Modular Gracious Labellings of Trees

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ABSTRACT

A *gracious labelling g* of a tree is a graceful labelling in which, treating the tree as a bipartite graph, the label of any edge (*d,u*) (*d* a 'down' and *u* an 'up' vertex) is $g(u) - g(d)$. A *gracious k-labelling* is one such that each residue class modulo *k* has the 'correct' numbers of vertex and edge labels - that is, the numbers that arise by interpreting the labels of a gracious labelling modulo *k*. In this paper it is shown that every non-null tree has a gracious *k*-labelling for each $k = 2, 3, 4, 5$.

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1. Introduction

Let $T = (V, E)$ be a tree with (non-empty) vertex set V and edge set E. The *size* of T is the number of edges in T, denoted by $q(T)$ (or by q where T is given by the context). We regard the trivial graph with just one vertex as a tree, the *null tree*, of size 0.

Throughout this paper, we use the term *labelling of T* to refer to a bijection from V to the set $\{0, 1, 2, \ldots, q\}$.

The main focus of this paper involves labellings that arise if, for some *k,* we (informally speaking) remember only the congruence class of each label modulo *k*. More precisely, let Z_k be the additive group of integer congruence classes modulo k and define the function $\phi_{k,q} : Z_k \to Z$ as follows: $\phi_{k,q}(x)$ is the number of integers in the set $\{0, 1, \ldots, q\}$ belonging to congruence class x. Then a *k-labelling of T* is a function $f : V(T) \to Z_k$ such that for each $x \in Z_k$ there are $\phi_{k,q}(x)$ vertices with the label x.

Graph labellings have been quite extensively studied. For a useful dynamic survey, see [2]. In particular, a *graceful labelling* of a tree T is a labelling such that the *induced edge labelling* defined by $\tilde{g}(e) = |g(v_1) - g(v_2)|$ $(e = (v_1, v_2) \in E)$ is a bijection: $E \to \{1, 2, \ldots, q\}$. The concept was introduced (under the name *β-valuation*) by Rosa [7] as an aid to decomposing the edges of the complete graph K_{2q+1} into $2q+1$ copies of a tree of size q (see [5]). The name *graceful* was coined by Golomb [3]; it is used also to describe any tree that possesses a graceful labelling. It is a long-standing conjecture, attributed to Kotzig (see [7]), that every tree is graceful. It is noted in [2] that Aldred and Mackay, in a preprint, show that every tree of size at most 27 is graceful.

In attempting to make progress on this conjecture, Huang, Kotzig and Rosa [4] defined an *α-labelling* of a tree to be a graceful labelling g having the additional property that there exists an integer l with the following property: for every edge $e = (v_1, v_2)$ with $g(v_1) < g(v_2)$, we have $g(v_1) \le l < g(v_2)$. Clearly, if g is such a labelling, then it is consistent with the bipartitioning of V induced by the edges of T, in the following sense. Let V be bipartitioned into D (the *down vertices*) and U (the *up vertices*) so that every edge of T joins a down vertex to an up vertex. Then the down vertices have the labels from 0 to l and the up vertices the labels from l+1 to q.

It is known (see [4], [7]) that not every tree has an α-labelling. A counter-example (and the smallest such) is the tree of size 6 formed by joining three paths of size 2 at one end of each; that is, the tree with vertex set $\{v_0, v_1, \ldots, v_6\}$ and edge set $\{(v_0, v_1), (v_0, v_2), (v_0, v_3), (v_1, v_4), (v_2, v_5), (v_3, v_6)\}$.

Throughout this paper, we shall assume for any tree T under discussion that the selection of the vertex sets D, U has been made, and we shall refer to the *state* of a vertex of T as 'down' or 'up' accordingly.

We now propose a relaxation of the concept of an α-labelling.

A *gracious labelling* of T is a bijection $g : V \to \{0, 1, \ldots, q)$ such that the *induced edge labelling* $\tilde{g}(e) = g(u) - g(d)$ $(e = (d,u) \in E, d \in D, u \in U)$ is a bijection: $E \to \{1, 2, \ldots, q\}$. A tree is *gracious* if it possesses a gracious labelling.

It is arbitrary which set of the vertex bipartition is regarded as D and which as U, since if g is a gracious labelling, then replacing g by $q - g$ and reversing the vertex states creates another such labelling.

We note, for interest, that the concept of a gracious labelling is capable of extension to the class of connected bipartite graphs, as follows. If B is a connected bipartite graph (with bipartition $D \cup U$ and q edges), then a *gracious labelling* of B is a labelling g: $V(G) \to \{0, 1, \ldots, q\}$, such that g is an injection while the induced edge labelling $\tilde{g} : E(G) \to \{1, 2, \ldots, q\}$ is a bijection.

The concept of a gracious labelling (of a tree) lies between those of an α-labelling and a graceful labelling, in that an α-labelling is a graceful labelling in which every down vertex has a lower label than *every* up vertex, whereas a gracious labelling is a graceful labelling in which every down vertex has a lower label than every *adjacent* vertex.

For example, consider the path of size 5 with vertex set $\{v_0, v_1, \ldots, v_5\}$ and edge set $\{(v_0, v_1), (v_1, v_2), \ldots, (v_4, v_5)\}$, and let $D = \{v_1, v_3, v_5\}$. Labelling the vertices (in order) 5, 0, 4, 1, 3, 2 produces an α-labelling, labelling them 2, 0, 5, 1, 4, 3 produces a gracious (but not α-) labelling, and labelling them 4, 1, 5, 0, 2, 3 produces a graceful (but not gracious) labelling.

An immediate question is whether every tree has a gracious labelling. One of the authors has verified by computer that every tree of size at most 20 possesses such a labelling. This prompts the following conjecture:

Conjecture 1. Every tree is gracious.

Conjecture 1 is difficult to tackle in full. The object of this paper is to study a 'modulo *k*' version of the conjecture and introduce techniques which in principle seem capable of extension (though the details become progressively more laborious as *k* increases).

Let $T = (V, E)$ be a tree, let k be an integer greater than 1, and let f be a k-labelling of T (so that for each $x \in Z_k$ there are $\phi_{k,q}(x)$ vertices labelled x). The *induced edge-labelling* $\widetilde{f}$ is defined as above; that is to say,

$$\widetilde{f}(e) = f(u) - f(d) \quad (e = (d,u), d \in D, u \in U)$$

where subtraction is understood modulo k. Then f is a *gracious k-labelling* of T if the edge-labels also are 'correctly' distributed over the congruence classes; that is, there are $\varphi_{k,q}(0) - 1$ edges labelled 0 and, for each non-zero $x \in Z_k$, $\varphi_{k,q}(x)$ edges labelled x. A tree is *k-gracious* if it possesses a gracious k-labelling.

In order to compare gracious labellings with gracious k-labellings, let ζ_k be the function from Z to Z_k that maps each integer to its congruence class modulo k and let ξ_k be the function from Z_k into Z that maps each congruence class to its representative in the set $\{0, 1, \ldots, k - 1\}$.

Proposition 1.

(i) Given any k greater than 1 and any gracious labelling g of a tree T, the labelling $\zeta_k \circ g$ is a gracious k-labelling.

(ii) Given any k greater than $2q$ and any gracious k-labelling f of T, the labelling $\xi_k \circ f$ is a gracious labelling.

Proof.

(i) This follows directly from the definition of gracious k-labelling.

