Designs having the parameters of projective and affine spaces

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Abstract

Two constructions are described that yield an improved lower bound for the number of 2-designs with the parameters of $PG_d(n,q)$, and a lower bound for the number of resolved 2-designs with the parameters of $AG_d(n,q)$.

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1 Introduction

Various lower bounds for the numbers of 2-designs with the parameters of $PG_{n-1}(n, q)$ and $AG_{n-1}(n, q)$ have been established in the papers [3, 5, 6, 8, 7]. In [4] these results were extended to $PG_d(n, q)$ and, more recently in [1], to $AG_d(n, q)$, for $2 \leq d \leq n-1$. In this paper we describe two constructions that, for sufficiently large n, facilitate an improved lower bound in the case of $PG_d(n, q)$, and a lower bound for the number of resolved 2-designs with the parameters of $AG_d(n, q)$.

Throughout the paper, q is taken to be a prime power. For $n \geq d \geq 1$, $\left[{n \atop d}\right]_q$ denotes the product

$$\prod_{i=1}^{d} \frac{q^{n-d+i}-1}{q^i-1}.$$

For $n > 0$, $\left[{n \atop 0}\right]_q$ is defined to take the value 1.

A *2-design* $2-(v, k, \lambda)$ is an ordered pair $(V, \mathcal{B})$ where V is a v-set (the *points*) and $\mathcal{B}$ is a collection of k-subsets of V (the *blocks*) such that every unordered pair of distinct points occurs in precisely λ blocks. A 2-design may have repeated blocks; it is said to be *simple* if it has no repeated blocks. It is said to be *resolvable* if the blocks may be partitioned (resolved) into *parallel classes* each of which consists of a set of disjoint blocks that collectively contain all the points of the design. Note that a resolvable 2-design may have more than one resolution into parallel classes. A resolvable 2-design *with a given resolution* is called a *resolved* 2-design. We will say that a resolved 2-design is *semi-simple* if no two of the parallel classes have all their blocks identical, that is to say no parallel classes are repeated. Plainly, a simple resolved 2-design is semi-simple.

A *transversal design* $TD(k, n)$ is an ordered triple $(V, \mathcal{G}, \mathcal{B})$, where V is a kn-set (the *points*), $\mathcal{G}$ is a partition of V into k disjoint n-subsets (the *groups*), and $\mathcal{B}$ is a collection of k-subsets of V (the *blocks*) such that every unordered pair of points from different groups occurs in precisely one block, and no block contains more than one point from any group. Such a design is said to be *resolvable* if the blocks may be partitioned (resolved) into n *parallel classes* each of which consists of a set of n disjoint blocks that collectively contain all the points of the design. Note that a $TD(k, n)$ design may have more than one resolution into parallel classes. A resolvable $TD(k, n)$ *with a given resolution* may be called a *resolved* transversal design, $RTD(k, n)$.

The set of d-flats of the affine geometry $AG(n, q)$ is denoted by $AG_d(n, q)$

and forms the block set of a $2-(q^n, q^d, {n-1 \brack d-1}_q)$ design. A design having these parameters will be called an $A(d,n,q)$ design. Thus an $A(d,n,q)$ design has q^n points, block length q^d, every pair of points appears ${n-1 \brack d-1}_q$ times, every point appears ${n \brack d}_q$ times, and the number of blocks is $q^{n-d}{n \brack d}_q$. A resolved $A(d,n,q)$ design, that is a resolvable design with a given resolution, may be called an $RA(d,n,q)$ design. If the design is semi-simple it may be called an $S^*RA(d,n,q)$ design, if it is also simple it may be called an $SRA(d,n,q)$ design. In the case of an $RA(d,n,q)$ design, the number of parallel classes is ${n \brack d}_q$ and each parallel class has q^{n-d} blocks.

