Pasch trades with a negative block

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Abstract

A Steiner triple system of order v, STS(v), may be called *equivalent* to another STS(v) if one can be converted to the other by a sequence of three simple operations involving Pasch trades with a single negative block. It is conjectured that any two STS(v)s on the same base set are equivalent in this sense. We prove that the equivalence class containing a given system S on a base set V contains all the systems that can be obtained from S by any sequence of well over one hundred distinct trades, and that this equivalence class contains all isomorphic copies of S on V. We also show that there are trades which cannot be effected by means of Pasch trades with a single negative block.

AMS Subject Classifications: primary 05B07.

Keywords: Pasch configuration, Steiner triple system, trade.

1 Introduction

This paper is concerned with the question of converting one specified Steiner triple system to another specified Steiner triple system of the same order and on the same base set by repeatedly applying a sequence of basic operations. Only three operations will be allowed: inflations, Pasch trades with negative blocks, and reductions. These are defined below. Two Steiner triple systems that can be converted, one to the other, by such a sequence will be said to be *equivalent*. P. J. Cameron [1] raised the question of whether any two Steiner triple systems of the same order on the same base set are equivalent. It was previously shown by M. T. Jacobson and P. Matthews [8] that the corresponding question for Latin squares has an affirmative answer. If a graph is formed with the Steiner triple systems of a given order on a given base set as the vertices, and vertices joined by an edge when the corresponding Steiner triple systems are equivalent, then Cameron's question is equivalent to asking if the graph is connected. We will call such a graph an *STS-graph*. For orders of Steiner triple systems up to and including 19, the STS-graphs are now known to be connected. In this paper we establish that any pair of isomorphic Steiner triple systems will lie in the same connected component of the graph, and that any pair of Steiner triple systems that can be converted, one to the other, by any sequence of well over a hundred specified trades will lie in the same component. At the very least, this suggests that constructing an example to disprove connectedness would be very difficult.

We start by recalling some basic terminology. A Steiner triple system of order v, STS(v), is an ordered pair $(V, \mathcal{B})$ where V is a v-element set (the *points*) and $\mathcal{B}$ is a set of 3-element subsets of V (the *blocks*), such that each 2-element subset of V appears in precisely one block. The necessary and sufficient condition for the existence of an STS(v) is that $v \equiv 1$ or 3 (mod 6) [11]; such values are said to be *admissible*. A partial Steiner triple system of order v, PSTS(v) is defined in the same way but with "precisely one" replaced by "at most one", and with the additional assumption that every point of V appears in some block, so that the order v is well defined. There is no restriction on v in a PSTS(v). We may denote a triple as a set $\{a, b, c\}$, or we may suppress the brackets and commas when no confusion is likely and write it as abc. We may also treat pairs in a similar fashion.

A *Pasch configuration* is a set of four triples on six distinct points having the form $\{xyz, abz, ayc, xbc\}$. If a Pasch configuration appears in an STS(v) or in a PSTS(v) then it may be replaced by the opposite Pasch configuration

$\{abc, xyc, xbz, ayz\}$ to give (respectively) another STS(v) or PSTS(v), the latter covering the same pairs as the original PSTS(v). This operation is described as a *Pasch trade*. The resulting system may or may not be isomorphic to the original.

More generally, if T_1 and T_2 are partial Steiner triple systems covering the same pairs, but without common blocks, then the pair $\mathcal{T} = \{T_1, T_2\}$ is called a *trade pair*, and T_1 and T_2 are called *tradeable configurations*. If an STS(v) or PSTS(v) contains a copy of T_1, then that copy may be replaced by a corresponding copy of T_2 to give (respectively) another STS(v) or PSTS(v), the latter covering the same pairs as the original PSTS(v). This operation is called a $\mathcal{T}$-trade. The set of points covered by T_1 and T_2 is called the *foundation* of the trade, and the number of blocks in each T_i ($i = 1, 2$) is called the *volume* of the trade. Thus a Pasch trade has foundation of cardinality 6 and volume 4. A comprehensive listing of trades of volume up to and including 10 is given by A. D. Forbes in his Ph.D. thesis [2]. Those of volume up to and including 9 are also given in Table 3.4 of [3]. Each tradeable configuration in these tables appears with an isomorphism class number in the column labelled "Config"; the meaning of the other entries should be clear.

