n)$; such a design comprises an ordered triple $(V, \mathcal{G}, \mathcal{B})$, where V is a $3n$-element set (the *points*), $\mathcal{G}$ is a partition of V into three disjoint sets (the *groups*) each of cardinality n, and $\mathcal{B}$ is a set of 3-element subsets of V (the *triples*), such that every unordered pair of elements from V is either contained in precisely one triple or one group, but not both. As with STSs, the vertices of the embedded graph $K_{n,n,n}$ form the points of the designs, the tripartition determines the groups, and the faces in each colour class form the triples of each design. The connection with Latin squares is that a TD$(3, n)$ determines a *main class* of Latin squares by assigning the three groups of the design as the identifiers of the rows, columns and entries (in any one of the six possible orders) of the Latin square. Thus a face 2-colourable triangulation of $K_{n,n,n}$ may be considered as a biembedding of two Latin squares.

Before proceeding, we here review some basic aspects of notation and terminology which are important for our discussion. For further details we refer the reader to texts such as [3, 9, 10]. Two Latin squares of the same order n will be regarded as isomorphic if they belong to the same main class, i.e. if there exist three bijections mapping the row, column and entry identifiers of the first square to those of the second (not necessarily in the same order) which map the first square to the second. We assume that the reader is familiar with the description of topological embeddings by means of rotation schemes. Our embeddings will always be in surfaces rather than pseudosurfaces (the latter result from surfaces by repeating a finite number of times the operation of identifying a finite number of points on a surface). Equivalently, in the description of an embedding by means of a rotation scheme, the rotation about each vertex comprises a single cycle. In Section 4 we make use of voltage graphs to construct certain embeddings. When discussing isomorphisms and automorphisms of embeddings, we allow in the orientable case mappings which reverse the orientation and, in the case of face 2-colourability, mappings which reverse the two colour classes.

An embedding of a graph G is said to be *regular* if and only if for every two *flags*, i.e. triples (v_1, e_1, f_1) and (v_2, e_2, f_2), where e_i is an edge incident with the vertex v_i and the face f_i, there exists an automorphism of the embedding which maps v_1 to v_2, e_1 to e_2 and f_1 to f_2. This definition of regularity is equivalent to requiring that the automorphism group of the embedding be as large as possible. Thus, in the case $G = K_{n,n,n}$, an embedding is regular if and only if its automorphism group has order $12n^2$; there being $3n^2$ edges each of which is incident with two vertices and two faces.

We point out that the definition of regularity varies somewhat between authors; see [1] (p.36) for a discussion of the terminology. The definition of regularity given above, called by some authors *reflexive regularity*, requires the admission of automorphisms which reverse the orientation of the surface in the orientable case. However, many authors require that any global orientation of the surface is preserved and this means that their regular embeddings may be less symmetric.

Our first observation is easily proved but, possibly, surprising.

Proposition 1 *A triangulation of $K_{n,n,n}$ is orientable if and only if it is face 2-colourable.*

Proof. Suppose that $K_{n,n,n}$ has tripartition $\{A, B, C\}$. If an orientable embedding is given, then triangles with clockwise orientation (A, B, C) may be coloured black and those with clockwise orientation (A, C, B) may be coloured white. Conversely, suppose that a face 2-colourable triangulation is given. If a black triangle of the embedding with vertices $a \in A, b \in B, c \in C$ is oriented, say clockwise, as (A, B, C), then all black triangles incident with a also have clockwise orientation (A, B, C), while the white triangles incident with a have orientation (A, C, B). Since the vertices of these triangles span $B \cup C$, all remaining black triangles have clockwise orientation (A, B, C) and all remaining white triangles have clockwise orientation (A, C, B). It follows that the rotation scheme for the embedding satisfies Ringel's rule R^* (see page 28 of [10]) and therefore represents an orientable embedding. □

Our second observation relates face 2-colourable embeddings of $K_{n,n,n}$ to certain orientable embeddings of $K_{n,n}$.

Proposition 2 *Given an orientable embedding of $K_{n,n}$ in which all face boundaries form Hamiltonian cycles, a face 2-colourable triangulation of $K_{n,n,n}$ may be constructed by inserting a new vertex into the interior of each face and joining it by new non-intersecting edges to all the vertices on the boundary of that face. Conversely, given a face 2-colourable triangulation of $K_{n,n,n}$ with tripartition $\{A, B, C\}$, by deleting all vertices in one of the sets A, B or C, together with* *all (open) incident edges, we may form an orientable embedding of $K_{n,n}$ in which all the face boundaries form Hamiltonian cycles.*

Proof. Given an orientable embedding of $K_{n,n}$ in which all the face boundaries form Hamiltonian cycles, by counting edges, the number N of such faces is given by $2n \times N = 2n^2$. Thus $N = n$. If the bipartition is $\{A, B\}$ then the Hamiltonian cycles alternate points of A with points of B. Inserting a new vertex c into the interior of a face and adding the edges as described results in oriented triangles incident with c which alternately have the forms (c, a_i, b_j) and (c, b_k, a_l) where $a_i, a_l \in A$ and $b_j, b_k \in B$. The former may all be coloured black and the latter white, giving a proper face 2-colouring of the resulting $K_{n,n,n}$ embedding. The converse follows immediately. □

Given an orientable embedding of $K_{n,n}$ in which all the face boundaries form Hamiltonian cycles, by assigning an orientation, we obtain a decomposition of the complete bipartite directed graph $K^*_{n,n}$ into directed Hamiltonian cycles. For any decomposition of $K^*_{n,n}$ into directed Hamiltonian cycles, we may define the *neighbourhood* of a vertex x to be the graph formed on the n vertices adjacent to x by joining two vertices with an edge if and only if they are both adjacent to x in the same Hamiltonian cycle. We say that a decomposition of $K^*_{n,n}$ into directed Hamiltonian cycles is *perfect* if the neighbourhood of every vertex is a single cycle. Plainly, a decomposition arising from an orientable embedding of $K_{n,n}$ is perfect. Conversely, a perfect decomposition gives an orientable embedding of $K_{n,n}$ in which all the face boundaries form Hamiltonian cycles because the neighbourhood of each point defines the rotation about that point. Thus we have established the following result.

Proposition 3 *There is a one-to-one correspondence between orientable embeddings of $K_{n,n}$ in which all the face boundaries form Hamiltonian cycles, and perfect decompositions of the complete bipartite directed graph $K^*_{n,n}$ into directed Hamiltonian cycles.*

As a consequence of the above propositions, we may search for biembeddings of Latin squares either directly or by first examining decompositions of $K^*_{n,n}$ into directed Hamiltonian cycles and then restricting attention to those decompositions which are perfect. In an attempt to verify our computational results, we have used both approaches, and then reconciled the corresponding outputs. In making this reconciliation, account has to be taken of the fact that up to three different (i.e. nonisomorphic) $K_{n,n}$ embeddings may result from deleting in turn each of the three sets of the tripartition in an embedding of $K_{n,n,n}$. The cases of $K_{1,1,1}$ and $K_{2,2,2}$ correspond respectively to a spherical embedding of a triangle and an octahedron. In order to avoid trivial cases, throughout the remainder of this paper we assume that $n \geq 3$.

2 Hamiltonian decompositions of $K^*_{n,n}$

Perhaps surprisingly, no enumeration of Hamiltonian decompositions of $K^*_{n,n}$ for small n seems to have been undertaken. This paper rectifies this deficiency for $n = 3, 4, 5$ and 6. We take the bipartition of $K^*_{n,n}$ to be $\{A, B\}$ where $|A| = |B| = n$. The points of A will be denoted by a_i and those of B by b_i. Given a decomposition D_1, we may form a decomposition D_2 by reversing all the cycles; we regard D_1 and D_2 as identical decompositions. The automorphism group of the decomposition D will be denoted by $Aut(D)$. A directed Hamiltonian cycle such as $(a_0, b_1, a_1, b_0, a_2, b_3, a_3, b_2)$ will be given more succinctly as 01102332; by convention all cycles start with a_0.

