On the minimal nonzero distance between triangular embeddings of a complete graph

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Abstract

Given two triangular embeddings f and f' of a complete graph K and given a bijection $\phi : V(K) \to V(K)$, denote by $M(\phi)$ the set of faces (x, y, z) of f such that $(\phi(x), \phi(y), \phi(z))$ is not a face of f'. The distance between f and f' is the minimal value of $|M(\phi)|$ as ϕ ranges over all bijections between the vertices of K. We consider the minimal nonzero distance between two triangular embeddings f and f' of a complete graph. We show that 4 is the minimal nonzero distance in the case when f and f' are both nonorientable, and that 6 is the minimal nonzero distance in each of the cases when f and f' are orientable, and when f is orientable and f' is nonorientable.

Running head:
Distances between embeddings

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1 Introduction

A triangular embedding of a graph is a 2-cell embedding of the graph in a surface (orientable or nonorientable) such that all the faces are triangular. The problem of constructing triangular embeddings of certain complete graphs was posed and solved in the course of the proof of the Map Colour Theorem [9]. Later it was shown [4, 8] that the set of triangular embeddings of a complete graph may contain many nonisomorphic embeddings.

Studying a set of triangular embeddings of a complete graph, we may introduce the distance between two embeddings of the set. We will consider the *face set* $F(f)$ of a triangular embedding f of a graph as the set of unordered triples (x, y, z), where a triple (x, y, z) denotes the face incident with the vertices x, y and z of the embedded graph. Let f and f' be two triangular embeddings of a complete graph K. For a bijection $\phi : V(K) \to V(K)$, denote by $M(f, f'|\phi)$ the set of faces (x, y, z) of f such that $(\phi(x), \phi(y), \phi(z))$ is not a face of f'. The *distance* $d(f, f')$ between the embeddings f and f' is the minimal value of $|M(f, f'|\phi)|$ as ϕ ranges over all bijections between the vertices of K.

Note that if f and f' are isomorphic then $d(f, f') = 0$. Also, if $|M(f, f'|\phi)| = \delta$, then there are faces $F_1, F_2, \ldots, F_\delta$ in $F(f)$ whose images under ϕ are not in $F(f')$. Consequently, there must be faces $F'_1, F'_2, \ldots, F'_\delta$ in $F(f')$ which are not the images under ϕ of faces in $F(f)$, while all other faces of $F(f')$ are images of faces in $F(f)$. It follows that $|M(f', f|\phi^{-1})| = \delta$, and so $d(f, f') = d(f', f)$. Moreover, the distance function d obeys the triangle inequality. This can be seen by taking ϕ, ψ such that $|M(f_1, f_2|\phi)| = d(f_1, f_2) = d_1$ and $|M(f_2, f_3|\psi)| = d(f_2, f_3) = d_2$. Then $|M(\psi(f_1), \psi(f_2)|\psi\phi\psi^{-1})| = d_1$ and putting $\mathcal{F}_1 = F(\psi\phi(f_1)), \mathcal{F}_2 = F(\psi(f_2)), \mathcal{F}_3 = F(f_3))$ and $N = |\mathcal{F}_1| = |\mathcal{F}_2| = |\mathcal{F}_3|$, gives $|\mathcal{F}_1 \cap \mathcal{F}_2| = N - d_1$ and $|\mathcal{F}_2 \setminus \mathcal{F}_3| = d_2$. But, for any sets A, B, C we have $A \cap B \cap C = A \cap B \setminus (B \setminus C)$ and so $|\mathcal{F}_1 \cap \mathcal{F}_3| \geq |\mathcal{F}_1 \cap \mathcal{F}_2 \cap \mathcal{F}_3| \geq N - d_1 - d_2$. Consequently $M(f_1, f_2|\psi\phi) \leq d_1 + d_2$ and hence $d(f_1, f_3) \leq d(f_1, f_2) + d(f_2, f_3)$.

In the present paper we consider the minimal nonzero value of $d(f, f')$ as f and f' range over all triangular embeddings of a complete graph. Three cases are considered:

(the *nn*-case) f and f' are nonorientable,

(the *oo*-case) f and f' are orientable, (the *on*-case) f is orientable and f' is nonorientable.

We show that 4 is the minimal nonzero distance in the *nn*-case and that 6 is the minimal nonzero distance in both the *oo*-case and the *on*-case. The arguments employed in this paper are based on consideration of topological aspects of the embeddings. An alternative approach is based on the fact that the faces of a triangular embedding of a complete graph determine a twofold triple system. The determination of $d(f, f')$ may thus be related to consideration of trades in the associated triple systems. We plan to investigate this aspect in a forthcoming paper [6].