The set of d-flats of the projective geometry $PG(n,q)$ is denoted by $PG_d(n,q)$ and forms the block set of a $2-({n+1 \brack 1}_q, {d+1 \brack 1}_q, {n-1 \brack d-1}_q)$ design. A design having these parameters will be called a $P(d,n,q)$ design. Thus a $P(d,n,q)$ design has ${n+1 \brack 1}_q$ points, block length ${d+1 \brack 1}_q$, every pair of points appears ${n-1 \brack d-1}_q$ times, every point appears ${n \brack d}_q$ times, and the number of blocks is ${n+1 \brack d+1}_q$. A simple $P(d,n,q)$ design may be called an $SP(d,n,q)$ design. Generally speaking, such designs are not resolvable because a necessary condition for resolvability is that ${d+1 \brack 1}_q$ divides ${n+1 \brack 1}_q$, that is $(q^{d+1}-1) \mid (q^{n+1}-1)$.

Two (unresolved) designs with the same parameters, on the same set of points and, in the case of transversal designs, with the same groups, are *distinct* if they have different blocks. When speaking of the *number of designs* with a given set of parameters, we mean the number of distinct designs. Two resolved designs are *distinct* if they have different blocks, or if they have identical blocks but different parallel classes. When discussing resolvable designs it is necessary to distinguish the number of designs from the number of resolutions of these designs. The latter quantity is called the *number of resolved designs*. We will try to be careful about this point. Two designs (or two resolved designs) are *isomorphic* if there is a permutation of the points that maps the blocks of one design to those of the other (and, in the case of resolved designs, preserves the parallel classes). If a given 2-design D is on v points then the largest possible isomorphism class of designs isomorphic to D on the same point set has cardinality $v!$, and this occurs when D has the trivial automorphism group. Thus the number of isomorphism classes in such cases (colloquially, the number of nonisomorphic designs) may be estimated by dividing the number of designs by $v!$. The same argument applies to isomorphism classes of resolved 2-designs.

The constructions presented below have some similarities to constructions presented in [10, 11, 4, 1]. In [4] and [1], new designs are created from

$PG_d(n,q)$ and $AG_d(n,q)$ by rearranging parts of these designs. In Constructions 2.1 and 2.2, we start with arbitrary constituent parts $P(d',n',q)$ and $RA(d',n',q)$ and combine these to form new designs $P(d,n,q)$ and $RA(d,n,q)$. An advantage of the former approach is that it permits a good estimation of the size of isomorphism classes of the designs produced. However, the latter approach permits a wider range of basic ingredients and allows the constructions to be used recursively. It will be shown in Section 3 that our bound for the number of nonisomorphic $P(d,n,q)$ designs is better than that of [4] when $n \geq 7$. We also obtain a considerably larger bound for the number of nonisomorphic *resolved* $A(d,n,q)$ designs than that given in [1] for *resolvable* $A(d,n,q)$ designs when $q \neq 2$ and $d \neq n-1$. The paper [1] also gives a construction for non-resolvable $A(d,n,q)$ designs, and a resulting bound on the number of isomorphism classes. In a future paper we intend to show how Construction 2.2 can be generalized to deal with non-resolvable $A(d,n,q)$ designs and that this leads to an improved bound for these designs.

2 Constructions

Construction 2.1

(A recursive construction for a $P(d,n,q)$ design)

Here $n > d > 1$ and the ingredients required are as follows:

(a) a $P(d-1,n-1,q)$, $\mathcal{D}$, on a point set V_1 of cardinality $\left[{n \atop 1}\right]_q$,

(b) a $P(d,n-1,q)$, $\mathcal{E}$, on the same point set V_1,

(c) an $RA(d,n,q)$, $\mathcal{F}$, on a disjoint point set V_2 of cardinality q^n.

The resulting $P(d,n,q)$ is realized on the point set $V = V_1 \cup V_2$ of cardinality $\left[{n \atop 1}\right]_q + q^n = \left[{n+1 \atop 1}\right]_q$.

Note first that the number of blocks of $\mathcal{D}$ is the same as the number of parallel classes of $\mathcal{F}$, namely $\left[{n \atop d}\right]_q$. To each parallel class $\mathcal{P}$ of $\mathcal{F}$, associate a distinct block D of $\mathcal{D}$ by appending a copy of D to each of the blocks of $\mathcal{P}$. Denote the resulting set of blocks by $\mathcal{B}_1$. Allowing possible multiplicity, $|\mathcal{B}_1| = q^{n-d}\left[{n \atop d}\right]_q$. Each block $B \in \mathcal{B}_1$ has length $\left[{d \atop 1}\right]_q + q^d = \left[{d+1 \atop 1}\right]_q$, and contains no repeated points.