The decompositions were found by an exhaustive backtracking program. Without loss of generality, it was assumed that each decomposition of $K^*_{n,n}$ contains the cycle $a_0b_0a_1b_1 \cdots a_{n-1}b_{n-1}$. Isomorphisms between decompositions were then easily determined because of the limited number of possibilities: the cycle $a_0b_0a_1b_1 \cdots a_{n-1}b_{n-1}$ can only map to one of n possible $2n$-cycles, and there are only $2 \times 2n$ ways of defining each such mapping. The automorphism group of each decomposition was also easily found in a similar manner.

$\boldsymbol{n = 3}$.
Up to isomorphism, there is a unique decomposition and this is perfect. The cycles are: 001122, 011220, 021021. The automorphism group has order 36; nine of these mappings preserve the bipartition and direction of the cycles, nine exchange A and B but preserve the direction, nine preserve the bipartition but reverse the direction of the cycles, and nine exchange A and B and reverse the direction.

$\boldsymbol{n = 4}$.
Up to isomorphism, there are four decompositions of which one is perfect. They are as follows.

1. 00112233, 01102332, 02132031, 03122130. $|Aut(D)| = 32$, all mappings preserve the bipartition and 16 preserve direction.

2. 00112233, 01102332, 02132130, 03122031. $|Aut(D)| = 16$, all mappings preserve the bipartition and eight preserve direction.

3. 00112233, 01122330, 02132031, 03102132. $|Aut(D)| = 64$, 16 mappings preserve the bipartition and directions, 16 reverse the bipartition but preserve the direction, 16 preserve the bipartition but reverse direction, and 16 reverse the bipartition and the direction. This decomposition is perfect.

4. 00112233, 01302312, 02132031, 03322110. $|Aut(D)| = 32$, of which 16 mappings preserve the bipartition. Each Hamiltonian cycle in this decomposition appears with its reverse so that each automorphism may be considered as both preserving and reversing the directions. We call such a decomposition *reversible*.

$n = 5$.

Up to isomorphism, there are 14 decompositions of which four are perfect. All 14 are shown in Table 1 with the notation $(M; m_1, m_2, m_3, m_4)$ denoting that $|Aut(D)| = M$ and that there are m_1 mappings which preserve direction and bipartition, m_2 mappings which preserve direction and reverse the bipartition, m_3 mappings which reverse direction and preserve the bipartition, and m_4 mappings which reverse direction and reverse the bipartition.

1	0011223344	0110233442	0213243041	0412203143	0314213240	(2; 1, 0, 1, 0)	
2	0011223344	0110233442	0312243041	0214203143	0413213240	(2; 1, 0, 1, 0)	
3	0011223344	0110233442	0324124031	0420413213	0221431430	(1; 1, 0, 0, 0)	
4	0011223344	0112233440	0213243041	0314203142	0410213243	(100; 25, 25, 25, 25)	perfect
5	0011223344	0112233440	0241302413	0314203142	0432104321	(20; 5, 5, 5, 5)	
6	0011223344	0112233440	0241302413	0320421431	0421431032	(20; 5, 5, 5, 5)	perfect
7	0011223344	0112243043	0214233140	0310213442	0413203241	(10; 5, 0, 5, 0)	perfect
8	0011223344	0112243043	0241341320	0423103142	0332144021	(2; 1, 0, 1, 0)	
9	0011223344	0112243043	0241341320	0442102331	0314214032	(2; 1, 0, 1, 0)	
10	0011223344	0112243043	0241342310	0332144021	0413203142	(2; 1, 0, 1, 0)	
11	0011223344	0113243042	0210342143	0412402331	0314413220	(1; 1, 0, 0, 0)	
12	0011223344	0113243042	0410322143	0214402331	0312413420	(2; 1, 1, 0, 0)	
13	0011223344	0113244032	0214302143	0312413420	0410422331	(4; 1, 1, 1, 1)	perfect
14	0011223344	0110322443	0240231431	0320413412	0421421330	(2; 1, 0, 1, 0)	

Table 1. All decompositions of $K^*_{5,5}$.

$n = 6$.

Up to isomorphism there are 18 969 decompositions. Of these 59 are perfect and a further seven are reversible. Reversible decompositions correspond to Hamiltonian decompositions of the undirected $K_{6,6}$. The 59 perfect decompositions are given in Table 2 in the same format as those for the $n = 5$ case. The seven Hamiltonian decompositions of $K_{6,6}$ are given in Table 3 with the notation $(M; m_1, m_2)$ denoting that $|Aut(D)| = M$ and that there are m_1 mappings which preserve the bipartition and m_2 which reverse the bipartition.