The examples of embeddings given in this paper make use of rotation schemes and index one current graphs. The reader is assumed to be familiar with these standard tools of topological graph theory.

This paper is concerned with the minimum value of $d(f, f')$ but an obviously related problem is the determination of the maximum value of $d(f, f')$ as f and f' range over all triangular embeddings of a complete graph.

2 Modifiable sets of faces

Given a triangular embedding f of a complete graph, by a *modifiable set* A of the embedding we mean a nonempty subset $A \subseteq F(f)$ such that there is a triangular embedding f' of the graph with the face set $(F(f) \setminus A) \cup A'$ where $A \cap A' = \emptyset$. To obtain f' from f, we remove from f the interior of all faces from A and then attach the new faces from A'. We indicate the modification operation by the notation $(f, A) \to (f', A')$, or $(f, A) \to f'$, for short.

Denote by N_{nn} (respectively N_{oo}, N_{on}) the minimum number t such that there is a modifiable set A of a triangular embedding f of a complete graph such that $|A| = t$ and $(f, A) \to f'$ in the *nn*-case (respectively the *oo*-case, the *on*-case). By an *nn*-minimal modifiable set we mean a modifiable set A of a triangular embedding f of a complete graph such that $|A| = N_{nn}$ and $(f, A) \to f'$ in the *nn*-case. Similar definitions apply to *oo*-minimal modifiable sets and to *on*-minimal modifiable sets.

Note that, given two triangular embeddings f and f' of a complete graph K and given a bijection $\phi : V(K) \to V(K)$, the set $M(f, f'|\phi) = A$ is either empty or is a modifiable set of f. In the latter case, take f' and rename each vertex $\phi(v)$ of K as v to get an embedding f'' of K such that

$(f, A) \to f''$. Hence, if A is an nn-minimal (respectively oo-minimal, on-minimal) modifiable set, then the minimal nonzero distance between two triangular embeddings of a complete graph in the nn-case (respectively the oo-case, the on-case) is at least $|A|$. If, in addition, $(f, A) \to f'$, where f and f' are nonisomorphic, then $|A|$ is the minimal nonzero distance.

In this section nn-minimal, oo-minimal and on-minimal modifiable sets are constructed. We show that there are no modifiable sets A with $|A| \leq 3$ and with $|A| = 5$. All modifiable sets A with $|A| = 4$ are shown to be nn-minimal. We construct oo-minimal and on-minimal modifiable sets A with $|A| = 6$. If $(f, A) \to f'$ in the on-case, then f and f' are nonisomorphic, hence 6 is the minimal nonzero distance in this case. We also show that 4 and 6 are minimal nonzero distances in the nn-case and oo-case respectively by giving examples of nonisomorphic embeddings f and f' such that $(f, A) \to f'$ where A is the constructed nn-minimal (or oo-minimal) modifiable set.

We need some terminology concerning modifiable sets. Given a modifiable set A, the faces from A are called *A-faces*, the vertices and edges incident with the A-faces are called the *vertices* and *edges of A* or the *A-vertices* and *A-edges*. By the *A-degree* of an A-vertex we mean the number of A-faces (<u>not</u> edges) incident with the vertex. An A-vertex of A-degree k is called a *k-vertex of A*. Denote by $V(A)$ the set of A-vertices.

A modifiable set A can be given as a picture. For example, Figure 1(a) shows a modifiable set; ignore for now the dashed cycle. The A-faces are depicted as shaded triangular regions. In Figure 1(a), the A-edges incident with an A-vertex v are depicted around v in a circular order. This circular order is induced by the circular order in accordance with which all edges of the graph incident with v are arranged around v on the surface. For example, if an A-vertex is depicted as in Figure 2(a), then the circular order of the embedded incident edges around v on the surface is as shown in Figure 2(b).