Let $\mathcal{B}_2$ denote the set of blocks of $\mathcal{E}$. Then $\mathcal{B}_1 \cap \mathcal{B}_2 = \emptyset$ and each block $B \in \mathcal{B}_2$ has length ${d+1 \brack 1}_q$. Put $\mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2$, so that $|\mathcal{B}| = q^{n-d}{n \brack d}_q + {n \brack d+1}_q = {n+1 \brack d+1}_q$, the number of blocks in a $P(d,n,q)$ design.

To prove that the blocks of $\mathcal{B}$ form a $P(d,n,q)$ design on the point set V, it remains only to show that every pair of distinct points $x, y \in V$ appears ${n-1 \brack d-1}_q$ times. This is certainly the case if $x, y \in V_2$ since this pair of points appears in ${n-1 \brack d-1}_q$ blocks of $\mathcal{F}$. If $x \in V_1$ and $y \in V_2$, then the pair x, y appears in a block $B \in \mathcal{B}_1$ whenever B is formed from a block of $\mathcal{D}$ containing x appended to a block of $\mathcal{E}$ containing y, and this occurs once for every block of $\mathcal{D}$ containing x, and hence ${n-1 \brack d-1}_q$ times in the blocks of $\mathcal{B}_1$. Finally, if $x, y \in V_1$ then the pair x, y appears ${n-2 \brack d-2}_q$ times in the blocks of $\mathcal{D}$ and hence $q^{n-d}{n-2 \brack d-2}_q$ times in the blocks of $\mathcal{B}_1$; but the same pair also appears ${n-2 \brack d-1}_q$ times in the blocks of $\mathcal{B}_2$. Hence, overall, the pair $x, y \in V_1$ appears in $q^{n-d}{n-2 \brack d-2}_q + {n-2 \brack d-1}_q = {n-1 \brack d-1}_q$ blocks of $\mathcal{B}$. $\square$

Example 2.1 below illustrates how Construction 2.1 may be used to obtain a lower bound on the number of $P(d,n,q)$ designs. It also improves the previously known lower bound on the number of nonisomorphic $P(2,4,2)$ designs.

Example 2.1 (The construction of $P(2,4,2)$ designs)

The ingredients required for each such design are:

(a) a $P(1,3,2)$ design, that is a $2-(15,3,1)$ design, also known as a Steiner triple system of order 15, necessarily simple,

(b) a $P(2,3,2)$ design, that is a $2-(15,7,3)$ design,

(c) an $RA(2,4,2)$ design, that is a resolved $2-(16,4,7)$ design.

The construction can be used to provide a lower bound on the number of distinct $P(2,4,2)$ designs and also on the number of isomorphism classes.

The association of blocks of the $2-(15,3,1)$ design with the parallel classes of the $2-(16,4,7)$ design may be carried out in $({4 \brack 2}_2)! = 35!$ ways. Provided that the $2-(16,4,7)$ design has no repeated parallel classes (i.e. it is semi-simple), no two of the resulting 35! designs are identical. Furthermore, varying any one of the three ingredient designs also results in different $P(2,4,2)$ designs. Thus the number of distinct $P(2,4,2)$ designs is at least

$35!N_1N_2N_3$, where N_1 is the number of distinct $2-(15,3,1)$ designs, N_2 is the number of distinct $2-(15,7,3)$ designs, and N_3 is the number of distinct semi-simple $2-(16,4,7)$ designs. In the latter case, distinct resolutions of a resolvable $2-(16,4,7)$ design count as distinct designs.

There are 80 nonisomorphic $2-(15,3,1)$ designs. These are listed along with the orders of their automorphism groups in [9] and in [2]. These orders and the numbers of designs having these orders are shown in Table 1. If a_i denotes the order of the automorphism group of the i^{th} system for $1 \le i \le 80$, then $N_1 = 15! \sum_{i=1}^{80} \frac{1}{a_i}$. This gives $N_1 = 60\,281\,712\,691\,200$.

order	1	2	3	4	5	6	8	12	21
number	36	6	12	8	1	1	2	3	1
order	24	32	36	60	96	168	192	288	20160
number	2	1	1	1	1	1	1	1	1

Table 1. Automorphism orders of the 80 $2-(15,3,1)$ designs.