1	001122334455	021021344553	011320354254	041225314350	031524324051	051423304152	(2; 1, 0, 1, 0)
2	001122334455	021021344553	011324354250	031425324051	041520314352	051223304154	(4; 2, 0, 2, 0)
3	001122334455	021021344553	014250351324	052043511432	031240542531	041541522330	(2; 1, 0, 1, 0)
4	001122334455	021021354354	011520344253	041223314550	051324324051	031425304152	(24; 12, 0, 12, 0)
5	001122334455	021021354354	031520344251	041325324150	051224314053	011423304552	(2; 1, 0, 1, 0)
6	001122334455	021021354354	051420324153	011325344052	031524304251	041223314550	(12; 6, 0, 6, 0)
7	001122334455	021021354354	052440325113	015315342042	041223314550	032514305241	(24; 6, 6, 6, 6)
8	001122334455	021023344551	031425304152	041521324053	011324354250	051220314354	(12; 6, 0, 6, 0)
9	001122334455	021023344551	041321354052	051224304153	031520314254	011425324350	(2; 1, 0, 1, 0)
10	001122334455	021023344551	053041522413	013240532514	043521431250	035420311542	(2; 1, 0, 1, 0)
11	001122334455	021023354154	052034425113	035045122431	041553213240	014314253052	(1; 1, 0, 0, 0)
12	001122334455	021023354154	052440325113	035042251431	014315342052	041245532130	(2; 1, 1, 0, 0)
13	001122334455	021024354153	051321304254	041523314052	031220344551	011425324350	(4; 2, 0, 2, 0)
14	001122334455	021024354153	032132504514	052051133442	042315524031	014354122530	(2; 1, 1, 0, 0)
15	001122334455	021024354351	042552411330	013253204514	053140541223	031521503442	(2; 1, 0, 1, 0)
16	001122334455	021025314354	011552233440	032412503541	042051324513	052153143042	(1; 1, 0, 0, 0)
17	001122334455	021320344551	033140251254	013542241053	052150431432	042352411530	(4; 2, 0, 2, 0)
18	001122334455	021320344551	033142251054	013540241253	052350411432	042152431530	(4; 2, 0, 2, 0)
19	001122334455	021320344551	033052254114	013523541240	051024314253	042150431532	(6; 3, 0, 3, 0)
20	001122334455	021320344551	035425311240	014253351024	041541305223	052150431432	(2; 1, 0, 1, 0)
21	001122334455	021320344551	035241351024	011254253043	052140531432	041542233150	(2; 1, 0, 1, 0)
22	001122334455	021324354051	041520314253	031021344552	011425324350	051223304154	(36; 18, 0, 18, 0)
23	001122334455	021324354051	041520314253	031425304152	051021324354	011223344550	(144; 36, 36, 36, 36)
24	001122334455	021324354051	041520314253	031041542532	011445522330	051243502134	(36; 9, 9, 9, 9)
25	001122334455	021324354051	052350311442	042145125330	013415522043	031025324154	(6; 3, 0, 3, 0)
26	001122334455	021325304154	033442511520	014514235032	054310522431	041240213553	(2; 1, 0, 1, 0)
27	001122334455	021325344150	041521304352	033551124024	054220531431	014510325423	(6; 3, 0, 3, 0)
28	001122334455	021325344150	052052431431	011253354024	035421451032	041551304223	(2; 1, 0, 1, 0)
29	001122334455	021420354153	011334422550	041531402352	031245243051	051043213254	(12; 6, 0, 6, 0)
30	001122334455	021420354153	011350452432	053440231251	041552432130	033142251054	(12; 12, 0, 0, 0)
31	001122334455	021420354153	011352452430	053140231254	041550432132	033442251051	(12; 6, 0, 6, 0)
32	001122334455	021425314053	041523513042	031250243541	011334455220	051032432154	(4; 2, 0, 2, 0)
33	001122334455	021524314053	041320354251	031025324154	051421304352	011223344550	(36; 18, 0, 18, 0)
34	001122334455	021524314053	041320354251	031241502534	011445522330	051043542132	(36; 18, 0, 18, 0)
35	001122334455	021024355143	041540532132	051254233041	011352452034	031442253150	(2; 1, 0, 1, 0)
36	001122334455	021024355143	052053124134	035225144031	041321504532	011542542330	(2; 1, 0, 1, 0)
37	001122334455	021024355143	013415235240	044513215032	031254253041	054220533114	(1; 1, 0, 0, 0)
38	001122334455	021324355140	033150422514	013412452053	044115302352	054310325421	(4; 1, 1, 1, 1)
39	001122334455	021423355140	052032411354	031531502442	043045125321	013410522543	(6; 3, 0, 3, 0)
40	001122334455	021425305143	032032541541	043112452350	051334402152	013553104224	(12; 3, 3, 3, 3)
41	001122334455	021425305143	044523311250	051354324021	031041355224	015320341542	(4; 1, 1, 1, 1)
42	001122334455	021425315043	013445231052	054254133021	035140351224	043241155320	(2; 1, 1, 0, 0)
43	001122334455	021024413553	042331154052	052043143251	014225135034	032154124530	(1; 1, 0, 0, 0)
44	001122334455	021024453153	031520543241	051421304352	011325425034	041223355140	(2; 1, 0, 1, 0)
45	001122334455	021324413550	031432452051	041540233152	051034214253	011225304354	(2; 1, 0, 1, 0)
46	001122334455	021425403153	051243502134	044115233052	014532541320	031042243551	(2; 1, 0, 1, 0)
47	001122334455	021425403153	051054324321	044113355220	013415235042	031251453024	(4; 1, 1, 1, 1)
48	001122334455	021425403153	041352413520	053442231051	011550432432	033021451254	(6; 3, 0, 0, 3)
49	001122334455	021425413053	043513204251	031245315024	011552233440	052143541032	(2; 1, 0, 1, 0)
50	001122334455	021025415334	053142205413	044035231251	011552432430	035045142132	(1; 1, 0, 0, 0)
51	001122334455	021325405431	014553341220	051421503243	031042513524	044115302352	(2; 1, 0, 1, 0)
52	001122334455	021425405331	043512432150	031551304224	052034411352	014532541023	(2; 1, 1, 0, 0)
53	001122334455	021425435130	014024351253	031521503442	041045522331	053241132054	(2; 1, 0, 1, 0)
54	001122334455	021524415330	011435522043	032132451054	041350253142	054034231251	(2; 1, 0, 1, 0)
55	001122334455	021421503543	041352402531	014510532432	052330411254	031551344220	(4; 2, 0, 2, 0)
56	001122334455	021435215043	032542541031	014512205334	053024411352	042315513240	(12; 3, 3, 3, 3)
57	001122334455	021051354324	031425304152	051354324021	014550341223	044253311520	(24; 12, 0, 0, 12)
58	001122334455	021054354321	052440325113	014553341220	035241302514	042315315042	(24; 12, 0, 0, 12)
59	001122334455	021351354024	014250341523	031425304152	044553311220	051054324321	(48; 12, 12, 12, 12)

Table 2. Perfect decompositions of $K^*_{6,6}$.

1	001122334455	021324354051	041520314253	(72; 36, 36)
2	001122334455	021425314053	043023154251	(4; 4, 0)
3	001122334455	021524314053	041320354251	(36; 36, 0)
4	001122334455	021425403153	043513205241	(2; 1, 1)
5	001122334455	021425413053	043513204251	(2; 2, 0)
6	001122334455	021423503541	043152402513	(6; 6, 0)
7	001122334455	021351354024	014250341523	(24; 12, 12)

Table 3. Hamiltonian decompositions of $K_{6,6}$.

3 Face 2-colourable triangulations of $K_{n,n,n}$

The principal method employed to tabulate face 2-colourable triangulations of $K_{n,n,n}$ was based on taking representatives of the main classes of Latin squares of order n. For $4 \leq n \leq 7$ these are given in [3], and for $n = 3$ there is just one main class. Having selected one such square, its triples are regarded as triangles with the common clockwise orientation (row, column, entry). In any biembedding containing this Latin square, the rotation about each point contains n known ordered pairs; what remains unknown is the ordering of these pairs. By considering all possible orderings and rejecting those which give rise to pseudosurfaces, all biembeddings containing the given square may be determined. Working through the main classes of Latin squares of order n, each new biembedding was checked for isomorphism with those found previously. Finally, the results for $n \leq 6$ were reconciled with those of Section 2. For $n = 7$ the large number of biembeddings to be checked required the use of an effective invariant in order to establish the isomorphism classes. The invariant used was as follows.

Consider a fixed embedding of $K_{n,n,n}$, and denote by ρ_u the rotation around a vertex u. Since ρ_u is a cyclic permutation, for each two neighbours v and w of u there are n_1 and n_2 such that $w = \rho_u^{n_1}(v)$ and $w = \rho_u^{-n_2}(v)$ (where $1 \leq n_1, n_2 \leq 2n-1$ and $n_1 + n_2 = 2n$). Denote by $d(u; v, w)$ the minimum of n_1 and n_2. Now if $d(u; x, v) = 1$ and $d(u; v, y) = 1$, $x \neq y$, then $d(v; x, y) = 2$. However if $d(u; x, v) = 3$ and $d(u; v, y) = 3$, $x \neq y$, then $d(v; x, y)$ can be any even number from 2 to n. (The number $d(v; x, y)$ is even because x and y belong to the same set of the tripartition.) For each vertex v, let I_v be the sum of $2n$ numbers given by

$$I_v = \sum_u (d(v; x, y) : \text{where } d(u; x, v) = d(u; v, y) = 3 \text{ and } x \neq y),$$

where the sum extends over all vertices u of $K_{n,n,n}$ for which uv is an edge. Then for $n = 7$, $\{I_v : v \in V(K_{n,n,n})\}$ is a satisfactory set of invariants.

For $n \leq 6$ we specify a representative biembedding from each isomorphism class by means of a vector (i, j, p_1, p_2, p_3) where i, j give the main class numbers of the two squares as in [3], and p_1, p_2, p_3 specify permutations applied respectively to the rows, columns and entries of the second square. From these the biembedding may be constructed as explained in the case $n = 3$ below. We use I to denote the identity permutation. In no case do we need to permute rows, columns and entries with each other. We also give information about the automorphism group of each biembedding E with a second vector $(M; m_1, m_2, m_3, m_4)$ denoting that $|Aut(E)| = M$ and that there are m_1 mappings which preserve orientation and colour classes, m_2 mappings which preserve orientation and reverse the colour classes, m_3 mappings which reverse orientation and preserve the colour classes, and m_4 mappings which reverse orientation and reverse the colour classes.