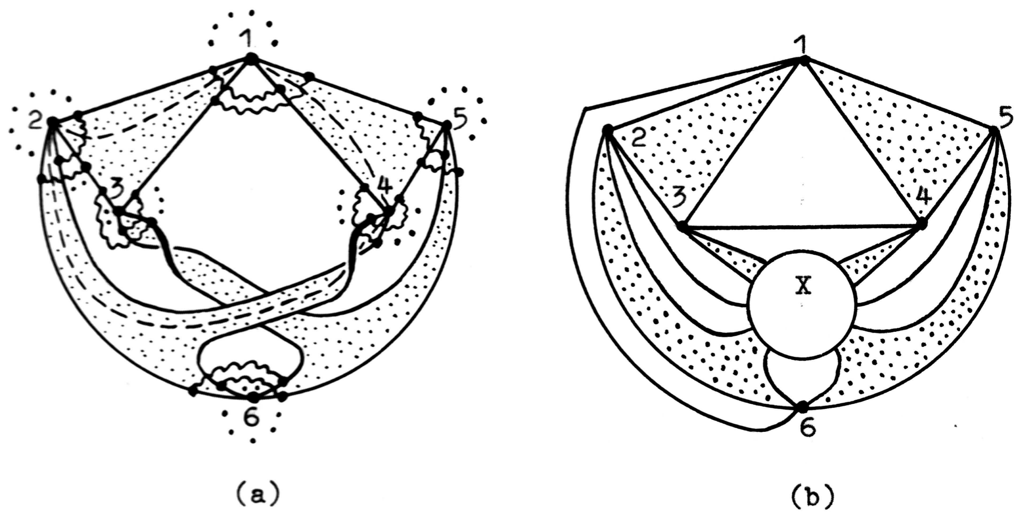


Figure 1: A modifiable set A and an embedding of K_6.

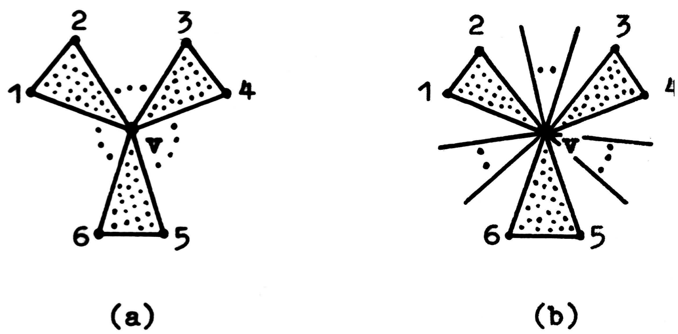


Figure 2: The order of incident edges.

At each A-vertex v, the A-edges incident with v and incident with the same new face are joined by a wavy line. The reader can check that after removing the shaded faces and attaching the new faces in accordance with the wavy lines, all new faces obtained are triangular and, at each vertex of the embedded graph, the faces incident with the vertex form exactly one disc; that is, we get an embedding of the graph in a surface and not in a pseudosurface.

Now we consider some properties of an arbitrary modifiable set A of an arbitrary triangular embedding of a complete graph. The following properties apply (M1 to M6).

(M1) A modifiable set A cannot have an A-vertex v of A-degree 1, 2 or 3 of the types shown in Figure 3 (a), (b) or (c) respectively. In each of these three cases, if we remove the interior of the A-faces incident with v then we cannot attach new triangular faces incident with v (and different from the ones removed) to get an embedding in a surface and not a pseudosurface.

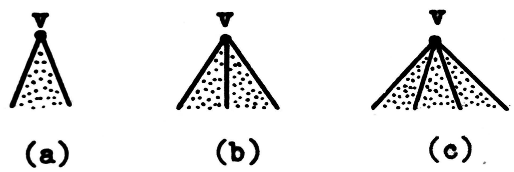


Figure 3: Infeasible configurations.

(M2) For a 2-vertex v of A, there is exactly one way to attach new faces incident with v and this is as shown in Figure 1. Consequently, in the case of a 2-vertex of A we do not need to indicate the wavy lines. From this and M1 we get the following.

(M3) If (v, w) is a common edge of two A-faces, then v and w have A-degrees at least 3.

(M4) Denote by $\Delta(A)$ the maximal A-degree of the A-vertices. For a $\Delta(A)$-vertex v of A, there are at least $\Delta(A)$ vertices of A adjacent to v. Taking M1 into account, we see that every A-vertex adjacent to v is incident with an A-face not incident with v. Every A-face not incident with v has no more than three vertices adjacent to v, hence there are no less than $\lceil \Delta(A)/3 \rceil$ faces from A not incident with v. Hence we have the inequality

$$\Delta(A) + \left\lceil \frac{\Delta(A)}{3} \right\rceil \leq |A| \qquad (1).$$

(M5) Suppose that v and w are 2-vertices incident with the same A-face F as shown in Figure 4. Since the A-edge (v, w) must lie on the boundary of a new face, and taking M2 into account, we see that v and w are incident with A-faces F_1 and F_2, respectively, different from F and such that F_1 and F_2 have a common vertex

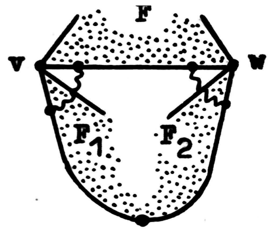


Figure 4: Two 2-vertices incident with a common face.