(ii) Let f be a gracious k-labelling of T, where $k > 2q$. Then the vertex labels of $\xi_k \circ f$ constitute the set $\{0, 1, \ldots, q\}$. If $(d, u) \in E(T)$, then $\xi_k \circ f(u) > \xi_k \circ f(d)$, since otherwise $\widetilde{f}(e) \in \{k - q, k - q + 1, \ldots, k - 1\}$, which does not intersect $\{1, \ldots, q\}$. Thus $\{\xi_k \circ \widetilde{f}(e) : e \in E(T)\} = \{1, \ldots, q\}$ as required. ■

It follows that a proof, for some k, that every tree is k-gracious would imply that every tree of size less than $\frac{k}{2}$ is gracious (and hence graceful).

The relation between gracious labellings and gracious k-labellings is somewhat similar to that between graceful and *equitable* labellings [1], which are defined as follows. Consider a function $f : V \to \{0, 1, \ldots, k\}$ (where $k \leq q$), and the induced edge labelling $\tilde{f}(v_1, v_2) = |f(v_1) - f(v_2)|$. For $x = 0, 1, \ldots, k$, let $v_f(x)$ and $e_f(x)$ be the numbers of vertices and edges respectively of T with label x. Such a labelling is (k+1)-*equitable* if $|v_f(i) - v_f(j)| \le 1$, $|e_f(i) - e_f(j)| \le 1$ $(i, j = 0, 1, \ldots, k)$. For a given labelling f of a tree T, f is graceful if and only if it is (q+1)-equitable.

Every tree has a 2-equitable labelling [1]. Recently, it has been proved that every tree is also 3-equitable [8]. However, for larger values of k the only result that seems to be known is that every path is k-equitable for every k [9]. The situation for gracious k-labellings seems to be slightly more tractable, in that we show that every tree has a gracious k-labelling for each k = 2, 3, 4, 5; and (see Lemma 1) every caterpillar is k-gracious for every $k \ge 2$.

The main technique used in this paper is as follows. We fix k and assume there is a tree T of smallest size that does not have a gracious k-labelling. We then show that for some integer p, a set of pk edges can be detached from T in such a way that any gracious k-labelling of the remaining tree S can be extended to a gracious k-labelling of T (occasionally with a slight adjustment to the labelling of S).

2. Caterpillars and forests

A *caterpillar* is a tree such that the deletion of each vertex of degree 1 together with the incident edge leaves a path. The following lemma produces a useful stock of small trees with gracious k-labellings for all k.

Lemma 1.

(i) Every tree of size less than 6 is a caterpillar.

(ii) Every caterpillar has a gracious k-labelling for all $k \ge 2$.

Proof.

(i) This follows by a simple examination of the trees of size less than 6.

(ii) By [7], every caterpillar has a labelling which is an α-labelling and hence a gracious labelling; the result follows from part (i) of Proposition 1. ■

Note 1. If f is a gracious k-labelling of a tree T of size congruent to r (mod k), then $r - f$ becomes a gracious k-labelling when the roles of D and U are interchanged. Thus, when considering the existence of gracious k-labellings, it is irrelevant which set of the bipartition is taken as D and which as U. Nevertheless, it will be convenient to use the notation $-T$ to refer to the result of exchanging the roles of D and U in T, and the notation $\pm T$ where both choices of D and U are required.

To formalise the 'detachment and re-attachment' technique described in Section 1, we proceed as follows.

A *rooted tree* (R, v) is a tree R rooted at a vertex v. Let S be any tree and $w \in V(S)$; we say that (R, v) is *attached to S at w* to form a tree T, if T is formed by identifying v

with w to form a vertex x. Conversely, we say that S is the *residual tree* formed as a result of *detaching* (R, v) *from* T at the vertex x of T.

More generally, a *rooted forest F of c components* is a list[1] $[(R_1, v_1), \ldots, (R_c, v_c)]$ of rooted trees. Such a forest may be *attached* to a tree S at a list $[w_1, \ldots, w_c]$ of vertices of S or *detached* from a tree T at a list $[x_1, \ldots, x_c]$ of vertices of T. We assume throughout that attachment and detachment respect vertex states.

For any integer $k > 1$, a *rooted k- tree* [2][resp *rooted k- forest*] is a rooted tree [resp rooted forest] whose size is a multiple of k. (In the case of a rooted forest, the individual component sizes are *not* restricted.) A *rooted k-labelling* of a rooted k-forest F of size pk is a vertex labelling of F such that, for each $x \in Z_k$, exactly p *non-root* vertices have the label x and exactly p edges have the induced label x.

The following useful symmetry principle minimises the labour of finding rooted k-labellings.

Lemma 2. Let F be a rooted k-forest with a rooted k-labelling f, and let λ be a bijection of Z_k of the form $z \mapsto az + b \ (z \in Z_k)$ where $a, b \in Z_k$ and a is a generator of Z_k. Then $\lambda \circ f$ is a rooted k-labelling, both of F and of $-F$.

Proof. It is immediate that $\lambda \circ f$ allocates the vertex labels to $\pm F$ satisfactorily; moreover, if μ is the bijection $z \mapsto az \ (z \in Z_k)$, then the edge-labelling of $\pm F$ induced by $\lambda \circ f$ is $\pm \mu \circ \tilde{f}$, and so the edge labels are also satisfactorily allocated. ■

Let S be a tree with a gracious k-labelling f and let F be a rooted k-forest with a rooted k-labelling g. If each component (R_i, v_i) of F is attached to S at a vertex that agrees with v_i in label and state, then it is immediate that the labelling that agrees with f on S and with g on F is a gracious k-labelling of the tree T so formed. If the labels and/or states do not agree, then no general conclusion can easily be drawn except in the case where F is a rooted k-tree, in which case we have the following result.

Lemma 3. If S is a k-gracious tree and (R, v) is a rooted k-tree possessing a rooted k-labelling, then attaching (R, v) to S at any vertex (exchanging the vertex states on (R, v) if necessary) will always result in a k-gracious tree T.

Proof. Let f be a gracious k-labelling of S, and let g be a rooted k-labelling of (R, v). Suppose $(\pm R, v)$ (as appropriate) is attached to S at a vertex u where $f(u) = a$ and $g(v) = b$; then, by Lemma 2, $g + a - b$ is a rooted k-labelling of $(\pm R, v)$, that extends f to a gracious k-labelling of T. ■

[1] By *list* we mean what is frequently called a *multiset*.
[2] Although this usage of the term *k-tree* is non-standard, it is natural in this context and other terminology would be awkward.

3. 2-labellings

In this section we show that every tree is 2-gracious. [3] First we establish that every tree of size at least 3 has a one-component detachment of size 2; the result then follows easily.

Lemma 4. Let T be a tree of size q and let c_i $(i \geq 1)$ be the number of vertices of degree i. Then

$$c_1 > \sum_{i \geq 3} c_i .$$

Proof. Counting vertices, $\sum_i c_i = q + 1$; counting ends of edges, $\sum_i ic_i = 2q$. Subtracting the second equation from twice the first gives

$$c_1 - \sum_{i \geq 3} (i - 2)c_i = 2,$$

and the result follows. ■

Lemma 5. Let T be a tree of size at least 3; then it has a one-component detachment of size 2.

Proof. As $|T| > 1$, every vertex of degree 1 (of which there are at least two) is adjacent to a vertex of degree > 1. The proof then falls into two cases.

Case 1: *T has a vertex x of degree* 1 *that is adjacent to a vertex y of degree* 2.

Let z be the other vertex of T to which y is adjacent. Then we can detach a path of size 2 from T at z.

Case 2: *Every vertex of T of degree* 1 *is adjacent to a vertex of degree greater than* 2.

By Lemma 3, there are more vertices of degree 1 than of degree greater than 2, and so there are vertices x, y of degree 1 adjacent to the same vertex z. Thus we can detach at z a tree of size 2, rooted at its vertex of degree 2 and having x, y as its other two vertices. ■

Theorem 1. Every tree is 2-gracious.