In the case $n = 7$ the number of biembeddings is so large that it is only feasible to summarize the results in the form of tables showing which pairs of squares biembed and how many nonisomorphic biembeddings there are for a given pair. We also give a little more information about this case in Section 4.

$\boldsymbol{n = 3}$.
There is just one biembedding given by $(1,1,I,I,201)$, $(108;27,27,27,27)$. This biembedding is regular. To obtain the biembedding, take main class #1 Latin square representative as $\begin{array}{|ccc|}\hline 0 & 1 & 2 \\ 1 & 2 & 0 \\ 2 & 0 & 1 \\ \hline\end{array}$. This forms the first square. To form the second, apply the permutation $\begin{pmatrix} 0\,1\,2 \\ 2\,0\,1 \end{pmatrix}$ to the entries to get $\begin{array}{|ccc|}\hline 2 & 0 & 1 \\ 0 & 1 & 2 \\ 1 & 2 & 0 \\ \hline\end{array}$. The rows and columns of each square are indexed by $0, 1$ and 2. Then take the nine points of $K_{3,3,3}$ to be $0_r, 1_r, 2_r, 0_c, 1_c, 2_c, 0_e, 1_e, 2_e$. Black triangles with clockwise orientation (r, c, e), are read from the first square so that, for example, the $(0, 2)$ entry 2 gives the triangle $(0_r, 2_c, 2_e)$. White triangles with clockwise orientation (r, e, c) are read from the second. The resulting rotation scheme is

$$\begin{array}{llllllll}
0_r: & 0_c & 0_e & 1_c & 1_e & 2_c & 2_e \\
1_r: & 0_c & 1_e & 1_c & 2_e & 2_c & 0_e \\
2_r: & 0_c & 2_e & 1_c & 0_e & 2_c & 1_e \\
0_c: & 0_e & 0_r & 2_e & 2_r & 1_e & 1_r \\
1_c: & 0_e & 2_r & 2_e & 1_r & 1_e & 0_r \\
2_c: & 0_e & 1_r & 2_e & 0_r & 1_e & 2_r \\
0_e: & 0_r & 0_c & 1_r & 2_c & 2_r & 1_c \\
1_e: & 0_r & 1_c & 1_r & 0_c & 2_r & 2_c \\
2_e: & 0_r & 2_c & 1_r & 1_c & 2_r & 0_c
\end{array}$$

This biembedding has its (full) automorphism group of order 108 with 27 automorphisms in each of the four classes described above. In this particular example, the embedding could equally well be specified by the vector $(1, 1, 120, I, I)$ since the second square may also be obtained from the first by applying the permutation $\begin{pmatrix} 0\,1\,2 \\ 1\,2\,0 \end{pmatrix}$ to the rows.

$\boldsymbol{n = 4}$.
There are two main classes of Latin squares but only one biembedding given by $(2, 2, I, 3201, I)$, $(192; 48, 48, 48, 48)$. This biembedding is regular.

$\boldsymbol{n = 5}$.
There are two main classes of Latin squares and three biembeddings.
1. $(1, 1, I, 40123, I)$, $(300; 75, 75, 75, 75)$, regular,
2. $(1, 1, I, 14023, I)$, $(20; 5, 5, 5, 5)$,
3. $(1, 1, 10234, 30412, 10234)$, $(12; 3, 3, 3, 3)$.

The second main class does not feature in any biembedding.

$\boldsymbol{n = 6}$.

There are 12 main classes of Latin squares and 29 biembeddings. These are given in Table 4. Main classes 3, 4, 7 and 10 do not feature in any biembedding.

1.	$(1, 1, I, 541032, I)$, $(432; 108, 108, 108, 108)$, regular,
2.	$(1, 1, 410325, 345012, 410325)$, $(108; 27, 27, 27, 27)$,
3.	$(1, 2, 103245, 450132, 013245)$, $(72; 36, 0, 36, 0)$,
4.	$(1, 2, 103245, 430512, 013245)$, $(24; 12, 0, 12, 0)$,
5.	$(1, 2, 301425, 210534, 031425)$, $(18; 9, 0, 9, 0)$,
6.	$(2, 2, I, 451023, I)$, $(48; 12, 12, 12, 12)$,
7.	$(2, 2, 154203, 045312, 021435)$, $(36; 9, 9, 9, 9)$,
8.	$(2, 2, 042135, 451023, I)$, $(24; 6, 6, 6, 6)$,
9.	$(2, 2, 042135, 153024, I)$, $(12; 3, 3, 3, 3)$,
10.	$(2, 8, 214035, 214350, 201345)$, $(3; 3, 0, 0, 0)$,
11.	$(5, 5, 013254, 013254, 240513)$, $(4; 1, 1, 1, 1)$,
12.	$(5, 5, 542310, 542310, 240513)$, $(4; 1, 1, 1, 1)$,
13.	$(5, 5, 043215, 102354, 240513)$, $(2; 1, 0, 0, 1)$,
14.	$(6, 6, 204135, 130425, 103254)$, $(4; 1, 1, 1, 1)$,
15.	$(8, 8, 041523, 354012, 240135)$, $(6; 3, 3, 0, 0)$,
16.	$(9, 9, I, 524031, I)$, $(12; 6, 0, 0, 6)$,
17.	$(9, 9, 041325, 520134, I)$, $(6; 3, 0, 0, 3)$,
18.	$(9, 9, 032154, 531024, I)$, $(4; 2, 0, 0, 2)$,
19.	$(9, 9, 032154, 324150, 402315)$, $(2; 1, 0, 0, 1)$,
20.	$(11, 11, 450213, 305124, 104325)$, $(6; 3, 0, 0, 3)$,
21.	$(11, 11, I, 520413, I)$, $(4; 2, 0, 0, 2)$,
22.	$(11, 11, I, 450213, I)$, $(2; 1, 0, 0, 1)$,
23.	$(11, 11, 305124, 450213, I)$, $(2; 1, 0, 0, 1)$,
24.	$(11, 11, 315042, 543210, 302145)$, $(2; 1, 0, 0, 1)$,
25.	$(11, 11, 315024, 421530, 102345)$, $(2; 1, 1, 0, 0)$,
26.	$(11, 11, 032145, 520431, 312045)$, $(2; 1, 1, 0, 0)$,
27.	$(11, 11, 105324, 520431, 304125)$, $(2; 1, 1, 0, 0)$,
28.	$(11, 12, 103245, 254013, 103245)$, $(3; 3, 0, 0, 0)$,
29.	$(12, 12, I, 534012, I)$, $(36; 18, 0, 0, 18)$.

Table 4. Biembeddings for $n = 6$.

$\boldsymbol{n = 7}$.

There are 147 main classes of Latin squares and 23 664 biembeddings. These are summarized in the 16 sub-tables of Table 5. The Latin squares partition into groups such that biembeddings only exist inside these groups. Each sub-table specifies the main class numbers of the squares along the top and left borders, and the entries in the body of each sub-table give the numbers of nonisomorphic embeddings. The number at the top left specifies the number of main classes included in that sub-table. It is interesting to note that inside each sub-table the number of zero entries (indicated by -) is small. Of the 23 664 biembeddings, 4 761 are biembeddings of a Latin square with itself. Although all 147 squares appear in Table 5, several squares do not biembed with themselves. Precisely one of the biembeddings is regular (see Section 4.2 for details).