There are five nonisomorphic $2-(15,7,3)$ designs. These are listed along with the orders of their automorphism groups in [6]. These orders are 96, 168, 168, 576 and 20160. Denoting these by b_i for $1 \le i \le 5$, we have $N_2 = 15! \sum_{i=1}^{5} \frac{1}{b_i}$. This gives $N_2 = 31\,524\,292\,800$.

To estimate N_3, note first that $AG_1(2,4)$ is, up to isomorphism, the unique $2-(16,4,1)$ design and it is resolvable with a unique resolution into five distinct parallel classes [2]. We may form an $S^*RA(2,4,2)$ design by combining seven copies of $AG_1(2,4)$ on the same point set, taking care to avoid repeated parallel classes. So let N^* be the size of the largest collection of $2-(16,4,1)$ designs (on a common point set) such that no two of these designs have a common parallel class. Then $N_3 \ge \binom{N^*}{7}$. We estimate N^* as follows.

Take a fixed copy $\mathcal{G}$ of the resolved $2-(16,4,1)$ design on a point set V. Let $\mathcal{G}_1$ be any copy of $\mathcal{G}$ on V. If π is a permutation on V such that $\pi(\mathcal{G})$ has a parallel class in common with $\mathcal{G}_1$, then call π *incompatible* with $\mathcal{G}_1$, otherwise call π *compatible* with $\mathcal{G}_1$. If the five parallel classes of $\mathcal{G}$ are $\mathcal{P}_i$ and those of $\mathcal{G}_1$ are $\mathcal{Q}_i$, $1 \le i \le 5$, where each class contain four blocks of length four, then for a given ordered pair $(\mathcal{P}_i, \mathcal{Q}_j)$, there are 4! ways to map the blocks of $\mathcal{P}_i$ to those of $\mathcal{Q}_j$, and within each block 4! ways of mapping the points. Hence there are at most $5^2(4!)^5$ permutations incompatible with $\mathcal{G}_1$, and consequently at least $16! - 5^2(4!)^5$ permutations π that are compatible with $\mathcal{G}_1$. Choose any one of these, say π_1, and put $\mathcal{G}_2 = \pi_1(\mathcal{G})$. Then $\mathcal{G}_1$ and $\mathcal{G}_2$ have no common parallel class.

By repeating this argument, it follows that there are at least $16! - 2 \times 5^2(4!)^5$ permutations π that are compatible with both $\mathcal{G}_1$ and $\mathcal{G}_2$. Choose any one of these, say π_2, and put $\mathcal{G}_3 = \pi_2(\mathcal{G})$. Then $\mathcal{G}_1$, $\mathcal{G}_2$ and $\mathcal{G}_3$ have no common parallel class. Clearly this process may be continued $16!/(5^2(4!)^5)$ times, so that $N^* \geq 16!/(5^2(4!)^5)$. This gives

$$N_3 \geq 28\,109\,060\,191\,179\,933\,806\,258\,464\,246\,200.$$

Combining the estimates gives the number of distinct $P(2,4,2)$ designs as at least 5.519629×10^{95}. There are 31 points in a $P(2,4,2)$ design, so dividing by $31!$, the largest possible size of an isomorphism class, gives the number of nonisomorphic $P(2,4,2)$ designs as at least 6.712559×10^{61}. This is a significant improvement on the bound of 5.125897×10^{28} given in [4].

Although individual examples like the one above require ad-hoc arguments that depend on existence results and known properties of the ingredients, it should be reasonably clear from Example 2.1 how to proceed in the general case. Bearing in mind that we wish to use the construction recursively, we make the following observations in the form of a lemma using the terminology of Construction 2.1.

Lemma 2.1 *Suppose that $\mathcal{E}$ is simple and that either (a) $\mathcal{D}$ is simple, or (b) $\mathcal{F}$ is simple. Then the resulting $P(d,n,q)$ design is simple. Furthermore, if $\mathcal{D}$ is simple and $\mathcal{F}$ is semi-simple, varying any one of the three ingredients and varying the associations of the blocks of $\mathcal{D}$ with the parallel classes of $\mathcal{F}$ results in distinct designs.*

Proof Since $\mathcal{E}$ is simple, there are no repeated blocks in $\mathcal{B}_2$. Since $\mathcal{B}_1 \cap \mathcal{B}_2 = \emptyset$, no block of $\mathcal{B}_2$ is repeated in $\mathcal{B}_1$.