Proof. By Lemma 1 there are gracious 2-labellings of the trees of sizes 0, 1 and 2; moreover, each of the rooted trees of size 2 with root state 'down' has a rooted 2-labelling (see Figure 1). Thus, by Lemmas 3 and 5, if we assume that T is a tree of minimum size that has no gracious 2-labelling, we immediately obtain a contradiction.

[3] Although this result follows almost immediately from the proof in Section 7 that all trees are 4-gracious, the proof forms a good introduction to the methods of this paper.

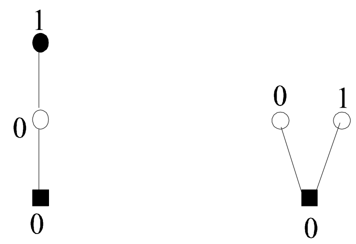

Figure 1 ■

Note 2. Here and throughout the paper, we adopt the graphical convention that the root vertex of a rooted tree is drawn as the lowest vertex in the picture; it is denoted by a square and any other vertex is denoted by a circle. A vertex is filled if it is ‘down’ and open if it is ‘up’.

In the next section we prepare for the cases $k = 3, 4, 5$ by setting up further terminology for trees.

4. Further terminology and notation

The object of this section is to define and use a class of trees which we call *quipus* (a quipu being an Inca device for conveying information by knotted strings). It is straightforward to find gracious k-labellings of many quipus, and so it is useful to investigate which quipus can be detached from a given tree.

Let S be a tree and $\mathbf{R} = [(R_1, v_1), \ldots, (R_p, v_p)]$ a list of disjoint rooted trees. It is possible to construct a tree T of size $|S| + \sum_{i=1}^{p} |R_i|$ formed by attaching the (R_i, v_i) to S at a given list of vertices of S.

In particular, let each (R_i, v_i) be a path of size q_i rooted at an end vertex, and let S be the null tree with just one vertex, w, to which all the (R_i, v_i) are attached. In this case T will be called a *quipu* and will be denoted by $Q[q_1\ q_2\ \ldots\ q_p]$. If some of the paths are of the same size, it is convenient to abbreviate the list in a standard way, so that (for example) the quipu formed from two paths of size 1 and three of size 2 is denoted by $Q[1^2\ 2^3]$. The vertex of a quipu to which the paths are attached is the *knot*; the paths are its *strings*. Thus a quipu has at most one vertex of degree greater than 2; if it has such a vertex, it must be the knot, but any vertex of a path graph P may be regarded as the knot, thus making P into a quipu of one or two strings.

If $Q[\mathbf{q}]$ is a quipu and $\mathbf{q}'$ is a sublist of $\mathbf{q}$, then $Q[\mathbf{q}']$ is a *subquipu* of $Q[\mathbf{q}]$. Most of the quipus considered in this paper will be rooted at the knot; we then use the notation $Q[\mathbf{q}]$ (or $+Q[\mathbf{q}]$) if the root is ‘down’, $-Q[\mathbf{q}]$ if the root is ‘up’. Occasionally, however, it will be useful to root a quipu at an end vertex adjacent to the knot; such a quipu is denoted by $\hat{Q}[\mathbf{q}]$ with the same sign convention as above. (See Figure 2 for various examples of rooted quipus.)

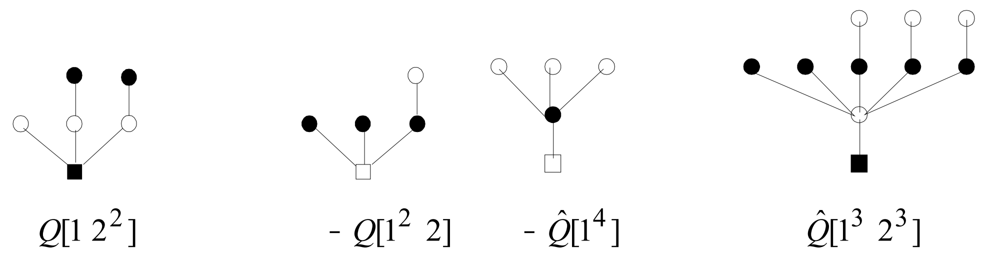

Figure 2: Examples of rooted quipus

Note 3. Later in the paper we describe a compact notation for labelling trees and forests. In preparation for this, we adopt the graphical convention that the strings of a quipu are always depicted in non-decreasing order of size from left to right, as above.

Let T be any tree. We call the vertices of degree 1 of T, the *tips*; those of degree 2, the *through-vertices,* and those of degree greater than 2, the *major vertices.* If a tree T [resp quipu Q] has at least one major vertex (i.e. is not a path graph), it is a *major tree* [resp *major quipu*]; a major vertex v of T is a *twig vertex* if there is a path from v to a tip t such that all intermediate vertices (if any) are through-vertices. Such paths are the *twigs* of T at v, and v is the *base* of each twig at v. A twig of size t is a *t-twig.*

Let T be a major tree, let v be a twig vertex of T, and let $Q(v)$ be the subtree of T whose edges are those of the twigs at v; this is the *quipu at v*, and is of the form $\pm Q[\mathbf{t}]$ where **t** is the list of twig sizes at v. The *quipu list* of T is the list of the quipus at the twig vertices and is denoted by $\mathbf{Q}(T)$. The size of the largest twig of T is denoted by $m(T)$.

The *core* $C(T)$ of a major tree T is the residual subtree which remains after detaching all twigs (or equivalently, all the quipus at twig vertices). In particular, the core of a major quipu has just one vertex, the knot. (If T is a path, then $C(T)$ is undefined.)

If T is not a quipu, then $C(T)$ is a non-null tree, whose tips we call the *crux vertices* of T. The *crux list* of T is the list of the quipus at the crux vertices and is denoted by $\mathbf{X}(T)$.

If v is a crux vertex of T, then it is adjacent to exactly one other vertex of $C(T)$, the *retract* of v, which we denote by v_ρ.

Lemma 6. Let T be any major tree. Then:

(i) for each i $(1 \le i \le m(T))$, $\pm Q[i]$ may be detached from T;

(ii) any subquipu of a quipu in $\mathbf{Q}(T)$ may be detached from T ;

(iii) if $\pm Q[\mathbf{q}] \in \mathbf{X}(T)$, then $\mp \hat{Q}[1\ \mathbf{q}]$ may be detached from T ;

(iv) there are at least two twigs at each crux vertex; that is, every quipu in $\mathbf{X}(T)$ has at least two strings.

Proof.

(i): detach at a suitable through-vertex on a twig of size at least i, or detach an i-twig at its base.

(ii): detach at the corresponding twig vertex.

(iii): detach at the retract of the corresponding crux vertex.

(iv): Suppose v is a twig vertex and P is a twig at v. As v is a major vertex, it is adjacent to at least two vertices x and y that do not lie on P. If neither of these is on a twig, then v cannot be a crux vertex of T. ■

5. 3-labellings

Lemma 7. Let T be a tree of size at least 4; then at least one of the following statements is true:

(i) $\pm\ T$ has one of the quipus $Q[1\ 2], Q[3], Q[1^3], \hat{Q}[1^3]$ as a detachment of size 3.

(ii) $\pm\ T$ has the quipu $Q[2^3]$ as a detachment of size 6.

(iii) $\pm\ T$ has one of the two-component forests $[Q[2^2], \pm\ \hat{Q}[1\ 2^2]]$ as a detachment of size 9.

Proof. We shall assume that T is a counter-example and obtain a contradiction.