1	6
6	284

1	78
78	65

1	87
87	284

2	105	136
105	248	353
136	353	372

3	1	3	7
1	249	8	49
3	8	3	2
7	49	2	27

3	2	4	5
2	6	7	3
4	7	91	61
5	3	61	87

3	90	124	125
90	190	114	119
124	114	88	97
125	119	97	249

6	52	76	112	141	143	147
52	3	2	28	10	28	16
76	2	-	7	-	5	6
112	28	7	61	65	180	106
141	10	-	65	11	68	34
143	28	5	180	68	82	141
147	16	6	106	34	141	12

6	71	81	108	109	121	140
71	3	14	19	9	29	21
81	14	86	78	80	81	143
108	19	78	36	38	115	68
109	9	80	38	45	75	86
121	29	81	115	75	42	88
140	21	143	68	86	88	104

8	8	10	46	77	84	129	135	146
8	-	-	38	12	4	-	9	-
10	-	-	-	2	4	-	-	-
46	38	-	107	24	21	194	51	80
77	12	2	24	6	6	45	7	22
84	4	4	21	6	8	30	7	18
129	-	-	194	45	30	131	57	46
135	9	-	51	7	7	57	19	52
146	-	-	80	22	18	46	52	59

8	12	15	51	65	68	79	97	130
12	-	-	2	2	-	3	5	-
15	-	-	12	16	3	20	17	-
51	2	12	1	11	9	8	6	43
65	2	16	11	12	23	44	39	46
68	-	3	9	23	4	19	27	10
79	3	20	8	44	19	28	42	56
97	5	17	6	39	27	42	65	39
130	-	-	43	46	10	56	39	-

9	57	63	66	82	86	92	119	120	122
57	22	26	39	68	57	52	47	68	29
63	26	4	15	24	18	21	24	14	5
66	39	15	32	42	50	44	42	42	19
82	68	24	42	30	64	98	57	99	39
86	57	18	50	64	46	105	50	89	19
92	52	21	44	98	105	60	35	88	19
119	47	24	42	57	50	35	43	48	28
120	68	14	42	99	89	88	48	47	17
122	29	5	19	39	19	19	28	17	-

18	29	36	38	43	45	50	55	60	91	93	103	104	107	113	116	123	126	142
29	4	7	13	26	5	6	7	16	20	14	39	18	28	34	7	7	4	5
36	7	-	5	15	3	5	10	4	18	8	15	5	13	18	4	11	5	9
38	13	5	9	22	7	14	11	8	21	14	36	9	20	19	8	20	4	12
43	26	15	22	34	22	22	34	18	63	27	47	33	62	86	19	30	12	58
45	5	3	7	22	11	9	8	22	24	10	38	20	18	23	8	13	4	4
50	6	5	14	22	9	14	18	9	27	10	31	22	25	36	6	16	-	25
55	7	10	11	34	8	18	10	15	37	10	40	24	31	37	5	16	1	11
60	16	4	8	18	22	9	15	5	35	16	15	5	26	37	6	16	22	25
91	20	18	21	63	24	27	37	35	27	32	48	30	71	65	15	46	16	21
93	14	8	14	27	10	10	10	16	32	9	34	20	20	40	17	14	3	25
103	39	15	36	47	38	31	40	15	48	34	33	21	69	96	28	45	24	55
104	18	5	9	33	20	22	24	5	30	20	21	4	36	35	11	31	15	17
107	28	13	20	62	18	25	31	26	71	20	69	36	40	60	14	41	14	51
113	34	18	19	86	23	36	37	37	65	40	96	35	60	43	21	26	22	37
116	7	4	8	19	8	6	5	6	15	17	28	11	14	21	11	15	4	3
123	7	11	20	30	13	16	16	16	46	14	45	31	41	26	15	14	7	16
126	4	5	4	12	4	-	1	22	16	3	24	15	14	22	4	7	-	2
142	5	9	12	58	4	25	11	25	21	25	55	17	51	37	3	16	2	10

19	16	17	32	41	42	48	49	56	83	85	89	94	101	106	117	118	127	131	133
16	14	19	9	13	3	12	3	4	3	13	20	13	13	22	21	30	2	6	9
17	19	8	6	9	8	13	2	13	18	5	17	22	8	7	14	16	7	3	9
32	9	6	6	12	1	15	4	10	7	3	15	16	13	5	11	10	3	4	10
41	13	9	12	24	17	27	2	23	25	13	52	21	10	20	34	28	25	11	18
42	3	8	1	17	6	6	3	6	10	11	13	6	9	13	15	20	4	6	5
48	12	13	15	27	6	18	8	11	7	17	31	37	14	28	35	30	17	27	11
49	3	2	4	2	3	8	2	8	7	2	5	8	8	1	4	14	3	5	-
56	4	13	10	23	6	11	8	4	11	10	13	16	26	13	20	20	3	11	2
83	3	18	7	25	10	7	7	11	15	13	22	27	1	15	18	22	9	6	11
85	13	5	3	13	11	17	2	10	13	4	22	14	11	1	12	31	8	5	2
89	20	17	15	52	13	31	5	13	22	22	36	43	12	19	72	39	15	20	16
94	13	22	16	21	6	37	8	16	27	14	43	25	10	23	50	23	20	17	14
101	13	8	13	10	9	14	8	26	1	11	12	10	15	15	17	7	6	6	6
106	22	7	5	20	13	28	1	13	15	1	19	23	15	14	22	14	12	19	11
117	21	14	11	34	15	35	4	20	18	12	72	50	17	22	38	48	11	15	26
118	30	16	10	28	20	30	14	20	22	31	39	23	7	14	48	40	34	33	26
127	2	7	3	25	4	17	3	3	9	8	15	20	6	12	11	34	11	9	8
131	6	3	4	11	6	27	5	11	6	5	20	17	6	19	15	33	9	15	13
133	9	9	10	18	5	11	-	2	11	2	16	14	6	11	26	26	8	13	10