In case (a) consider a block $B \in \mathcal{B}_1$. Put $D = B \cap V_1$, so that D is a block of $\mathcal{D}$ that appears only once in $\mathcal{D}$ because $\mathcal{D}$ is simple. Put $F = B \setminus D$, so that F is a block of $\mathcal{F}$. Then F may appear more than once in $\mathcal{F}$, but at most once in any given parallel class of $\mathcal{F}$. The only blocks of $\mathcal{B}_1$ that contain D are those formed from blocks in a single parallel class $\mathcal{P}$ of $\mathcal{F}$ combined with the block D, and hence one and only one of the blocks of $\mathcal{B}_1$ that contain D also contains F. Thus the block $B = D \cup F$ cannot be repeated in $\mathcal{B}_1$. So in case (a) the resulting design is simple.

In case (b) again consider a block $B \in \mathcal{B}_1$. Put $F = B \cap V_2$, so that F is a block of $\mathcal{F}$ that appears only once in $\mathcal{F}$ because $\mathcal{F}$ is simple. Put $D = B \setminus F$, so that D is a block of $\mathcal{D}$. Then D may appear more than once in $\mathcal{D}$, but D will only be appended once to F since F lies in one and only one parallel class of $\mathcal{F}$. Thus the block $B = D \cup F$ cannot be repeated in $\mathcal{B}_1$. So in case (b) the resulting design is simple.

Now consider two designs, $\mathcal{R}$ and $\mathcal{R}'$, resulting from the construction, the first formed from $\mathcal{D}$, $\mathcal{E}$ and $\mathcal{F}$, and the second from $\mathcal{D}'$, $\mathcal{E}'$ and $\mathcal{F}'$. If $\mathcal{E} \neq \mathcal{E}'$, or if $\mathcal{D} \neq \mathcal{D}'$, then $\mathcal{R}$ and $\mathcal{R}'$ contain different blocks and so are distinct. If $\mathcal{E} = \mathcal{E}'$ and $\mathcal{D} = \mathcal{D}'$, then deleting the blocks of $\mathcal{E}$, and deleting the blocks of $\mathcal{D}$ from the blocks of $\mathcal{R}$ and $\mathcal{R}'$ that contain them, gives respectively $\mathcal{F}$ and $\mathcal{F}'$, each arranged in parallel classes. So, if these two differ, then $\mathcal{R} \neq \mathcal{R}'$. The only remaining possibility for duplication is that $\mathcal{R}$ and $\mathcal{R}'$ are both formed from the same three ingredients, but with different associations of the blocks of $\mathcal{D}$ with the parallel classes of $\mathcal{F}$. However, if no block of $\mathcal{D}$ is repeated and no parallel class of $\mathcal{F}$ is repeated, then different associations result in different blocks. $\square$

Denote by $NSP(d,n,q)$ and $NS^*RA(d,n,q)$ respectively the numbers of distinct $SP(d,n,q)$ and $S^*RA(d,n,q)$ designs. Lemma 2.1 gives the following inequality.

$$\begin{aligned} NSP(d,n,q) \;\; &\geq \;\; \left(\begin{bmatrix} n \\ d \end{bmatrix}_q\right)! \times NSP(d-1,n-1,q) \\ &\qquad \times NSP(d,n-1,q) \times NS^*RA(d,n,q). \qquad (1) \end{aligned}$$