(M6) Given a modifiable set A, denote by n_j the number of j-vertices of A. Then $\sum_{j\geq 2} n_j = |V(A)|$ and we also have

$$3|A| = \sum_{j\geq 2} j n_j \qquad (2).$$

Now we are ready to construct *nn*-minimal, *oo*-minimal and *on*-minimal modifiable sets.

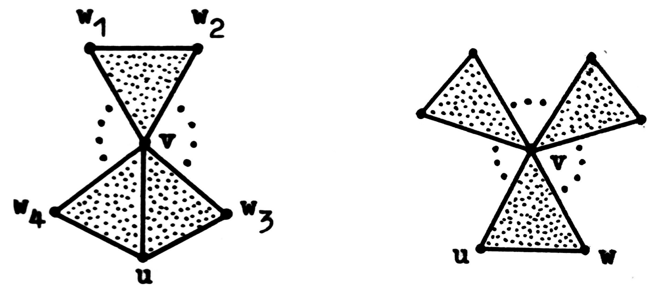
Lemma 1 *There is no modifiable set A with $|A| \leq 3$.*

Proof. If $|A| \leq 3$ then, by (1), $\Delta(A) \leq 2$. But then, by M1, all A-vertices must be 2-vertices. One can easily see that such an A does not exist. □

Lemma 2 *There are modifiable sets A with $|A| = 4$. They are all nn-minimal.*

Proof. Figure 1(a) shows a modifiable set A with $|A| = 4$. We prove that any modifiable set with $|A| = 4$ has this form. To do this, let A be any modifiable set with $|A| = 4$. Then, by (1), we have $\Delta(A) \leq 3$. We first show that there are no 3-vertices by supposing (*reductio ad absurdum*) that A has such a vertex v. By M1, there are just two possibilities; these are shown in Figure 5 and in either case it follows that $|V(A)| \geq 6$. But, by (2), there must be an even number of 3-vertices and then from (2) it follows that $|V(A)| \leq 5$, a contradiction. Hence all vertices of A are 2-vertices. Then by M1 and M5 it follows that A must be of the form shown in Figure 1(a).

A triangular embedding of K_6 in the projective plane is shown in Figure 1(b), where the crosscap is depicted as a circle with an $\times$ inside. This embedding contains a modifiable set of the form shown in Figure 1(a), thereby establishing the existence of triangular embeddings of complete graphs which contain such modifiable sets.



(a) (b)

Figure 5: Possible 3-vertices.

It remains to show that if $(f, A) \to (f', A')$ where A is of the form shown in Figure 1(a), then f and f' are nonorientable embeddings. If $(f, A) \to (f', A')$, then $(f', A') \to (f, A)$, and so A' is a modifiable set with 4 faces and is therefore of the form shown in Figure 1(a). But if an embedding has a modifiable set of the form shown in Figure 1(a), then the surface contains a Möbius strip (this strip includes the dashed cycle shown in Figure 1(a)). Hence f and f' are nonorientable embeddings. $\square$

Note that K_6 has, up to isomorphism, just one embedding in the projective plane [1]. Hence, for the embedding f and the modifiable set A shown in Figure 1(b), if $(f, A) \to f'$, then f and f' are isomorphic and so $d(f, f') = 0$.

Lemma 3 *There is no modifiable set A with $|A| = 5$.*

Proof. Suppose (*reductio ad absurdum*) that there is a modifiable set A with $|A| = 5$. Then, by (1), $\Delta(A) \leq 3$, and, by (2) there is an odd number of 3-vertices. Let v be a 3-vertex. Either Figure 5(a) or Figure 5(b) applies.

In the case of Figure 5(a), by M3, the vertex u is a 3-vertex, hence A has at least three 3-vertices. If all vertices of A are 3-vertices then, by (2), we get $|V(A)| = 5$, but Figure 5(a) shows that $|V(A)| \geq 6$. Hence A has exactly three 3-vertices and three 2-vertices, and so v is adjacent to all other A-vertices. Since u is a 3-vertex, there must be a face (u, w_i, w_j), but then M1 gives $i, j \notin \{3, 4\}$ and $\{i, j\} \neq \{1, 2\}$, a contradiction.