A trade of particular interest to us is known as an n-cycle trade. The trade pair has the form given by $T_1 = \{ax_1x_2, bx_2x_3, ax_3x_4, bx_4x_5, \ldots bx_nx_1\}$, $T_2 = \{bx_1x_2, ax_2x_3, bx_3x_4, ax_4x_5, \ldots ax_nx_1\}$ (where all the points are distinct), so that T_1 and T_2 are n-cycles. This trade has foundation $\{a, b, x_1, x_2, \ldots, x_n\}$ and volume n. Note that n is necessarily even and that $n \geq 4$. A Pasch trade is simply a 4-cycle trade. Every STS(v) contains n-cycles for some values of n; simply choose any block abc (which is then discarded) and then any other block containing a, say ax_1x_2, then take the block containing the pair bx_2, say bx_2x_3, then the block containing the pair ax_3, say ax_3x_4, and so on until finally, for some n, the block bx_nx_1 is encountered. The resulting cycle of length n is said to be a cycle on the pair $\{a, b\}$. Thus every STS(v) may, for appropriate values of n, be subjected to n-cycle trades.

We next describe the three basic operations mentioned at the start of this section and as applied to Steiner triple systems. In order to do this we make the following definition.

Definition An *improper Steiner triple system of order* $v \equiv 1$ or $3 \pmod 6$, denoted by ISTS(v), is an ordered pair $(V, \mathcal{B})$, where V is a v-element set (the points) and $\mathcal{B}$ is a set of 3-element subsets of V (the blocks), such that exactly three of the 2-element subsets of V each appear in precisely two blocks and the remaining 2-element subsets of V each appear in precisely one block. Note that $\mathcal{B}$ is required to be a set rather than a multiset (so that repeated blocks are not allowed) and that $|\mathcal{B}| = v(v-1)/6 + 1$. In Lemma 1.1 below, we show that the three exceptional pairs necessarily have the form ab, bc, ca. The triple abc, which may or may not lie in $\mathcal{B}$, will be referred to as the *negative block*. If the negative block abc lies in $\mathcal{B}$, then $(V, \mathcal{B} \setminus \{abc\})$ forms an STS(v) which we shall call the *reduction* of the ISTS(v). Conversely, if $(V, \mathcal{B}')$ is an STS(v) and abc is any triple of points which is not in $\mathcal{B}'$, then $(V, \mathcal{B}' \cup \{abc\})$ forms an ISTS(v) with negative block abc; such an ISTS(v) will be called an *inflation* of the original STS(v).

Suppose that $(V, \mathcal{B})$ is an ISTS(v) with negative block abc. Denote by B_1, B_2, B_3 blocks of $\mathcal{B}$ which are distinct from abc and which contain respectively the pairs bc, ac, ab, so that $B_1 = \alpha bc, B_2 = a\beta c, B_3 = ab\gamma$ for some α, β, γ distinct from a, b, c. Form $\mathcal{B}'$ by deleting B_1, B_2, B_3 from $\mathcal{B}$ and replacing them with the triples $B_1' = a\beta\gamma$, $B_2' = \alpha b\gamma$, $B_3' = \alpha\beta c$. Then $(V, \mathcal{B}')$ is an ISTS(v) with negative block $\alpha\beta\gamma$. In effect, a Pasch trade is implemented from the triples B_1, B_2, B_3, $\alpha\beta\gamma$ to the triples B_1', B_2', B_3', abc.

The operation described above will be called a *Pasch trade with a negative block*, or *PN-trade* for short. We will say that two STS(v)s, two ISTS(v)s, or an STS(v) and an ISTS(v), on the same base set V, are *equivalent* if one may be obtained from the other by some finite sequence of inflations, PN-trades and reductions.

Conjecture 1.1 Any two STS(v)s on the same base set are equivalent.

Before addressing this conjecture we establish the result used in the definition of the negative block of an ISTS(v).

Lemma 1.1 *In an ISTS(v), the three exceptional pairs, that is those appearing in precisely two blocks, are of the form ab, bc, ca.*

Proof. Suppose that ab is an exceptional pair. The number of points other than a and b which must appear in blocks with a is $v-2$, which is odd. There are two blocks containing the pair ab, and every other block containing a must contain two points other than a and b. So by a parity argument, at least one of the other points, say c, must appear more than once with a, and hence exactly twice with a. Thus there is an exceptional pair ac where $c \neq a, b$. Similarly there must be an exceptional pair bd where $d \neq a, b$.