26	9	14	19	20	26	27	31	34	35	37	40	47	54	61	62	67	70	72	80	88	95	99	100	128	132	134
9	15	10	12	17	16	13	10	10	14	12	8	17	14	7	18	18	6	3	12	15	5	5	25	13	12	18
14	10	8	2	11	7	7	4	6	6	10	4	6	8	8	7	5	-	1	11	3	5	4	6	9	4	2
19	12	2	11	22	20	11	21	9	21	27	7	5	19	28	26	10	3	4	9	35	19	1	30	22	20	18
20	17	11	22	25	12	18	17	7	20	16	19	19	17	27	16	17	8	9	6	32	6	7	18	23	22	17
26	16	7	20	12	5	20	17	7	16	21	12	21	14	12	20	11	10	3	6	22	6	13	20	25	17	20
27	13	7	11	18	20	22	18	6	23	15	20	26	10	19	13	17	6	4	6	17	16	14	20	16	19	19
31	10	4	21	17	17	18	22	12	17	25	25	16	10	11	13	24	17	6	8	19	6	23	21	26	11	19
34	10	6	9	7	7	6	12	6	7	8	9	8	9	4	4	5	7	4	4	8	16	10	11	6	4	13
35	14	6	21	20	16	23	17	7	13	8	17	25	14	20	15	19	12	5	9	18	7	7	20	25	29	37
37	12	10	27	16	21	15	25	8	8	16	14	30	13	14	9	11	11	4	12	13	9	10	15	20	19	25
40	8	4	7	19	12	20	25	9	17	14	8	19	13	13	15	14	5	4	5	19	10	8	19	17	20	11
47	17	6	5	19	21	26	16	8	25	30	19	11	21	23	17	16	2	7	6	30	17	7	39	27	17	22
54	14	8	19	17	14	10	10	9	14	13	13	21	19	26	19	20	6	9	9	9	15	13	26	20	16	36
61	7	8	28	27	12	19	11	4	20	14	13	23	26	10	29	17	14	7	18	21	7	12	12	15	19	25
62	18	7	26	16	20	13	13	4	15	9	15	17	19	29	15	22	6	7	11	17	14	13	25	26	16	20
67	18	5	10	17	11	17	24	5	19	11	14	16	20	17	22	14	12	3	14	17	5	9	16	22	23	27
70	6	-	3	8	10	6	17	7	12	11	5	2	6	14	6	12	2	8	5	15	4	1	7	8	5	11
72	3	1	4	9	3	4	6	4	5	4	4	7	9	7	7	3	8	2	3	2	6	3	3	7	8	4
80	12	11	9	6	6	6	8	4	9	12	5	6	9	18	11	14	5	3	10	17	10	6	14	7	3	14
88	15	3	35	32	22	17	19	8	18	13	19	30	9	21	17	17	15	2	17	18	10	14	22	30	29	36
95	5	5	19	6	6	16	6	16	7	9	10	17	15	7	14	5	4	6	10	10	7	8	5	16	11	16
99	5	4	1	7	13	14	23	10	7	10	8	7	13	12	13	9	1	3	6	14	8	3	10	6	15	8
100	25	6	30	18	20	20	21	11	20	15	19	39	26	12	25	16	7	3	14	22	5	10	36	31	11	23
128	13	9	22	23	25	16	26	6	25	20	17	27	20	15	26	22	8	7	7	30	16	6	31	28	18	32
132	12	4	20	22	17	19	11	4	29	19	20	17	16	19	16	23	5	8	3	29	11	15	11	18	33	37
134	18	2	18	17	20	19	19	13	37	25	11	22	36	25	20	27	11	4	14	36	16	8	23	32	37	31

33	11	13	18	21	22	23	24	25	28	30	33	39	44	53	58	59	64	69	73	74	75	96	98	102	110	111	114	115	137	138	139	144	145
11	-	-	-	-	-	-	1	-	1	-	-	-	-	-	-	1	-	-	7	-	5	1	2	1	1	2	1	2	-	-	-	2	-
13	-	-	3	3	-	-	1	1	5	1	4	2	3	1	1	2	4	4	3	1	1	8	6	1	5	1	5	-	-	3	1	-	2
18	-	3	4	6	-	3	9	1	3	4	18	8	22	3	6	8	6	4	4	2	4	21	16	10	8	1	18	-	-	4	-	15	-
21	-	3	6	12	3	-	7	-	2	19	14	12	22	4	9	9	7	13	3	3	2	26	14	9	3	4	23	2	1	10	6	8	8
22	-	-	-	3	6	2	2	-	-	6	11	2	7	2	3	2	1	-	1	4	1	15	9	4	5	7	8	1	-	5	3	1	2
23	-	-	3	-	2	4	4	4	2	-	1	4	2	-	2	3	-	3	7	1	7	5	5	3	7	1	12	3	-	2	-	4	2
24	1	1	9	7	2	4	2	-	-	11	27	5	19	1	5	5	11	6	1	1	-	31	17	5	1	2	15	2	2	4	1	5	2
25	-	1	1	-	-	4	-	-	-	1	2	-	1	-	4	1	2	-	-	1	-	-	-	-	-	1	1	-	-	1	5	1	-
28	1	5	3	2	-	2	-	-	2	4	3	1	7	2	6	7	2	2	9	2	6	28	13	4	2	3	11	8	-	4	2	6	1
30	-	1	4	19	6	-	11	1	4	8	16	6	5	-	1	7	8	15	5	-	3	17	19	8	10	1	24	1	-	7	3	7	4
33	-	4	18	14	11	1	27	2	3	16	30	18	17	1	11	22	19	15	9	5	4	30	37	17	24	4	42	3	3	15	10	18	4
39	-	2	8	12	2	4	5	-	1	6	18	6	20	1	13	3	9	3	2	1	1	18	24	4	4	2	24	4	2	1	3	12	3
44	-	3	22	22	7	2	19	1	7	5	17	20	21	4	22	19	16	20	7	2	1	23	35	8	11	5	36	9	4	12	2	11	5
53	-	1	3	4	2	-	1	-	2	-	1	1	4	2	2	1	-	1	-	2	-	6	2	1	2	1	4	-	4	-	5	4	-
58	-	1	6	9	3	2	5	4	6	1	11	13	22	2	11	5	8	2	4	2	3	24	18	7	12	-	16	5	-	8	5	12	2
59	1	2	8	9	2	3	5	1	7	7	22	3	19	1	5	8	10	7	7	4	1	20	24	10	14	2	14	2	1	12	2	13	3
64	-	4	6	7	1	-	11	2	2	8	19	9	16	-	8	10	12	13	-	3	3	16	20	13	11	2	19	4	3	6	4	12	4
69	-	4	4	13	-	3	6	-	2	15	15	3	20	1	2	7	13	3	3	4	2	20	18	11	3	5	11	2	1	7	5	8	2
73	7	3	4	3	1	7	1	-	9	5	9	2	7	-	4	7	-	3	5	3	3	14	7	5	2	1	2	1	4	5	3	2	3
74	-	1	2	3	4	1	1	1	2	-	5	1	2	2	2	4	3	4	3	2	-	10	1	3	3	1	6	2	1	5	2	3	1
75	5	1	4	2	1	7	-	-	6	3	4	1	1	-	3	1	3	2	3	-	4	6	7	3	1	1	1	1	3	4	4	2	-
96	1	8	21	26	15	5	31	-	28	17	30	18	23	6	24	20	16	20	14	10	6	63	49	34	11	25	45	6	6	17	6	30	7
98	2	6	16	14	9	5	17	-	13	19	37	24	35	2	18	24	20	18	7	1	7	49	24	21	21	9	50	7	7	12	1	28	13
102	1	1	10	9	4	3	5	-	4	8	17	4	8	1	7	10	13	11	5	3	3	34	21	18	11	2	27	4	1	4	4	13	8
110	1	5	8	3	5	7	1	-	2	10	24	4	11	2	12	14	11	3	2	3	1	11	21	11	10	3	12	1	3	4	1	15	1
111	2	1	1	4	7	1	2	1	3	1	4	2	5	1	-	2	2	5	1	1	1	25	9	2	3	1	10	2	-	6	2	4	1
114	1	5	18	23	8	12	15	1	11	24	42	24	36	4	16	14	19	11	2	6	1	45	50	27	12	10	43	3	4	13	7	28	2
115	2	-	-	2	1	3	2	-	8	1	3	4	9	-	5	2	4	2	1	2	1	6	7	4	1	2	3	3	-	2	4	3	2
137	-	-	-	1	-	-	2	-	-	-	3	2	4	4	-	1	3	1	4	1	3	6	7	1	3	-	4	-	-	2	1	6	4
138	-	3	4	10	5	2	4	1	4	7	15	1	12	-	8	12	6	7	5	5	4	17	12	4	4	6	13	2	2	7	3	8	-
139	-	1	-	6	3	-	1	5	2	3	10	3	2	5	5	2	4	5	3	2	4	6	1	4	1	2	7	4	1	3	-	1	2
144	2	-	15	8	1	4	5	1	6	7	18	12	11	4	12	13	12	8	2	3	2	30	28	13	15	4	28	3	6	8	1	18	5
145	-	2	-	8	2	2	2	-	1	4	4	3	5	-	2	3	4	2	3	1	-	7	13	8	1	1	2	2	4	-	2	5	7

Table 5. Summary of $K_{7,7,7}$ results.