In the case of Figure 5(b), we have $|V(A)| \geq 7$, hence v is the unique 3-vertex. Then, by (2), A has one 3-vertex and six 2-vertices, and so v

is adjacent to all other A-vertices. If (v, u, w) is an A-face, then u and w are 2-vertices and, by M5, u and w must be incident with faces F_1 and F_2 respectively, such that these two faces are not incident with v and have a common vertex x adjacent to v. Thus x is a 3-vertex, a contradiction. □

In order to prove the main theorem, we use index one current graphs with the current group Z_p. Such a current graph can be described pictorially where each pair of reverse arcs is represented by one of these arcs, and arcs are labelled with nonzero elements of Z_p. Black vertices indicate a clockwise rotation and white vertices indicate an anticlockwise rotation. The rotations and arcs of the current graph yield exactly one (up to reversal) circuit whose *log* is the resulting cyclic permutation of the nonzero elements of Z_p obtained in the manner described in [9]. The current graph generates a cellular embedding of the complete graph K_p whose vertices are identified with the elements of Z_p. The face set of the embedding consists of the faces induced by the vertices of the current graph. In the cases we consider, the following properties (P1 and P2) apply.

(P1) Each vertex of the current graph has degree 3 and if $(\alpha_1, \alpha_2, \alpha_3)$ is the rotation of a vertex v of the current graph, where the arc α_i carries the current c_i for $i = 1, 2, 3$, then $c_1 + c_2 + c_3 = 0$ in Z_p and the vertex v induces p triangular faces $(x, x + c_1, x + c_1 + c_2)$, $x \in Z_p$. (See Figure 6.)

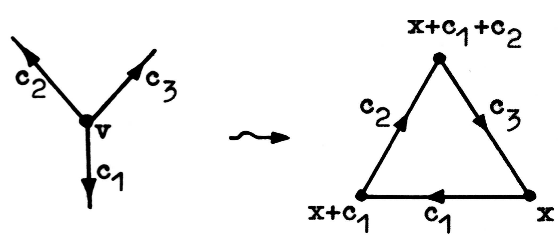


Figure 6: A vertex of the current graph and induced faces.

(P2) If the log is $(g_1, g_2, \ldots, g_{p-1})$, then for each vertex x of the embedded graph K_p, the edges incident with x are arranged around x on the surface in the circular order given by $(g_1+x, g_2+x, \ldots, g_{p-1}+x)$ which specifies the circular order of the neighbouring vertices. (See Figure 7.)

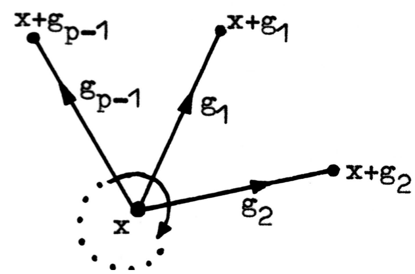


Figure 7: The circular order of the edges incident with a vertex x of K_p.

If $(f, A) \to f'$, then in order to prove that f and f' are nonisomorphic, it suffices to show that exactly one of the embeddings is face 2-colourable. Here we need the following definitions. Two faces of an embedding are called *adjacent* if they share a common edge. By a *band* of length t of an embedding, we mean a sequence $F_1, F_2, \ldots, F_t$ of faces of the embedding such that F_i and F_{i+1} are adjacent for $i = 1, 2, \ldots, t-1$. If $F_1 \neq F_t$, then the faces F_1 and F_t are called the *end faces of the band.* If $F_1 = F_t$, then the band is called a *strip of length* $t-1$. We now state and prove our main result.

Theorem 1 *In the nn-case, the minimum nonzero distance is 4. In the oo-case and in the on-case, the minimum nonzero distance is 6.*

Proof.

The *nn*-case. Figure 8 shows an index one current graph generating a triangular embedding f of K_{37}. Taking P1 and P2 into account, the reader can check the following claims. The embedding contains the modifiable set A_1 shown in Figure 9(a) (the faces of the modifiable set are induced by the starred vertices of the current graph in Figure 8), and hence, by Lemma 2, the embedding is nonorientable.