Statement (i) is clearly true if T is a path of size > 3, so assume that T is a major tree.

If $m(T) \geq 3$, then by Lemma 6(i), $\pm\ Q[3]$ may be detached from T, counter to assumption. Thus, $m(T) \leq 2$. Now every major quipu Q with $|Q| \geq 4$, $m(Q) \leq 2$ has as a subquipu one of $\pm\ Q[1^3]$, $\pm\ Q[1\ 2]$ or $\pm\ Q[2^3]$.

Thus T is not a quipu, and so has at least two crux vertices, v and w; moreover, $m(T) \leq 2$. By Lemma 6(ii), then, statement (i) or statement (ii) is true unless each of $Q(v)$ and $Q(w)$ has at most two strings of size 1 and at most two of size 2, but does not have both a string of size 1 and a string of size 2.

Thus each of $Q(v)$ and $Q(w)$ is $\pm\ Q[1^2]$ or $\pm\ Q[2^2]$. But if $Q(v) = \pm\ Q[1^2]$, then $\pm\ Q[1^2] \in \mathbf{X}(T)$ and, by Lemma 6(iii,) we may detach $\mp\ \hat{Q}[1^3]$ from v_ρ. Thus, $Q(v) = \pm\ Q(w) = \pm\ Q[2^2]$, and it follows from Lemma 6 that $\pm\ [Q[2^2], \pm\ \hat{Q}[1\ 2^2]]$ may be detached from T, contrary to assumption. ■

Theorem 2. Every tree is 3-gracious.

Proof. Suppose the contrary, and let T be a tree of minimum size having no gracious 3-labelling.

By Lemma 1, T has size at least 6. Thus, by Lemma 7, $\pm\ T$ either has one of a list of five one-component detachments (each of size 3 or 6) or has $[Q[2^2], \hat{Q}[1\ 2^2]]$ or $[Q[2^2], -\ \hat{Q}[1\ 2^2]]$ as a two-component detachment. Now

Figure 3 below exhibits rooted 3-labellings of each of the possible one-component detachments.

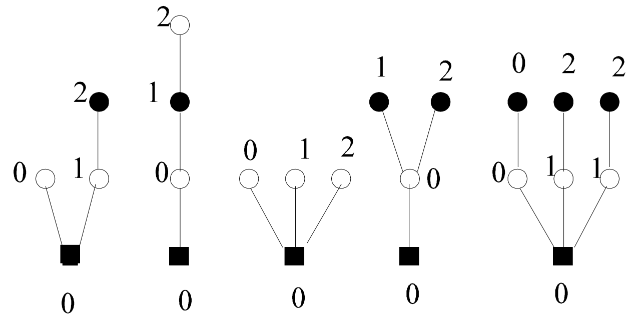

Figure 3: Rooted 3-labellings of $Q[1\ 2]$, $Q[3]$, $Q[1^3]$, $\hat{Q}[1^3]$, $Q[2^3]$.

After making any of these possible detachments, we obtain a tree S where $|S| < |T|$. By hypothesis S has a gracious 3-labelling, and so (by Lemma 3) we may re-attach the relevant quipu and conclude that T has a gracious 3-labelling, contrary to hypothesis. Thus the only possibility remaining is that $\pm\ T$ has one of the two-component detachments $[Q[2^2], \hat{Q}[1\ 2^2]]$ or $[Q[2^2], -\hat{Q}[1\ 2^2]]$.

Now let v, w be the vertices of $\pm\ T$ at which $Q[2^2]$ and $\pm\ \hat{Q}[1\ 2^2]$ respectively are detached.

Assume first that v and w have the same state. Figure 4 exhibits rooted 3-labellings of $[Q[2^2], \hat{Q}[1\ 2^2]]$; the first 3-labelling allocates label 0 to both roots and the second allocates 0 to the root of $Q[2^2]$ and 1 to the root of $\hat{Q}[1\ 2^2]$.

Figure 4: Rooted 3-labellings of the forest $[Q[2^2], \hat{Q}[1\ 2^2]]$

Note 4. In order to save space, we shall for the remainder of the paper depict a tree or forest at most once. We shall describe any subsequent labellings by giving an ordered list of vertex labels, starting at the root(s), proceeding from left to right within each level and upwards through the levels, with a semicolon between levels. For example, the labelling of $Q[1\ 2]$ depicted in Figure 3 is described by the list [0; 0,1; 2], while the labellings of $[Q[2^2], \hat{Q}[1\ 2^2]]$ in Figure 4 are described by the lists [0,0; 0,1,0; 0,2,1,1; 2,2] and [0,1; 1,0,1; 2,2,1,2; 0,0].

It follows from Lemma 2 that, given any gracious 3-labelling *f* of the residual tree $\pm\ S$, there is a rooted 3-labelling of $[Q[2^2],\ \hat{Q}[1\ 2^2]]$ that will extend *f* to a gracious 3-labelling of $\pm\ T$, contrary to assumption.

Assume now that *v* and *w* have opposite states. Then the lists [0,0; 1,0,0; 2,2,0,1; 1,2] and [0,1; 1,0,2; 2,2,0,1; 0,1] describe rooted 3-labellings of $[Q[2^2], -\hat{Q}[1\ 2^2]]$, and the above argument again shows that for any gracious 3-labelling *f* of $\pm\ S$ there is a rooted 3-labelling of $Q[2^2], -\hat{Q}[1\ 2^2]$ that extends *f* to a gracious 3-labelling of $\pm\ T$, contrary to assumption. ■

6. Rooted 4-labellings of certain 4-forests

We begin this section by depicting three rooted trees and a rooted 4-forest that cannot be described using the notation of Section 4; we denote the trees by *A, B, C* and the forest by *D*. (Figure 5.)

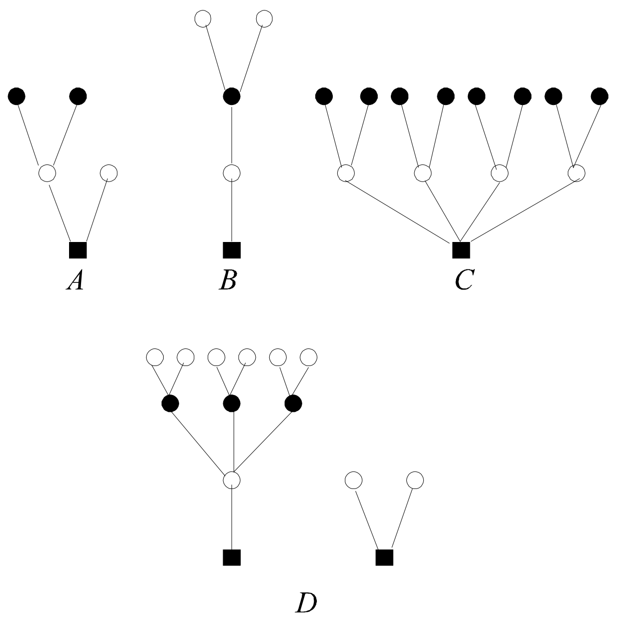

Figure 5: The rooted 4-forests A, B, C and D

We now build a stock of rooted 4-labellings of rooted 4-trees and 4-forests.

Lemma 8. The rooted 4-trees $Q[4]$, $Q[1^4]$, $\hat{Q}[1^4]$, *A*, *B*, *C* have rooted 4-labellings.

Proof. Suitable 4-labellings are described by:

For $Q[4]$: [0; 0; 1; 3; 2]; for $Q[1^4]$: [0; 0,1,2,3]; for $\hat{Q}[1^4]$: [0; 0; 1,2,3].