Now suppose that $c \neq d$. There must be an exceptional pair cf where $f \neq a, c$. Then this pair is distinct from ab, ac and bd and so there are at least four exceptional pairs, a contradiction. Hence $c = d$. $\square$

2 Trading n-cycles

We would like to prove or disprove Conjecture 1.1. As an initial step we will prove the following theorem.

Theorem 2.1 *Suppose that S is an STS(v) containing the n-cycle $T_1 = \{ax_1x_2, bx_2x_3, ax_3x_4, bx_4x_5, \dots bx_nx_1\}$. Then S is equivalent to S', where S' is formed from S by applying the n-cycle trade $\{T_1, T_2\}$ where $T_2 = \{bx_1x_2, ax_2x_3, bx_3x_4, ax_4x_5, \dots ax_nx_1\}$.*

Proof. As noted above, $n \geq 4$ and n is even. Arithmetic will be performed on the subscripts of x_i modulo n. We show how to obtain a sequence of equivalent systems, starting with S and ending with S' by applying the following steps. The initial step 0 inflates S and the final step $\frac{n}{2} - 1$ reduces an ISTS(v) to S'. We specify the general step i for $i = 1, 2, \dots, \frac{n}{2} - 1$, dealing separately with the cases i odd and i even. Note that ax_ix_{i+1} is a block of T_1 if i is odd, and bx_ix_{i+1} is a block of T_1 if i is even. To assist the reader we also give the specific cases $i = 1$ and $i = 2$.

Step 0. Inflate S using the triple ax_2x_3 to obtain an ISTS(v) which we denote by S_0.

Step 1. Apply a PN-trade to S_0 (which has negative block ax_2x_3) to get S_1 by deleting the triples $ax_1x_2, ax_3x_4, bx_2x_3$ and replacing these by the triples $bx_1x_2, bx_3x_4, ax_1x_4$. Then S_1 has negative block bx_1x_4. If $n = 4$ then $bx_1x_4 = bx_nx_1$ and reduction gives the required S'; otherwise proceed to step 2.

Step 2. (Assuming $n > 4$.) Apply a PN-trade to S_1 (which has negative block bx_1x_4) to get S_2 by deleting the triples $bx_nx_1, bx_4x_5, ax_1x_4$ and replacing these by the triples $ax_nx_1, ax_4x_5, bx_nx_5$. Then S_2 has negative block ax_nx_5. If $n = 6$ then $ax_nx_5 = ax_nx_{n-1}$ and reduction gives the required S'; otherwise proceed to step 3.

Step i. (Assuming $n > 2i$ and that i is odd.) Apply a PN-trade to S_{i-1} (which has negative block $ax_{3-i}x_{2+i}$) to get S_i by deleting the triples $ax_{2-i}x_{3-i}$, $ax_{2+i}x_{3+i}, bx_{3-i}x_{2+i}$ and replacing these by the triples $bx_{2-i}x_{3-i}, bx_{2+i}x_{3+i}$, $ax_{2-i}x_{3+i}$. Then S_i has negative block $bx_{2-i}x_{3+i}$. If $i = \frac{n}{2}-1$ then $bx_{2-i}x_{3+i} = bx_{2-i}x_{n+1-i}$ and reduction gives the required S'; otherwise proceed to step $i+1$.

Step i. (Assuming $n > 2i$ and that i is even.) Apply a PN-trade to S_{i-1} (which has negative block $bx_{3-i}x_{2+i}$) to get S_i by deleting the triples $bx_{2-i}x_{3-i}, bx_{2+i}x_{3+i}, ax_{3-i}x_{2+i}$ and replacing these by the triples $ax_{2-i}x_{3-i}$, $ax_{2+i}x_{3+i}, bx_{2-i}x_{3+i}$. Then S_i has negative block $ax_{2-i}x_{3+i}$. If $i = \frac{n}{2} - 1$ then $ax_{2-i}x_{3+i} = ax_{2-i}x_{n+1-i}$ and reduction gives the required S'; otherwise proceed to step $i+1$. □

Each n-cycle in an STS(v) may be treated successively and independently, so the following corollary is immediate.