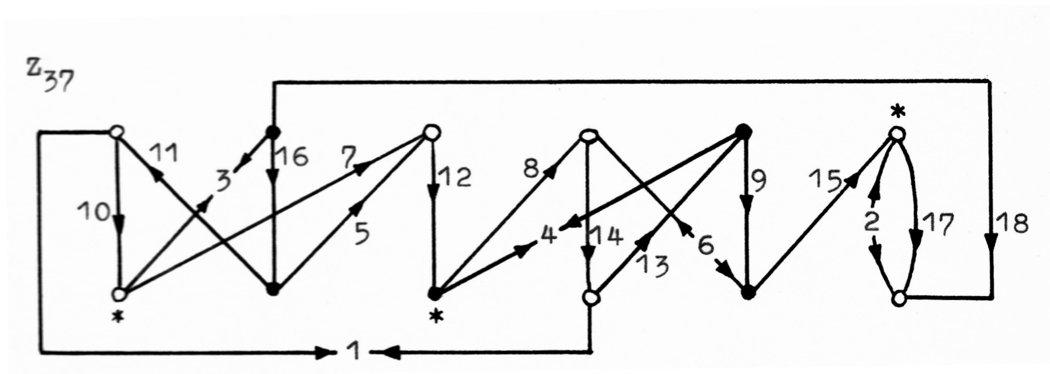


Figure 8: An index one current graph for K_{37}.

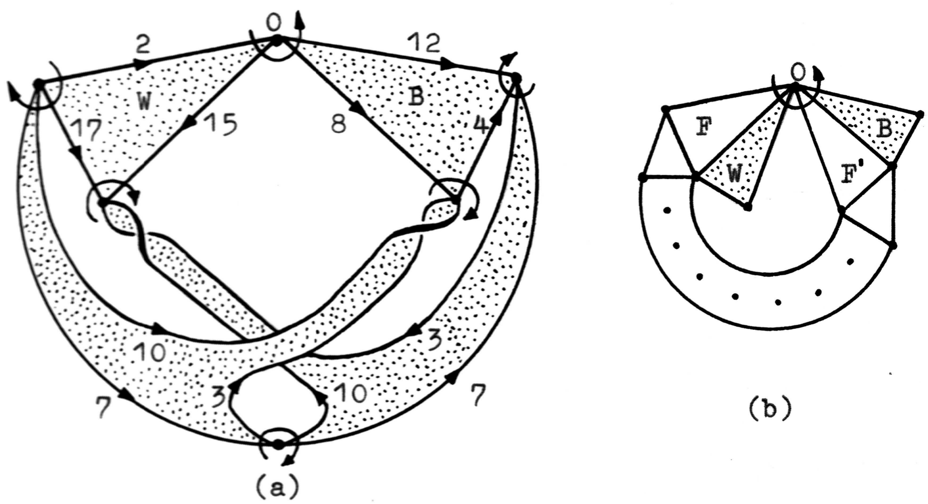


Figure 9: The modifiable set A_1 and an associated band.

Regarding Figure 8, the log of the circuit (up to reversal) is

$$(8, 12, \ldots, 2, 17, -2, 15, \ldots, -4, -12, -7, 3, \ldots,$$
$$-10, -3, \ldots, -17, -15, \ldots, -8, 4, \ldots, 10, 7, \ldots)$$

and for each vertex of A_1, this log determines the circular order of incident A_1-edges around that vertex as indicated in Figure 9(a).

Now consider the modification $(f, A_1) \to f'$. The current graph is bipartite and so f is a face 2-colourable embedding. The faces of f induced by the upper six vertices in Figure 8 will be called the white faces of f and those induced by the lower six vertices will be called the black faces. To prove that f and f' are nonisomorphic, we show that f' is not face 2-colourable. The modifiable set A_1 shown in Figure 9(a) contains a white face W and a black face B. The face W is adjacent to a black face F, and the face B is adjacent to a white face F' (see Figure 9(b)), such that neither F nor F' are faces of A_1. We leave it to the reader to show, using P1, that f contains a band of even length whose end faces are F and F', and none of whose faces are A_1-faces. Now, performing the modification, we add a new face adjacent to F and F', and as a result we obtain a strip of f' with an odd length. Hence f' is not face 2-colourable. It follows that 4 is the minimal nonzero distance in the *nn*-case.

There are smaller examples than the K_{37} example given above. Many further pairs of embeddings with distance 4 are given in [2] based on nonorientable triangular embeddings of K_{15} obtained from [3]. Although these involve smaller graphs than the K_{37} embedding, the embeddings are not index one and cannot be compactly represented using a current graph.