For *A*: [0; 0,2; 1,3]; for *B*: [0; 0; 2; 1,3]; for *C*: [0; 0,1,2,3; 0,1,2,3,0,1,2,3]. ■

Lemma 9. Let F_+ be the rooted forest $[Q[1], Q[1], Q[2]]$; F_- the rooted forest $[Q[1], Q[1], -Q[2]]$; and G the rooted forest $[Q[1^2], -Q[1^2]]$. Then each of F_+, F_-, G, D has a rooted 4-labelling for every choice of labels for the roots.

Proof. Consider the seven rooted 4-labellings of F_+ described by the lists:

[0,0,0; 1,3,0; 2]; [1,1,0; 2,3,0; 1]; [2,2,0; 1,3,0; 2]; [0,1,0; 0,3,1; 2]; [0,2,0; 2,3,0; 1]; [1,2,0; 0,2,1; 3]; [1,3,0; 1,0,2; 3].

Then by Lemma 2, F_+ has a rooted 4-labelling for every labelling of the roots.

Consider next the seven rooted 4-labellings of F_- described by the lists:

[0,0,0; 1,3,0; 2]; [1,1,0; 0,1,2; 3]; [2,2,0; 1,3,0; 2]; [0,1,0; 0,3,1; 2]; [0,2,0; 0,1,2; 3]; [1,2,0; 1,0,3; 2]; [1,3,0; 3,2,0; 1].

Again by Lemma 2, F_- has a rooted 4-labelling for every root labelling.

Consider finally the three rooted 4-labellings of G described by the lists [0,0; 0,2,1,3]; [0,1; 0,1,2,3]; [0,2; 0,2,1,3] and the three rooted 4-labellings of D described by the lists [0,0; 0,1,3; 0,0,2; 1,2,2,3,1,3]; [0,1; 2,1,2; 0,0,2; 1,3,0,3,1,3]; [0,2; 0,1,3; 0,0,1; 2,2,1,3,2,3]. Lemma 2 again implies the existence of rooted 4-labellings of G and D for every choice of root labelling. ■

7. Every tree is 4-gracious

Throughout this section, we let T denote a supposed tree of minimum size that has no 4-gracious labelling. Our technique is to derive a contradiction.

We say that a tree is *short-twigged* if $m(T) = 1$ (that is, every quipu in the quipu list is $\pm Q[1^p]$ for some p).

Lemma 10. T must be a short-twigged major tree, and every crux vertex must have the same state and have exactly two 1-twigs.

Proof. By Lemma 1, T cannot be a caterpillar (and so in particular cannot be a path). By Lemma 6(i), if $m(T) > 3$ then T has $\pm Q[4]$ as a detachment, and Lemmas 3 and 8 may be applied to give a contradiction.

Thus T is a major tree, with $m(T) \le 3$. Suppose there is a twig of size greater than 1; then no two other twigs can have tips of the same state, for otherwise one of the forests F_+, F_- could be detached from $\pm T$, and a contradiction could be deduced from Lemma 9. Thus, if $m(T) > 1$, then T has at most three tips; as T is major, it thus has exactly three tips and so is $\pm Q[a\ b\ c]$ where each of a, b, c is at most 3. By hypothesis at least one of a, b, c exceeds 1, and (as the other two tips must then be of opposite state) for any $x \in \{a, b, c\}$ with $x > 1$ the other two of a, b, c differ by 1. The only possibility is $\pm Q[1\ 2\ 2]$, which is a caterpillar and thus has a gracious 4-labelling, contrary to hypothesis.

Thus, T is short-twigged.

Now if T had a unique crux vertex v, then T would be $\pm\ Q[1^p]$ for some p, and this is a caterpillar, so that Lemma 1 contradicts the hypothesis. Thus there are at least two crux vertices; let two of them be v and w. By Lemma 6(iv), they must each have at least two 1-twigs. If they are of opposite state, then $\pm\ T$ has G as a detachment. This is impossible because Lemma 9 would then give a contradiction. Moreover, if any crux vertex has more than two 1-twigs, then by Lemma 6(ii), (iii), $\pm\ T$ has $Q[1^4]$ or $\hat{Q}[1^4]$ as a detachment, and Lemma 8 then provides a contradiction. ■

Since T is a major tree, $C(T)$ is defined. Moreover, $C(T)$ must itself be a major tree, otherwise T (being short-twigged) would be a caterpillar, and Lemma 1 would give a contradiction. Hence $C(C(T))$ is defined.

Lemma 11. $C(T)$ is itself short-twigged, and has at least two crux vertices; furthermore, each crux vertex of $C(T)$ is adjacent to exactly two 1-twigs of $C(T)$.

Proof. If $C(T)$ were not short-twigged, then (being a major tree) it would have a vertex w of degree 2 (in $C(T)$) adjacent to a tip v of $C(T)$.

If there were a 1-twig of T adjacent to w, then T would have $\pm\ A$ as a detachment at w. If there were no such twig, then there would be a vertex $x \neq v$ of $C(T)$ adjacent to w, and T would have $\pm\ B$ as a detachment at x. Both possibilities are contradicted by Lemma 8. Thus $C(T)$ is short-twigged.

If $C(T)$ had just one crux vertex, then $C(T)$ would be $Q[1^r]$ for some r, so that T would consist of r copies of $\hat{Q}[1^3]$ with their roots identified. If $r = 1$ or 2 then T would be a caterpillar; if $r > 3$ then T would either be $\pm\ C$ or have $\pm\ C$ as a detachment, and Lemma 8 would once more provide a contradiction.

Finally, if $r = 3$ then T would be as in Figure 6, and here the 4-gracious labelling shown in the figure provides a contradiction.

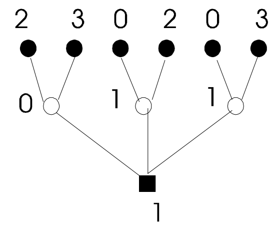

Figure 6

Thus $C(T)$ has at least two crux vertices. If v is such a vertex, it must have at least two 1-twigs of $C(T)$, by Lemma 6(iv). If v had exactly three such twigs, then a rooted tree isomorphic to the larger component of D could be detached from v_ρ, and $\pm\ Q[1^2]$ could be detached from a crux vertex of T adjacent to another crux vertex of $C(T)$. Furthermore, by Lemma 10 the roots of the two components must have the same state. Thus, $\pm\ D$ can be detached from T, and Lemma 9 provides a contradiction. Finally, if v had more than three 1-twigs (of $C(T)$), then $\pm\ C$ could be detached from T at v; here Lemma 8 supplies a contradiction. ■

Theorem 3. Every tree is 4-gracious.

Proof. Let T be a counter-example of minimum size. Let v and w be two crux vertices of $C(T)$ (which must exist by Lemma 11). As each of these is adjacent to exactly two 1-twigs of $C(T)$, the tips of which are each adjacent to exactly two 1-twigs of T, it follows that a rooted tree (which we denote by E) may be detached from each of v and w, thus detaching a rooted 4-forest $[E, E]$ to leave a residual tree S of which v and w are tips. Moreover, Lemma 10 implies that v and w must have the same state; let us assume without loss of generality that this state is 'up' (Figure 7.)

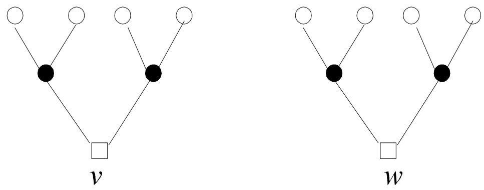

Figure 7: the rooted 4-forest $[E, E]$

Let f be a gracious 4-labelling of S (which must exist, by the minimality assumption). We now study two cases (the second of which splits into four subcases).