4 Remarks on the computational results

4.1 Reconciling the results of Sections 2 and 3

For each face 2-colourable triangulation of $K_{n,n,n}$ with $3 \leq n \leq 6$, three perfect decompositions of $K^*_{n,n}$ may be obtained as described in Proposition 2. For $n = 3$, the three decompositions obtained from the unique biembedding given in Section 3 are isomorphic with the unique perfect decomposition of $K^*_{n,n}$ given in Section 2. The same is true for $n = 4$. The decompositions of $K^*_{5,5}$ obtained by deleting respectively $R = \{0_r, 1_r, 2_r, 3_r, 4_r\}$, $C = \{0_c, 1_c, 2_c, 3_c, 4_c\}$, $E = \{0_e, 1_e, 2_e, 3_e, 4_e\}$ from each embedding of $K_{5,5,5}$ are shown in Table 6 with the embeddings numbered as in Section 3, and the decompositions as in Section 2 (Table 1).

$K_{5,5,5}$ embedding	Deleted set and $K^*_{5,5}$ decomposition R	C	E
1	4	4	4
2	7	6	7
3	13	13	13

Table 6. $K_{5,5,5}$ and $K^*_{5,5}$ correspondence.

Note that each of the perfect decompositions of $K^*_{5,5}$ appears in precisely one row, as predicted by Proposition 2.

Table 7 gives the corresponding results for $n = 6$ with the embeddings numbered as in Section 3 (Table 4), and the decompositions as in Section 2 (Table 2).

$K_{6,6,6}$ embedding	Deleted set and $K^*_{6,6}$ decomposition R	C	E
1	23	23	23
2	24	24	24
3	57	57	57
4	30	58	30
5	48	48	48
6	4	59	4
7	56	56	56
8	7	31	31
9	27	27	40
10	43	43	43
11	35	35	38
12	1	1	47
13	21	53	26
14	28	28	41
15	52	52	52
16	6	29	8
17	25	19	39
18	18	55	17
19	51	46	20
20	15	15	15
21	13	32	2
22	5	49	9
23	36	54	10
24	44	45	3
25	14	37	37
26	50	12	50
27	11	11	42
28	16	16	16
29	22	34	33

Table 7. $K_{6,6,6}$ and $K^*_{6,6}$ correspondence.

Again note that each of the perfect decompositions of $K^*_{6,6}$ appears in precisely one row, as predicted by Proposition 2.

4.2 Regular biembeddings

Our computational results show that for each $n \in \{3, 4, 5, 6, 7\}$ there exists precisely one regular face 2-colourable triangular embedding of $K_{n,n,n}$. It will be shown in [4] that this uniqueness result in fact holds for *every* n. In each case this embedding consists of a biembedding of two cyclic Latin squares of order n. Such biembeddings are not new. They may be constructed for each $n \geq 3$

directly from cyclic Latin squares or from voltage graphs. The former approach has the advantage of easily establishing the regularity of these embeddings. Take two isomorphic squares L_1 and L_2 whose rows, columns and entries are indexed by the group $\mathcal{Z}_n$ and whose entries in row i, column j are given respectively by $L_1(i,j) \equiv i+j \pmod n$ and $L_2(i,j) \equiv i+j-1 \pmod n$. Proceeding as in the $n=3$ case of Section 3, it is easy to see that we may obtain a rotation scheme for a biembedding of L_1 and L_2. Furthermore, this biembedding has n^2 automorphisms of the form $\phi_{\alpha,\beta} : (i_r, j_c, k_e) \to ((i+\alpha)_r, (j+\beta)_c, (k+\alpha+\beta)_e)$, and these all preserve the colour classes, the orientation, and the rows, columns and entries. In addition, the mapping $\chi : (i_r, j_c, k_e) \to (i_c, -j_e, -k_r)$ gives an automorphism of order 3 which permutes rows, columns and entries, but preserves the colour classes and the orientation. The mapping $\mu : (i_r, j_c, k_e) \to (i_c, j_r, k_e)$ gives an automorphism of order 2 which preserves the colour classes but reverses orientation, and the mapping $\nu : (i_r, j_c, k_e) \to (-i_c, -j_r, (-k-1)_e)$ gives an automorphism of order 2 which reverses the colour classes but preserves the orientation. It follows that the group of automorphisms generated by these mappings has order at least $12n^2$. Since this is the maximum possible order, we deduce that this group is the full automorphism group of the biembedding and that the biembedding is regular.

An alternative description given in [11] uses the voltage graph shown in Figure 1. The graph has two vertices labelled C and E connected by n arcs directed from C to E and labelled with elements of the group $\mathcal{Z}_n$ in order, as shown. The resulting regions are labelled 0 to $n-1$, with region i bordered by the arcs labelled $i-1$ and i, as shown. Clockwise rotations are imposed at C and E.

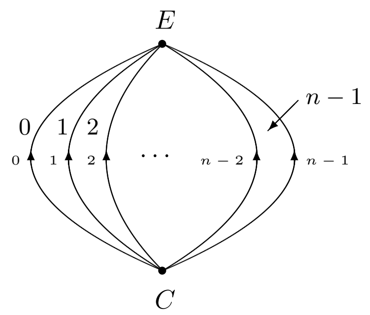

Figure 1: Voltage graph for regular biembedding.

The lifted graph has $2n$ vertices which may be taken as $C_0, C_1, \ldots, C_{n-1}$, $E_0, E_1, \ldots, E_{n-1}$, and n regions $r_0, r_1, \ldots, r_{n-1}$ whose clockwise boundaries are given by

$$\begin{array}{rllllllll} r_0: & C_0 & E_0 & C_1 & E_1 & \ldots & C_{n-1} & E_{n-1} \\ r_1: & C_0 & E_1 & C_1 & E_2 & \ldots & C_{n-1} & E_0 \\ \vdots & & & & & \vdots & & \\ r_{n-1}: & C_0 & E_{n-1} & C_1 & E_0 & \ldots & C_{n-1} & E_{n-2} \end{array}$$

Inserting a vertex R_i into the interior of each region r_i and joining it with new non-intersecting edges to all the vertices on the boundary of r_i gives the partial rotation scheme (rotations about R_i)

$$\begin{array}{rllllllll} R_0: & C_0 & E_0 & C_1 & E_1 & \dots & C_{n-1} & E_{n-1} \\ R_1: & C_0 & E_1 & C_1 & E_2 & \dots & C_{n-1} & E_0 \\ \vdots & & & & & \vdots & & \\ R_{n-1}: & C_0 & E_{n-1} & C_1 & E_0 & \dots & C_{n-1} & E_{n-2} \end{array}$$

It is then a routine matter to complete this scheme (rotations about C_i and E_i) and to verify that it gives the same biembedding as the previous method (by mapping R_i to i_r, etc.).

The paper [4] gives a systematic investigation of voltage graphs similar to that shown in Figure 1, but with varying distributions of the voltages. An alternative voltage graph construction for the dual of the regular biembeddings is given in [7].

4.3 Recursive constructions

In [7] a variety of recursive constructions for embeddings are presented. These require, as a basic ingredient, face 2-colourable triangulations of $K_{n,n,n}$ in which there are parallel classes of triangular faces in one, or preferably both, of the two colour classes. By a *parallel class* we mean a set of vertex-disjoint triangular faces whose vertices collectively cover the complete vertex set of the graph $K_{n,n,n}$. Such a set of triangles forms a 2-factor of the graph and is referred to as a *patchwork* by Gross and Tucker [9] (p.155). Since the regular embeddings described above and used in [7] involve cyclic Latin squares, these embeddings do indeed have parallel classes, in both colour classes, whenever n is odd because the Latin squares have transversals. But cyclic Latin squares of even order do not possess transversals and so the resulting regular embeddings do not have parallel classes when n is even. However, by examining the biembeddings listed in Section 3 for $n = 6$ we can identify suitable triangulations of $K_{6,6,6}$, such as #6 which biembeds two representatives of main class #2. This Latin square has 32 transversals and so triangulation #6 has 32 parallel classes of triangular faces in each of the two colour classes. By using this triangulation, we may extend the results of [7] significantly. For example, we may give a non-orientable version of Constructions 4 and 5 (alluded to in the Concluding Remarks) of that paper but now with $m = 6$ or, by making use of a slight generalization of Construction 2, with $m = 6^r s$ where $r > 0$ and s is odd.