Corollary 2.1.1 *Suppose that S is any STS(v) and that S^* is obtained from S by applying any sequence of n-cycle trades (possibly with differing values of n). Then S is equivalent to S^*.*

The next corollary is also a simple consequence of Theorem 2.1.

Corollary 2.1.2 *Suppose that S is any STS(v) and that S^* is obtained from S by applying the transposition $(a\ b)$, where $a, b \in V$ are any two distinct points. Then S is equivalent to S^*.*

Proof. The pair $\{a, b\}$ lies in some triple abc in $\mathcal{B}$, and this triple is invariant under $(a\ b)$. The other triples containing a or b partition into some number, say k, of n-cycles (possibly with differing values of n). Treating each such n-cycle in turn, a sequence $S_0, S_1, \ldots, S_k$ of equivalent STS(v)s is thereby obtained where $S_0 = S$ and $S_k = S^*$. Thus S is equivalent to S^*. □

An interesting consequence of the previous corollary is the following.

Corollary 2.1.3 *Suppose that S is any STS(v) and that S^* is an isomorphic copy of S on the same base set V. Then S is equivalent to S^*.*

Proof. The system S^* may be obtained from S by applying some sequence of transpositions. By the previous corollary, we obtain a sequence of equivalent STS(v)s starting with S and ending with S^*, so that S and S^* are equivalent. □

Remark Theorem 2.1 and its corollaries establish that for each of $v = 3, 7, 9, 13, 15$ and 19, all STS(v)s are equivalent. This is easy to prove for $v = 3, 7, 9$ and 13. In the case $v = 15$ there are 80 nonisomorphic STS(15)s. It was shown by P. B. Gibbons in [5] that 79 of these may be obtained from one initial system by suitable sequences of 4-cycle trades, and in [6] that the remaining system may be obtained by a 6-cycle trade. More recently, it was shown in [10] that the 11 084 874 829 non-isomorphic STS(19)s may also be obtained from one initial system by suitable sequences of n-cycle trades.

The situation for $v \geq 25$ is certainly more complicated. It is known that there exist so-called *perfect* STS(v)s in which all the cycles are of the greatest possible length $v - 3$. Trading a $(v-3)$-cycle on the pair $\{a, b\}$ is equivalent to transposing the two points a and b. Consequently every n-cycle trade on a perfect STS(v) (necessarily with $n = v - 3$) leads to an isomorphic system. This raises the question of whether or not perfect systems are only equivalent to isomorphic copies of themselves. Only a finite number of perfect Steiner triple systems are known, see [7, 4]. It was shown by P. Kaski [9] that, apart from the trivially perfect STS(7) and STS(9), there are no perfect STS(v)s for $v \leq 21$. However, a perfect STS(25) is known; it is #3 of the three systems found by V. D. Tonchev [12] that are invariant under the group $\mathbb{Z}_5 \times \mathbb{Z}_5$. We have tested this system and find that it is indeed equivalent to a non-perfect STS(25). In fact this perfect system contains other tradeable configurations apart from 22-cycles, and in the next section we turn our attention to a comprehensive set of small trades. We will show that most, but not all, of these can be effected by a suitable sequence of our three basic operations. Using these results, we will show how to convert the perfect STS(25) into a non-perfect STS(25).

3 Small trades

Theorem 2.1 asserts that n-cycle trades can be effected by means of inflations, PN-trades and reductions. Moreover, pairs of points not contained in the n-cycle played no role in the proof. Thus in a sense, which we will now make precise, the PN-trades used in the proof are PN-trades on the PSTS($n + 2$) defined by the n-cycle.