Case 1: $f(w) - f(v) = 0$ or 2

Now $[E, E]$ has the rooted 4-labellings described by the lists [0,0; 0,2,1,3; 0,2,1,3,0,2,1,3] and [0,2; 0,2,1,3; 0,2,1,3,0,2,1,3]. Thus by Lemma 2, f extends to a gracious 4-labelling of T.

In any rooted 4-labelling g of $[E, E]$, the sum of the induced edge labels is 2 and a parity argument then shows that $g(w) - g(v) = 0$ or 2. Therefore, Case 2 requires a more subtle argument.

Case 2: $f(w) - f(v) = 1$

Note that, as we may exchange v and w, this case also covers the possibility that $f(w) - f(v) = 3$.

Let $f(v) = a$, $f(w) = a+1$. Let x and y be the vertices of S adjacent respectively to v and w, and let $[\hat{E}, \hat{E}]$ be the rooted forest of size 14 obtained by detaching trees containing E at x and y (Figure 8).

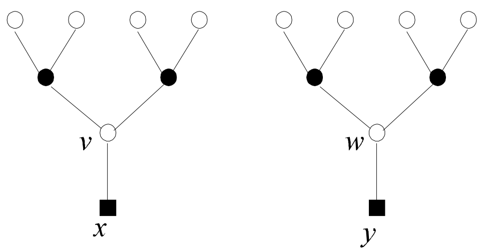

Figure 8: the rooted forest $[\hat{E}, \hat{E}]$

Let $\hat{S}$ be the corresponding residual tree (a subtree of S), and assume $f(x) = 0$. (We show later that this does not lose generality.)

Subcase 2(a): $f(y) = 0$

Consider the labelling of $[\hat{E}, \hat{E}]$ given by the list [0,0; 0,0; 0,1,1,1; a,a+1,2,3,2,3,2,3].

If this labelling is used to extend to T the restriction $\hat{f}$ of f to $\hat{S}$, then the resulting labelling has three more of each of 0,1,2,3 on the vertices, and also on the edges, than does the labelling f of S, and is therefore gracious. (Note, however, that the labels a and a+1, and the corresponding induced edge labels, have 'migrated' from their original positions on S.)

Subcase 2(b): $f(y) = 1$

This case is dealt with by the labelling of $[\hat{E}, \hat{E}]$ given by the list [0,1; 0,2; 1,1,0,3; a+1,1,2,3,a,2,0,3] (used in the same way as the labelling for Subcase 2(a)).

Subcase 2(c): $f(y) = 2$

This case is similarly dealt with using the labelling given by the list [0,2; 0,2; 1,3,1,2; a+1,3,0,2,a,3,0,1].

Subcase 2(d): $f(y) = 3$

This case is dealt with using the labelling given by the list

[0,3; 1,3; 0,3,3,2; a,0,2,1,a+1,2,1,0].

Thus in every case where $f(x) = 0$, there is a gracious 4-labelling of T, contrary to assumption. Suppose now that $f(x) = b$; then by adding b to each vertex label of an appropriate labelling from one of the above subcases, we again produce a gracious 4-labelling of T. ■

8. Every tree is 5-gracious

Lemma 12. Each of the three rooted 5-forests

$$[[Q[1], Q[2], Q[2]], \quad [Q[1], -Q[2], -Q[2]], \quad [Q[1], Q[2], -Q[2]]$$

has a rooted 5-labelling for every choice of root labels.

Proof. For all non-zero $a \in Z_5$ and for all $b \in Z_5$, the mapping $x \mapsto ax + b$ is a bijection on Z_5. Thus, by using Lemma 2, the existence of the following rooted 5-labellings establishes the result.

For [[Q[1], Q[2], Q[2]]: the labellings given by the lists [0,0,0; 0,1,4; 3,2]; [0,0,1; 0,4,3; 1,2]; [0,1,1; 2,1,0; 3,4]; [0,1,2; 0,2,4; 3,1]; [0,1,4; 0,2,3; 4,1].

For $[[Q[1], -Q[2], -Q[2]]$: the labellings given by the same lists as above except that the third list is replaced by [0,1,1; 2,1,3; 0,4].

For $[[Q[1], Q[2], -Q[2]]$: the labellings given by the lists [0,0,0; 2,3,0; 4,1]; [0,0,1; 1,0 4; 2,3]; [0,1,0; 0,3,1; 2,4]; [0,1,1; 3,1,2; 0,4]; [0,1,2; 0,2,4; 3,1]; [0,1,3; 0,2,1; 3 4]; [0,1,4; 0,3,1; 4,2]. ■

Lemma 13. The two 5-forests $[Q[1^2], Q[3]]$ and $[Q[1^2], -Q[3]]$ have rooted 5-labellings for every choice of root labels.

Proof. Using Lemma 2, the result follows from the following 5-labellings.

For $[Q[1^2], Q[3]]$: the labellings given by the lists [0,0; 0,1,2; 4; 3] and [0,1; 0,1,3; 4; 2].

For $[Q[1^2], -Q[3]]$: the labellings given by the lists [0,0; 0,1,3; 2; 4] and [0,1; 0,1,4; 2; 3]. ■

Lemma 14. The rooted 5-trees $Q[1^3, 2]$, $\hat{Q}[1^3\ 2]$ and the rooted 5-forests $[\hat{Q}[1^2\ 2], \pm Q[1]]$ have rooted 5-labellings for all choices of root labels.

Proof. This follows from Lemma 2 and rooted 5-labellings described as follows.

For $Q[1^3\ 2]$: [0; 0,2,4,1; 3].

For $\hat{Q}[1^3\ 2]$: [0; 0; 2,4,1; 3].

For $[\hat{Q}[1^2\ 2], Q[1]]$: [0,0; 1,0; 3,2; 4] and [0,1; 0,2; 3,1; 4].

For $[\hat{Q}[1^2\ 2], -Q[1]]$: [0,0; 1,0; 3,2; 4] and [0,1; 2,1; 3,4; 0]. ■

Lemma 15. *T* has at most one twig of size > 1, and that twig cannot be at a crux vertex.

Proof. By Lemma 1, *T* cannot be a path and so must have at least three tips (hence, at least three twigs). By Lemma 12, at most one of these twigs can have size > 1. Thus *T* cannot be a quipu (as a quipu that is not a caterpillar has at least three twigs of size greater than 1).

Assume now that *T* does have one twig of size $r > 1$, and that it is at the crux vertex v of *T*. Then $Q(v) = Q[1^p\ r]$ for some $p > 0$. Now if $p > 1$ then $m(T) \le 2$ by Lemmas 6 and 13, and so $r = 2$. Thus (using Lemma 6 again) $Q[1^3\ 2]$ can be detached from $\pm T$ at v if $p > 2$, while $\hat{Q}[1^3\ 2]$ can be detached at v_p if $p = 2$, contradicting Lemma 14.

The only remaining possibility is that $r = 2$ and $p = 1$; but then $\hat{Q}[1^2\ 2]$ may be detached from v_p and $\pm Q[1]$ from elsewhere on *T*, so that the 5-forest $[\hat{Q}[1^2\ 2], \pm Q[1]]$ may be detached from *T*. Once again this contradicts Lemma 14. ■

Lemma 16.

(i) The rooted 5-trees $Q[1^5]$ and $\hat{Q}[1^5]$ have rooted 5-labellings;

(ii) the rooted 5-forests $[\hat{Q}[1^3], \pm Q[1^2]]$ have rooted 5-labellings for every choice of root labels.

Proof. This follows from Lemma 2 and the rooted 5-labellings described as follows.