Construction *Suppose that* $n \equiv 1$ *or* 3 *(mod* 6*) with* $n \geq 7$*. Suppose also that we have* k *differently labelled face 2-colourable triangulations of* $K_{m,m,m}$*, where* $m = 6^r s$*,* $r > 0$ *and* s *is odd, all of which have a common parallel class of black triangular faces. Then we may construct* $k^{(n-1)(n-3)/6}$ *differently labelled face 2-colourable non-orientable triangulations of* $K_{m(n-1)+1}$*.*

By saying that two embeddings are "differently labelled" we mean that there is a face boundary in one embedding that is not a face boundary in the other even though the underlying graphs are identical. The proof follows the discussion in [7] but needs as ingredients a face 2-colourable triangulation of K_n (either orientable or nonorientable) and a face 2-colourable triangulation of K_{2m+1} (necessarily nonorientable). These ingredients may be found in [8, 10, 12].

4.4 Automorphisms of $K_{7,7,7}$ embeddings

Observe that all the face 2-colourable triangulations of $K_{n,n,n}$, where $3 \leq n \leq 5$, have automorphism group of order at least n. As regards the 29 face 2-colourable triangulations of $K_{6,6,6}$, exactly 14 of them have $|Aut(E)| \geq 6$. However, automorphism groups of order at least 7 are rare among the face 2-colourable triangulations of $K_{7,7,7}$. Whenever $|Aut(E)| \geq 7$, then the Latin squares involved in the biembedding are either #3 (the square generated by the STS(7)) or #7 (the cyclic square) of [3]. As there are only 32 biembeddings in which only #3 and #7 appear (see Table 5), we list all these embeddings in Table 8, in the same format as those of $K_{6,6,6}$ in Table 4.

1.	$(3, 3, 1603452, 5416023, 2031456)$,	$(28; 7, 7, 7, 7)$,
2.	$(3, 3, 1560234, 1560234, 3012456)$,	$(4; 1, 1, 1, 1)$,
3.	$(3, 3, 2651304, 6013452, 3120456)$,	$(1; 1, 0, 0, 0)$,
4.	$(3, 7, 4253610, 0231456, 0421356)$,	$(21; 21, 0, 0, 0)$,
5.	$(3, 7, 5142360, 0351426, 2403156)$,	$(3; 3, 0, 0, 0)$,
6.	$(7, 7, 1234560, I, I)$,	$(588; 147, 147, 147, 147)$, regular
7.	$(7, 7, 1256340, I, I)$,	$(28; 7, 7, 7, 7)$,
8.	$(7, 7, 1534620, I, I)$,	$(28; 7, 7, 7, 7)$,
9.	$(7, 7, 1436520, I, I)$,	$(28; 7, 7, 7, 7)$,
10.	$(7, 7, 1564230, I, I)$,	$(28; 7, 7, 7, 7)$,
11.	$(7, 7, 1352640, I, I)$,	$(28; 7, 7, 7, 7)$,
12.	$(7, 7, 1235640, I, I)$,	$(28; 7, 7, 7, 7)$,
13.	$(7, 7, 1456230, I, I)$,	$(28; 7, 7, 7, 7)$,
14.	$(7, 7, 5246310, I, I)$,	$(14; 7, 0, 7, 0)$,
15.	$(7, 7, 4265310, I, I)$,	$(14; 7, 0, 7, 0)$,
16.	$(7, 7, 1536240, I, I)$,	$(14; 7, 0, 7, 0)$,
17.	$(7, 7, 1254630, I, I)$,	$(14; 7, 0, 7, 0)$,
18.	$(7, 7, 1546320, I, I)$,	$(14; 7, 0, 7, 0)$,
19.	$(7, 7, 5341260, 0534126, 4230156)$,	$(12; 3, 3, 3, 3)$,
20.	$(7, 7, 1253460, 0234516, 0134256)$,	$(12; 3, 3, 3, 3)$,
21.	$(7, 7, 4512360, 0451236, 3401256)$,	$(12; 3, 3, 3, 3)$,
22.	$(7, 7, 5612340, 0312456, 1203456)$,	$(6; 3, 0, 3, 0)$,
23.	$(7, 7, 1425360, 0142536, 0314256)$,	$(6; 3, 0, 3, 0)$,
24.	$(7, 7, 5234610, 0135426, 3042156)$,	$(4; 1, 1, 1, 1)$,
25.	$(7, 7, 4532610, 0145236, 0341256)$,	$(4; 1, 1, 1, 1)$,
26.	$(7, 7, 5234610, 0145236, 0341256)$,	$(4; 1, 1, 1, 1)$,
27.	$(7, 7, 5613420, 0251346, 1402356)$,	$(2; 1, 0, 1, 0)$,
28.	$(7, 7, 5461230, 0421356, 2130456)$,	$(2; 1, 0, 1, 0)$,
29.	$(7, 7, 3456120, 0234516, 1234056)$,	$(2; 1, 0, 1, 0)$,
30.	$(7, 7, 2361540, 0534126, 4230156)$,	$(2; 1, 0, 1, 0)$,
31.	$(7, 7, 2456130, 0235416, 1243056)$,	$(2; 1, 0, 0, 1)$,
32.	$(7, 7, 4623510, 0235146, 1240356)$,	$(2; 1, 0, 1, 0)$.

Table 8. Biembeddings of #3 and #7.

Among these biembeddings, only #4 and #6 are vertex-transitive. The 12 embeddings #7-#18 are not transitive on vertices, but they can be obtained by the voltage graph construction, assigning voltages to a dipole. However, the voltages assigned are different from those depicted on Figure 1. We discuss these embeddings in [4].

To complete the information, in Table 9 we present the numbers of biembeddings according to the order of $Aut(E)$.

$\|Aut(E)\|$	number of embeddings
1	22 114
2	1 270
3	166
4	59
6	37
≥ 7	18

Table 9. Embeddings by order of automorphism group.

4.5 Biembeddings of Latin squares

Our computational results show that not every main class of Latin square features in a biembedding. For $n \leq 6$, those that do not are the Cayley table of the Klein group of order 4, the Latin square of order 5 which does not come from the cyclic group, and four of the 12 main classes of Latin square of order 6. In fact, the first of these exceptions can easily be established by hand calculation. However, for $n = 7$ all the main classes of Latin square feature in biembeddings. It is a very interesting question whether the six small exceptions identified above are the only ones.

More startling are the partitioning results given in Table 5 for the case $n = 7$, namely that the 147 main classes of Latin square partition into 16 subsets respectively containing 1, 1, 1, 2, 3, 3, 3, 6, 6, 8, 8, 9, 18, 19, 26 and 33 classes such that the biembeddings of Latin squares exist only when both squares belong to the same subset of the partition. Similar partitioning can be identified for $n = 6$ and, trivially, for $n = 3, 4$ and 5. In our opinion this is the most interesting and perhaps unexpected property to emerge from our computations. It raises the problem of providing a mathematical explanation for these results. We hope to return to this in a future paper.

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