Definition An *improper partial Steiner triple system of order* v, denoted by IPSTS(v), is an ordered triple $(V, \mathcal{B}, N)$ with the following properties. V is a

v-element set (the *points*), $\mathcal{B}$ is a set of 3-element subsets of V (the *blocks*), and N (the *negative block*) is a 3-element subset of V. The three pairs of points from N each appear in either one or two blocks of $\mathcal{B}$, the remaining 2-element subsets of V each appear in at most one block of $\mathcal{B}$, and every point of V appears in at least one block of $\mathcal{B}$. Note that $\mathcal{B}$ is required to be a set rather than a multiset (so that repeated blocks are not allowed). The triple N may or may not lie in $\mathcal{B}$; if it does lie in $\mathcal{B}$, then $(V, \mathcal{B} \setminus \{N\})$ forms a PSTS(v) which we shall call the *reduction* of the IPSTS(v). Conversely, if $(V, \mathcal{B}')$ is a PSTS(v) and the pairs of points ab, bc and ca lie in distinct blocks of $\mathcal{B}'$, then with $N = abc$, $(V, \mathcal{B}' \cup \{N\}, N)$ forms an IPSTS(v) with negative block N; such an IPSTS(v) will be called an *inflation* of the original PSTS(v).

Comparing this with our earlier definition of an ISTS(v), it will be seen that in an IPSTS(v) there is no restriction on v and that we now make the explicit assumption about the form of the three exceptional pairs, namely that they cover just three points. In the definition of an inflation, we make explicit that the three pairs of exceptional points must be covered by the blocks of the ISTS(v). It should also be clear that if an STS(v) is regarded as a PSTS(v), then the two definitions of an inflation are effectively identical.

We can now define a Pasch trade with a negative block (PN-trade) on an IPSTS(v) exactly as we did previously on an ISTS(v). As before, we will say that two PSTS(v)s, two IPSTS(v)s, or an PSTS(v) and an IPSTS(v), on the same base set V, are *equivalent* if one may be obtained from the other by some finite sequence of inflations, PN-trades and reductions. We will now apply this definition to trade pairs $\{T_1, T_2\}$ with the aim of transforming the PSTS(v) represented by T_1 to the PSTS(v) represented by T_2 by means of a suitable sequence of the three basic operations. When such a transformation can be effected, then T_1 and T_2 are equivalent in the sense just defined, and any Steiner triple system $(V, \mathcal{B})$ containing a copy of T_1 is equivalent to one containing a corresponding copy of T_2, with the remaining blocks of $\mathcal{B} \setminus T_1$ unaltered.

Forbes' table [2] gives all the 124 pairwise nonisomorphic trade sets of volume up to and including 10. The largest foundation amongst these has 14 points. Most of these trade sets are trade pairs as we defined them above. In a few cases a tradeable configuration appears in more than one trade pair and for this reason, six of the trade sets contain three tradeable configurations and one contains four, while the remaining 117 contain just two. If it is possible to transform T_1 to both T_2 and T_3 by means of inflations, PN-trades and reductions, then it is clearly possible to transform T_2 to T_3 by a suitable sequence of the same three operations. We have therefore examined 132=117+12+3 trade pairs $\{T_1, T_2\}$ to see if they can be effected by means of inflations, PN-trades and reductions. In some cases, an intermediate step converts T_1 to a PSTS(v), say T_1', which has blocks in common with the targeted T_2. In such cases the common blocks were removed from both T_1' and T_2, thereby resulting in a smaller trade pair (by volume).

The examination was undertaken by a computer program. In each case T_1 was inflated at random, the PN-trades were selected at random, and after each PN-trade, the resulting IPSTS(v) was examined for a possible reduction. In all but one of the 132 cases T_1 was quickly transformed to the targeted T_2. The exceptional case is #68 in Forbes' listing which has volume 10 and foundation cardinality 10. This exceptional trade pair is isomorphic to one given by $T_1 = \{013, 124, 235, 346, 457, 568, 679, 780, 891, 902\}$ and $T_2 = \{023, 134, 245, 356, 467, 578, 689, 790, 801, 912\}$, the isomorphism being carried by the permutation (2 3)(4 9 7 8 6 5). It will be seen that this T_1 and T_2 may be obtained by developing, respectively, the starters 013 and 023 cyclically modulo 10.

In the case of this exceptional trade it is not difficult to work through by hand all the possibilities for transforming T_1 to T_2 by a sequence of inflations, PN-trades and reductions. Without loss of generality, one of the pairs covered by the negative block in the initial inflation may be taken as 01, 03 or 13, and in each of these cases there is a limited choice for the other two pairs. After the initial inflation, however it is chosen, and the first PN-trade, there is only ever one way of carrying out each subsequent PN-trade that does not reverse the previous step. This exhaustive analysis establishes that the PSTS(10) represented by T_1 is not equivalent to the PSTS(10) represented by T_2. Note however that this is not quite the same as saying that, as part of an STS(v), T_1 cannot be traded for T_2 by a suitable sequence of the three basic operations, since it may be possible to use blocks of the system that are not present in T_1 or T_2 in the sequence.