(i) For $Q[1^5]$: [0; 0,1,2,3,4], and for $\hat{Q}[1^5]$: [0; 0; 1,2,3,4].

(ii) For $[\hat{Q}[1^3], Q[1^2]]$: [0,0; 0,2,3; 1,4] and [0,1; 0,2,4; 1,3];

for $[\hat{Q}[1^3], -Q[1^2]]$: [0,0; 0,3,4; 1,2] and [0,1; 2,0,1; 3,4]. ■

Lemma 17. There are exactly three 1-twigs at each crux vertex of T, and all crux vertices have the same state.

Proof. As previously observed, T cannot be a quipu. Thus T has at least two crux vertices (since $C(T)$ has at least two tips).

Lemma 6, 15 and and 16(i) rule out the possibility that any of these crux vertices have more than three 1-twigs. Lemmas 6, 15 and 16(ii) rule out the possibility that there can be exactly two 1-twigs at any crux vertex v (since there must be another crux vertex w, and one could then detach $\pm \hat{Q}[1^3]$ from v_p and $\pm Q[1^2]$ from w). Thus the only remaining possibility is that there are exactly three 1-twigs at each crux vertex.

If two crux vertices had opposite state, then $[Q[1^3], -Q[1^2]]$ could be detached from $\pm T$. But Lemmas 2, together with the rooted 5-labellings described by the lists [0,0; 0,2,3,1,4] and [0,1; 2,3,4,0,1], contradicts this. ■

Lemma 18. T is short-twigged.

Proof. By Lemmas 15 and 17, T is short-twigged apart possibly from one twig of size greater than 1 at a non-crux vertex, and there are three 1-twigs at each crux vertex. Thus $[Q[1^3], \pm Q[2]]$ may be detached from T. But Lemma 2 and the following rooted 4-labellings of $[Q[1^3, Q[2]]$ and $[Q[1^3], -Q[2]]$ contradict this:

For $[Q[1^3], Q[2]]$: [0,0; 0,1,2,3; 4] and [0,1; 2,3,4,1; 0];

for $[Q[1^3], -Q[2]]$: [0,0; 1,2,3,0; 4] and [0,1; 0,1,3,2; 4]. ■

Theorem 4. Every tree is 5-gracious.

Proof. We proceed once again to hound T out of existence, on the assumption that it is a tree of minimum size that does not possess such a labelling.

First, Figure 9 below establishes that the rooted trees formed by rooting $Q[1^3\ 2]$ at the vertex of degree 2 and at the tip of the path of size 2 have rooted 5-labellings.

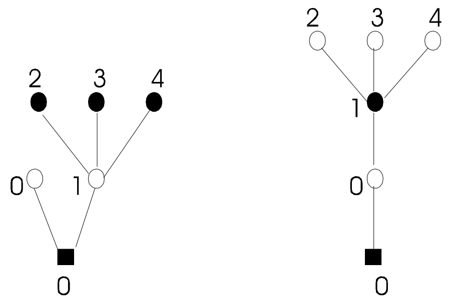

Figure 9

Now let v be a crux vertex of T, and consider the degree of v_ρ in $C(T)$. If v_ρ were of degree 2 in $C(T)$, then either it would be of degree 2 in T or (because T is short-twigged) there would be at least one 1-twig of T at v_ρ. In either case, one of the rooted trees of Figure 9 can be detached from T, and Lemma 3 can be applied to give a contradiction. Thus, $C(T)$ must itself be short-twigged. Now, T is not a caterpillar; hence (by Lemma 18) $C(T)$ is not a path, and so has at least three tips.

Arguing as in the discussion just prior to Lemma 11, $C(C(T))$ is defined. Now if $C(C(T))$ were the single-vertex tree, then by Lemmas 17 and 18 (and the fact that T is not a caterpillar) T would consist, for some $p \geq 3$, of p copies of $\hat{Q}[1^4]$ with the roots identified. The cases $p = 3, 4, 5$ are given gracious 5-labellings in Figure 10 below; as the case $p = 5$ is a 5-tree, this also deals with $p > 5$ (via Lemma 3).

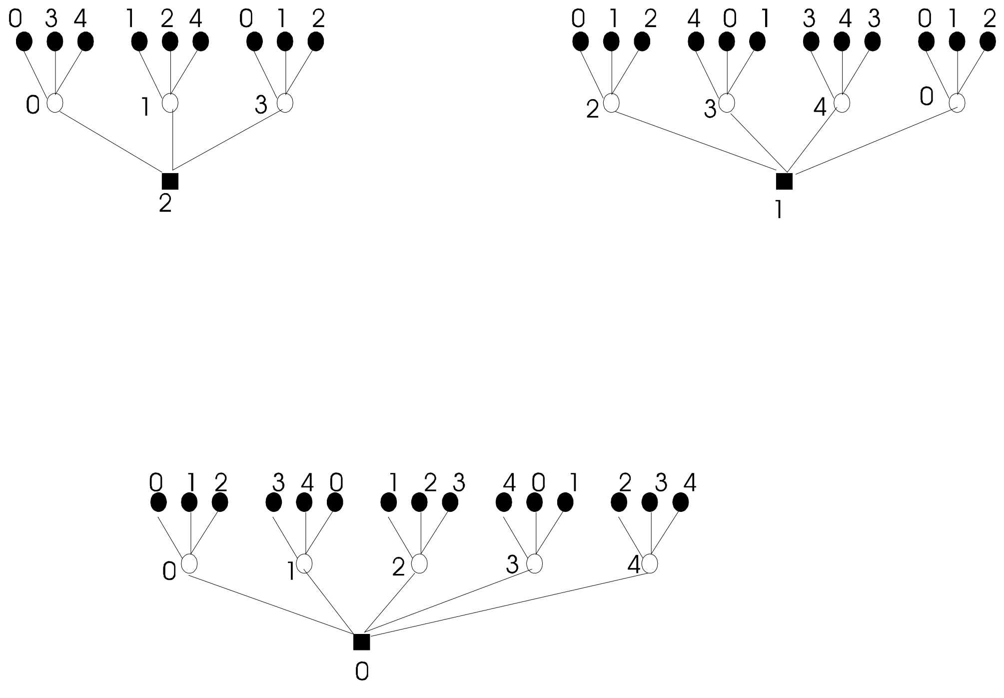

Figure 10

Thus $C(C(T))$ has at least two vertices. Let v be a crux vertex of $C(T)$. We now consider the possibilities for the number of 1-twigs of $C(T)$ at v (which must of course be at least two).

Case 1: there are exactly two 1-twigs of $C(T)$ at v.

Then the rooted forest H depicted in Figure 11 below (where the roots have opposite state, by Lemma 17) can be detached from T, at vertices v and w where w is a crux vertex of T.

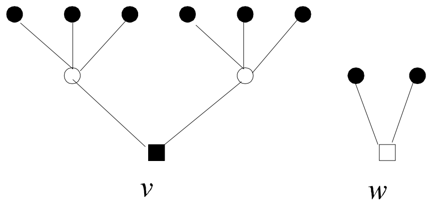

Figure 11: the rooted 5-forest H

Now H is a 5-forest, having the rooted 5-labelling defined by the list [0,0; 1,4,0,0; 2,3,4,1,2,3]. Let f be a gracious 5-labelling of the residual tree, S; it follows from Lemma 2 that, if $f(v) = f(w)$, then f extends to a gracious 5-labelling of T.

Thus we now assume $f(v) - f(w) = a$ where $a \neq 0$.

Now the non-root 'down' vertices of H have degree 1 while the non-root 'up' vertices have degree congruent to -1 (mod 5). It follows that in any rooted 5-labelling g of H, $g(v) = g(w)$. Thus, as in the proof of Theorem 3, we must extend H in order to finish the proof.