The *oo*-case. Figure 10 shows an index one current graph generating a triangular orientable embedding of K_{19}. Since the current graph is bipartite, the embedding is face 2-colourable. Taking P1 into account, the reader can easily check that the embedding contains a modifiable set A_2 shown in Figure 11 and consisting of three pairs of adjacent faces (initially ignore the dashed lines). Diagonal flips are now performed on these three pairs, that is, diagonals depicted as solid lines are replaced by the diagonals depicted as dashed lines. We get a new orientable triangular embedding of K_{19}. Hence this modifiable set is *oo*-minimal. All vertices of the modifiable set are 3-vertices. The reader can easily check that the resulting embedding has a strip of odd length, hence the modified embedding is not face 2-colourable. It follows that 6 is the minimal nonzero distance in the *oo*-case.

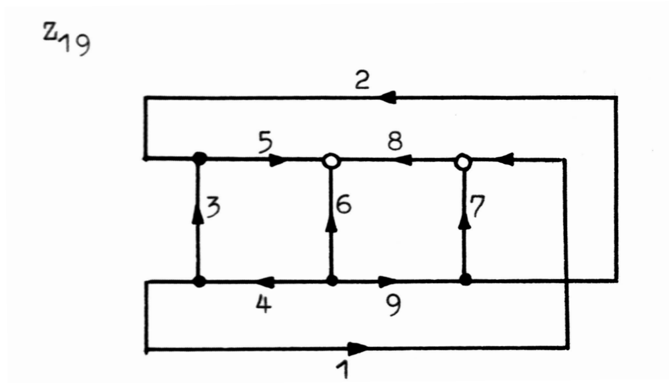


Figure 10: An index one current graph for K_{19}.

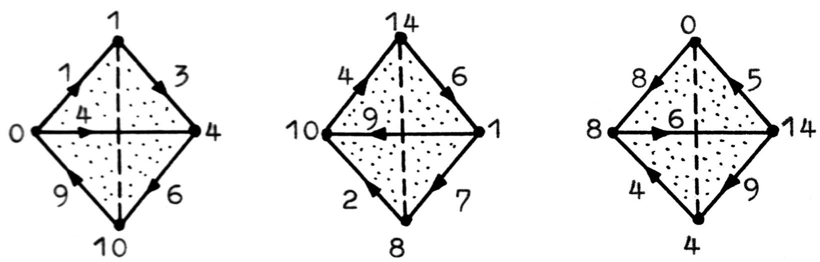


Figure 11: The modifiable set A_2.

The *on*-case. Figure 12 shows an index one current graph generating a triangular orientable embedding of K_{19}. Taking P1 and P2 into account, the reader can check the following claims. The embedding contains the modifiable set A_3 shown in Figure 13.

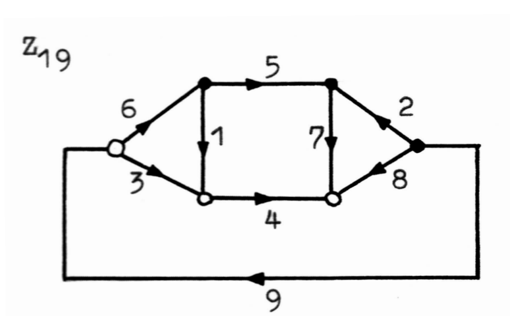


Figure 12: An index one current graph for K_{19}.