Although trade #68 is exceptional amongst the trades of volume up to 10, it generalizes to larger pairs of PSTS(v)s that are also not equivalent. To show this, let $\langle a, b, c\rangle_v$ denote the orbit of distinct blocks on the point set $\mathbb{Z}_v$ obtained from the starter $\{a, b, c\}$ by applying the mappings $\phi_i : x \mapsto x + i$ $(0 \le i \le v-1)$. Since we are only interested in partial Steiner triple systems we will only consider *suitable* orbits $\langle a, b, c\rangle_v$, that is to say orbits where a, b, c

are distinct points and no pair of points is repeated amongst the distinct blocks of the orbit. Thus trade #68 can be written as $\{\langle 0,1,3\rangle_{10}, \langle 0,2,3\rangle_{10}\}$. It is easy to see that whenever $\langle 0,a,b\rangle_v$ is a suitable set of v distinct blocks, then $\mathcal{T}_v(a,b) = \{\langle 0,a,b\rangle_v, \langle 0,b-a,b\rangle_v\}$ is a trade pair. We will now show that in most cases the two PSTS(v)s defined by such a trade pair are not equivalent.

Lemma 3.1 *Suppose that $T_1 = \langle 0,a,b\rangle_v$ is a suitable orbit of v distinct blocks and that $T_2 = \langle 0,b-a,b\rangle_v$, so that $\{T_1,T_2\}$ is a trade pair. Suppose also that none of the following relationships hold in $\mathbb{Z}_v$:*

$$3a = 0, 3b = 0, b = -2a, b = 3a, a = -2b, a = 3b, 2b = 3a, 2a = 3b, 3a = 3b.$$

Then the PSTS(v)s represented by T_1 and T_2 are not equivalent.

Proof. Note first that $2a \neq 0$ because suitability requires that if $a = -a$ then the blocks $\{0,a,b\}$ and $\{-a,0,b-a\}$ must be identical and so there are not v distinct blocks (indeed, suitability would also require $b = b-a$, giving $a = 0$). It is also the case that $a \neq 2b$ because if $a = 2b$ then we have blocks $\{0,b,2b\}$ and $\{-b,0,b\}$ which is again precluded for the same reasons. Likewise, $a \neq -b$ because $a = -b$ leads to blocks $\{0,b,-b\}$ and $\{-b,0,-2b\}$. By symmetry, we also have $2b \neq 0$ and $b \neq 2a$.

Now consider an inflation of T_1. Three pairs of points that appear in T_1 in distinct blocks are required. Without loss of generality, one of these can be taken to be $0a$. The remaining two pairs must be of the form $0x$ and ax, where $x \neq 0,a,b$. By considering the blocks of T_1 that contain 0, we see that x is one of $-a, b-a, -b, a-b$; similarly by considering blocks of T_1 that contain a, we see that x is one of $2a, a+b, a-b, 2a-b$. The conditions of the lemma ensure that $\{-a,b-a,-b\} \cap \{2a,a+b,2a-b\} = \emptyset$, so the only possibility is $x = a-b$.

If T_1 is inflated using $N = \{0,a,a-b\}$ as the negative block then the only possible PN-trade replaces the blocks $\{0,a,b\}, \{0,a-b,-b\}, \{a,a-b,2a-b\}$ with the blocks $\{b,-b,0\}, \{b,2a-b,a\}, \{-b,2a-b,a-b\}$. The new negative block is $\{b,-b,2a-b\}$. In order to carry out any further PN-trades or a reduction, it is necessary to determine which other blocks, if any, contain the pairs $\{b,-b\}, \{-b,2a-b\}, \{b,2a-b\}$. These pairs have differences $2b, 2a, 2a-2b$ respectively, so each of these pairs occur in another block if and only if the difference is one of $\pm a, \pm b, \pm(b-a)$. But again the conditions of the lemma ensure that this cannot happen. Thus the only blocks containing the pairs

$\{b,-b\},\{-b,2a-b\},\{b,2a-b\}$ are the ones generated by the initial PN-trade. Consequently the only possibility for a further PN-trade is to reverse this initial PN-trade, and the reduction returns to T_1. $\square$

Lemma 3.1 provides sufficient conditions to ensure that $\langle 0,a,b\rangle_v$ is not equivalent to $\langle 0,b-a,b\rangle_v$. However, Forbes' trade #68 shows that the conditions are not necessary since in that case $a=1$ and $b=3=3a$.