Let $\hat{H}$ be the rooted forest having H as a subforest and obtained by detaching at $x = v_\rho$ rather than at v (Figure 12), leaving the residual tree $\hat{S}$. We denote by $\hat{f}$ the restriction of f to $\hat{S}$.

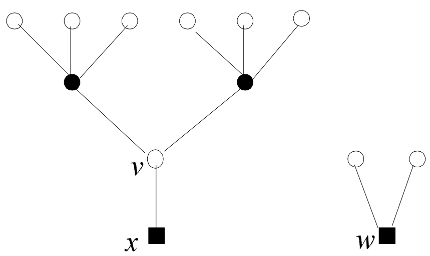

Figure 12: $\hat{H}$

As in the proof of Theorem 3, we may select one of x and w and assume without loss of generality that its label is 0.

For the remainder of our consideration of Case 1, then, we assume $f(v) = a \neq 0$, $f(w) = 0$.

The roots, x and w, have the same state; we may assume that this is 'down'. Suppose that $f(x) = a - b$ (so that the induced label on the edge xv is b). We now consider the five possible values of b. These are dealt with by the 5-labellings of $\hat{H}$ given by the lists below, each of which extends $\hat{f}$ to a gracious 5-labelling of T, contrary to assumption. (Note that the label on v and the induced label on xv 'migrate' as in the proof of Theorem 3.)

If $b = 0$, the list is [a,0; 0,2a,3a; a,4a; a,a,3a,0,2a,4a].

If $b = a$, the list is [0,0; 0,a,a; 3a,4a; 0,a,2a,2a,3a,4a].

If $b = 2a$, the list is [4a,0; 0,0,4a; 3a,4a; a,2a,3a,a,a,2a].

If $b = 3a$, the list is [3a,0; 0,a,4a; a,2a; a,3a,4a,0,2a,3a].

If $b = 4a$, the list is [2a,0; 0,0,2a; a,4a; a,2a,4a,a,3a,3a].

Case 2: there are exactly four 1-twigs of $C(T)$ at v.

Then a rooted tree may be detached at $x = v_\rho$, and $Q[1^3]$ at some tip w of $C(T)$ (which must have the same state as x), to form the rooted 5-forest shown in Figure 13.

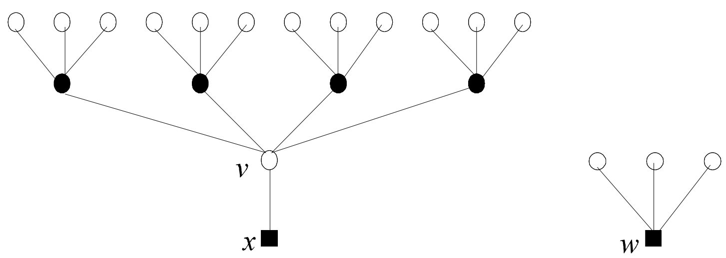

Figure 13

Two rooted 5-labellings of this forest are defined by the lists
[0,0; 0,0,1,2; 1,2,3,4; 3,4,0,1,2,3,4,0,1,2,3,4] and
[0,1; 2,1,1,2; 1,0,3,4; 2,4,0,0,2,3,4,0,1,3,3,4].

By Lemma 2, there is a rooted 5-labelling for each choice of root labels, and so T has a gracious 5-labelling, contrary to assumption.

Case 3: there are more than four 1-twigs of $C(T)$ at v.

Then the rooted tree of size 20 shown in Figure 10 may be detached at v, and has the rooted 5-labelling shown. By Lemma 3, T has a gracious 5-labelling, contrary to assumption.

Case 4: there are exactly three 1-twigs of $C(T)$ at v.

In view of Cases 1 – 3, we may assume that *every* crux vertex of $C(T)$ has exactly three 1-twigs of $C(T)$. We have already seen that there must be at least two such vertices, say v and w; since the crux vertices of T all have the same state (by Lemma

17), it follows that v and w have the same state. Thus, a 5-forest can be detached from T using detachment vertices $x = v_\rho$ and w (see Figure 14).

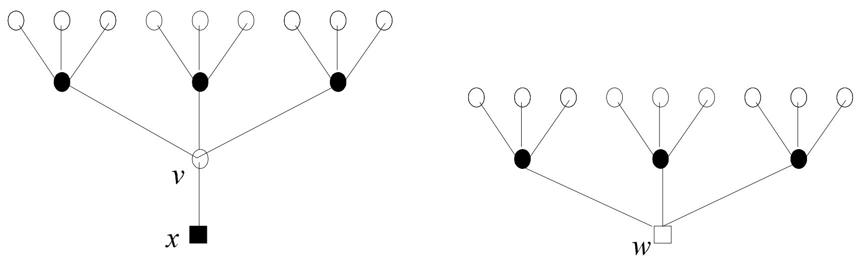

Figure 14

This forest has the rooted 5-labellings defined by the lists
[0,0; 0,3,4,0; 1,2,3,1,2,3,0,0,1,2,3,4; 4,0,1,2,3,4,4,1,2] and
[0,1; 4,3,4,0; 0,2,3,1,2,3,4,0,1,2,3,4; 0,0,1,2,3,4,1,1,2].

A final application of Lemma 2 shows again that T has a gracious 5-labelling, contrary to assumption. ■

9. Gracious and bigraceful labellings

In Section 1 we discussed the connection between the graceful tree conjecture and the problem of cyclic edge-decomposition of K_{2q+1} into $2q+1$ copies of an arbitrary tree of size q. There are other graph labellings that correspond to the existence of decompositions. For example (see [6]), a *bigraceful labelling* of a tree T of size q (with the vertex bipartition into partite sets D and U) is a function from V to $\{1, 2, \ldots, q\}$ that is injective when restricted to D and to U separately, and such that the edge labelling (induced as for the definition of gracious labelling) is a bijection from E to $\{0, 1, \ldots, q-1\}$. It is shown in [6] that the existence of a bigraceful labelling of T corresponds to the existence of an edge-decomposition of the complete bipartite graph $K_{q,q}$ into q copies of T.

Now every gracious labelling of T may be converted into a bigraceful labelling simply by adding 1 to each 'down' vertex label, and thus Conjecture 1, if true, shows that $K_{q,q}$ may be edge-decomposed into q copies of any tree of size q. It is reasonable to ask whether Conjecture 1 also corresponds to a decomposition result that does not arise from the 'bigraceful connection'; we offer the following theorem concerning the graph $K_q^{[2]}$, the multigraph obtained by doubling each edge of K_q (considered as a graph with vertex set Z_q).

Theorem 5. If Conjecture 1 is true, then $K_q^{[2]}$ has a cyclic edge-decomposition into q copies of any tree of size $q-1$.

Proof. Let T be any tree of size $q-1$. Assuming Conjecture 1, let g be a gracious labelling of T. For each $i \in Z_q$ let $\mu_i : V(T) \to Z_q$ be the bijection defined by

$$\mu_i(v) = \zeta_q \circ g(v) + i \quad (v \in V(T)).$$

Each μ_i gives rise to an injection $\tilde{\mu}_i : E(T) \to E(K_q^{[2]})$. If e is an edge of T such that $\tilde{g}(e) = p$, then the set $\{\tilde{\mu}_i(e) : i \in Z_q\}$ constitutes exactly the edges (i, j) of $K_q^{[2]}$ such that $j - i = \zeta_q(p)$. Thus the images of $E(T)$ under the $\tilde{\mu}_i$ constitute a cyclic edge-decomposition of $K_q^{[2]}$. ■

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