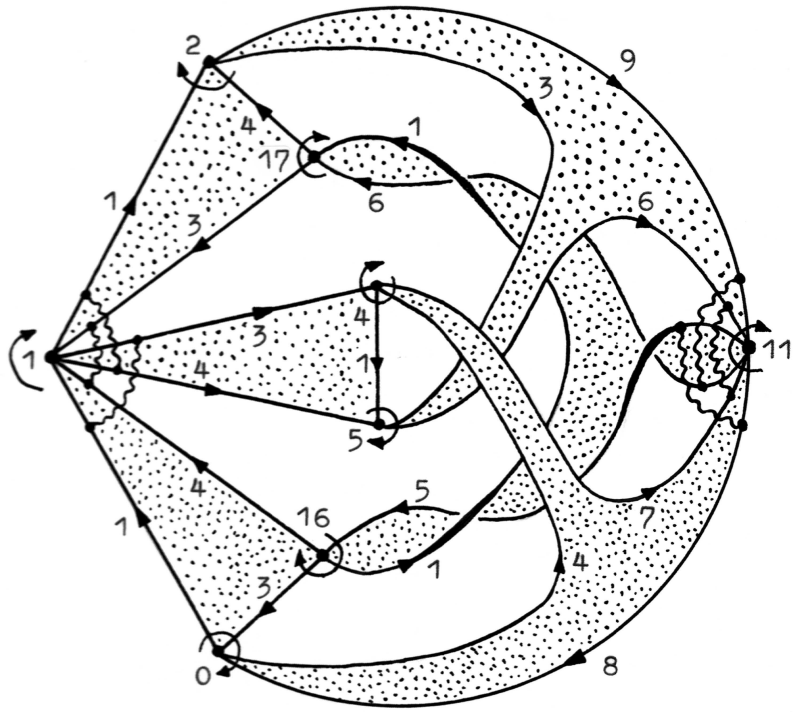


Figure 13: The modifiable set A_3.

Regarding Figure 12, the log of the circuit (up to reversal) is

$$(1, -3, 6, 5, -2, 9, 3, 4, -8, 2, 7, -4, -1, -6, -9, 8, -7, -5)$$

and for each vertex of A_3, this log determines the circular order of incident A_3-edges around that vertex as indicated in Figure 13. For each vertex x of K_{19}, the embedding contains a band as shown in Figure 14(a) and consisting of three faces. The band may be represented in abbreviated form as shown in Figure 14(b). The embedding has the faces shown in Figure 15, where only two faces belong to A_3. If we attach the new face incident with the vertex 0 in accordance with the indicated wavy line, then this new face and the depicted faces not in A_3 form a Möbius strip, and so the resulting modified embedding is nonorientable. Hence the modifiable set A_3 is *on*-minimal. It follows that 6 is the minimal nonzero distance in the *on*-case.

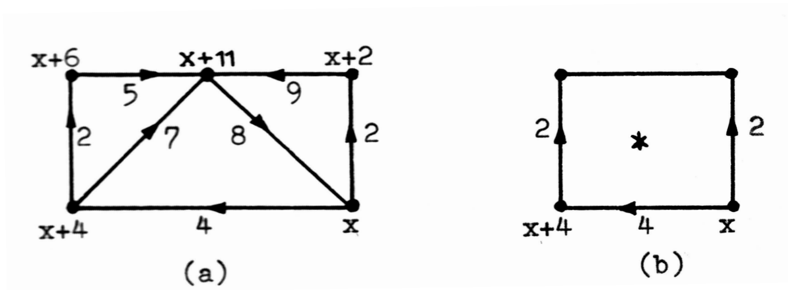


Figure 14: A band in the embedding.

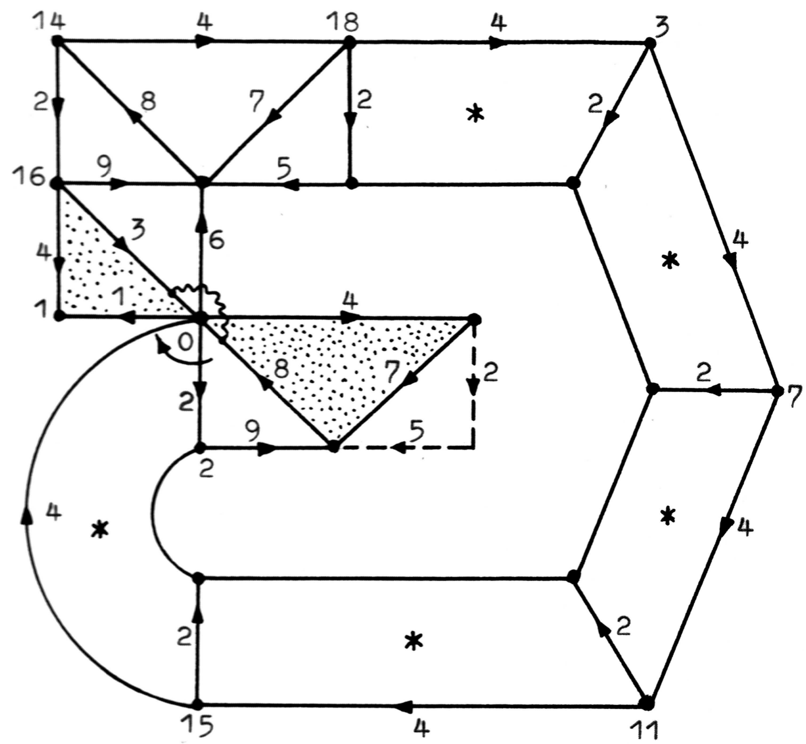


Figure 15: A Möbius strip in the modified embedding.

□

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