Finally in this section we return to the perfect STS(25) mentioned in the previous section. Although this system is perfect, it does contain small trades. The system can be represented on the point set $\mathbb{Z}_5\times\mathbb{Z}_5$. It has 100 blocks and these can be obtained from the following four starter blocks, where a pair such as $(2,3)$ is recorded as 23:

$$\{00,01,10\},\{00,02,21\},\{00,11,23\},\{00,13,33\}.$$

The 100 blocks of the design are formed by applying the 25 mappings $\phi_{i,j}:(x,y)\mapsto(x+i,y+j)$ $(0\le i,j\le 4)$ to these starter blocks. One tradeable configuration in this system is

$$\begin{aligned} T_1 &= \{\{00,01,10\},\{00,20,42\},\{00,22,30\},\{01,41,42\},\{02,10,30\},\\ &\qquad \{02,22,44\},\{10,41,44\},\{20,22,41\}\}. \end{aligned}$$

This may be traded with

$$\begin{aligned} T_2 &= \{\{00,01,42\},\{00,10,30\},\{00,20,22\},\{01,10,41\},\{02,10,44\},\\ &\qquad \{02,22,30\},\{20,41,42\},\{22,41,44\}\}. \end{aligned}$$

Forbes' trade #16 is isomorphic to this trade pair $\{T_1,T_2\}$ under the mapping

$$\begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 02 & 30 & 10 & 22 & 44 & 00 & 41 & 01 & 20 & 42 \end{pmatrix}.$$

It is easy to show by hand calculation that T_1 may be transformed to T_2 by an inflation, three PN-trades, and a reduction. The resulting STS(25) then contains the 6-cycle

$$\{\{20,22,00\},\{11,00,23\},\{20,23,44\},\{11,44,24\},\{20,24,34\},\{11,34,22\}\},$$

and therefore is not perfect. Thus our three basic operations can transform the perfect STS(25) into a nonisomorphic system.

4 Concluding remarks

An obvious variation of the problem is to allow two or more negative blocks in an ISTS(v) or IPSTS(v), with appropriate modifications to their definitions. If the number of negative blocks is restricted to two, then it is possible to effect Forbes' trade #68 by inflating $\langle 0,1,3\rangle_{10}$ with blocks 012 and 345, then performing the ten PN-trades shown in Table 1, and finally reducing to $\langle 0,2,3\rangle_{10}$. (In Table 1, an entry $[X,Y]$ means that the negative block X is replaced by the negative block Y, so that the first PN-trade replaces the negative block 345 and the blocks $346, 352, 457$ by the negative block 267 and the blocks $263, 275, 674$.)

Trade	$[X,Y]$	Trade	$[X,Y]$
1	$[345,267]$	6	$[027,358]$
2	$[012,349]$	7	$[358,026]$
3	$[267,359]$	8	$[126,345]$
4	$[349,126]$	9	$[026,389]$
5	$[359,027]$	10	$[389,012]$

Table 1: PN-trades with two negative blocks for $\mathcal{T}_{10}(1,3)$.

However, we believe that permitting two negative blocks is not generally sufficient to effect all trades of the form $\mathcal{T}_v(a,b)$. Whether some fixed number of negative blocks would suffice in all such cases is an open question.

Finally, we review the current state of Conjecture 1.1 (allowing only one negative block) in the light of our results. Firstly, given a Steiner triple system containing a small tradeable configuration, applying the trade generally (although not always) results in a nonisomorphic system. Secondly, trying to find a Steiner triple system of large order that lacks most of the trades listed by Forbes appears to be ferociously difficult, and may even be impossible. While these observations do not preclude the possibility that the STS-graph is disconnected, they do make this seem very unlikely.

Acknowledgements We thank Dr. A. D. Forbes for supplying a copy of his table and the referees for helpful comments.

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