The number of isomorphism classes may be estimated by dividing by $(\left[{n+1 \atop 1}\right]_q)!$. In connection with applying (1) recursively, note that a $P(n,n,q)$ design consists of a single block, so that $NSP(n,n,q) = 1$. Also, a $P(1,n,q)$ design is a $2-(\frac{q^{n+1}-1}{q-1}, q+1, 1)$ design, which is a Steiner system on $\frac{q^{n+1}-1}{q-1}$ points with block length $q+1$, and $NSP(1,n,q)$ is just the number of such systems. Use can also be made of the fact that for $1 \leq d \leq n-1$, $PG_d(n,q)$ has full automorphism group $P\Gamma L(n{+}1,q)$, and so $NSP(d,n,q) \geq (\left[{n+1 \atop 1}\right]_q)!/|P\Gamma L(n+1,q)|$. Furthermore, if $q = p^s$, where p is prime, then $|P\Gamma L(n+1,q)| = sq^{\frac{n(n+1)}{2}} \prod_{i=2}^{n+1}(q^i - 1)$. To apply (1) recursively, we also need a construction for $RA(d,n,q)$ designs, and we now present such a construction.

Construction 2.2

(A recursive construction for an $RA(d,n,q)$ design)

The resulting $RA(d,n,q)$ design will be realized on the point set $V = \bigcup_{i=1}^{q} V_i$, where $V_i = \{1_i, 2_i, \dots, q_i^{n-1}\}$. Here $n > d > 1$ and the ingredients required are as follows:

(a) an $RA(d-1, n-1, q)$, $\mathcal{D}_i$, on each point set V_i for $i = 1, 2, \dots, q$,

(b) an $RA(d, n-1, q)$, $\mathcal{E}_i$, on each point set V_i for $i = 1, 2, \dots, q$,

(c) $\left[{n-1 \atop d-1}\right]_q$ resolved transversal designs $RTD(q, q^{n-d})$.

Where several designs with the same parameter set are used, it does not matter whether or not these designs are isomorphic. In connection with item (c) we will specify the point sets and groups of the designs below. Each such design has q groups, each of size q^{n-d} and each of the q^{n-d} parallel classes contains q^{n-d} blocks, giving a total of $q^{2(n-d)}$ blocks. Each pair of points from different groups appears in precisely one block. Such designs are related to sets of $q-1$ MOLS of side q^{n-d} and they exist whenever $n > d$.

Suppose that $\{\mathcal{P}_{j,i} : 1 \le j \le \left[{n-1 \atop d-1}\right]_q\}$ is the set of parallel classes of the design $\mathcal{D}_i$. The order in which these parallel classes of each $\mathcal{D}_i$ are listed is arbitrary. Let the set of blocks forming the parallel class $\mathcal{P}_{j,i}$ be $\{B_{k,j,i} : 1 \le k \le q^{n-d}\}$. For each value of j, $1 \le j \le \left[{n-1 \atop d-1}\right]_q$, let $\mathcal{T}_j$ be an $RTD(q, q^{n-d})$, having point set $\{B_{k,j,i} : 1 \le i \le q, 1 \le k \le q^{n-d}\}$, and groups $\mathcal{P}_{j,i}$ for $i = 1, 2, \dots, q$. Each block $\{B_{k_1,j,1}, B_{k_2,j,2}, \dots, B_{k_q,j,q}\}$ of this transversal design forms a system of distinct representatives from the parallel classes $\mathcal{P}_{j,i}$ $(1 \le i \le q)$. For each such block, form the block $B^* = \bigcup_{i=1}^{q} B_{k_i,j,i}$. For each fixed j there will be $q^{2(n-d)}$ blocks of the form B^*, each of length $q \cdot q^{d-1} = q^d$, arranged in parallel classes $\mathcal{P}^*_{l,j}$, $1 \le l \le q^{n-d}$, on the point set V, and each parallel class will contain q^{n-d} blocks. Figure 1 illustrates this part of the construction. Let $\mathcal{B}_a$ be the collection of all the blocks of the form B^* taken over all values of j from $j = 1$ to $j = \left[{n-1 \atop d-1}\right]_q$, that is $\mathcal{B}_a = \bigcup_{l,j} \mathcal{P}^*_{l,j}$ where l runs from 1 to q^{n-d} and j runs from 1 to $\left[{n-1 \atop d-1}\right]_q$. Then $\mathcal{B}_a$ comprises $q^{n-d}\left[{n-1 \atop d-1}\right]_q$ parallel classes, each containing q^{n-d} blocks of length q^d on the point set V.

Suppose next that $\{\mathcal{Q}_{j,i} : 1 \le j \le \left[{n-1 \atop d}\right]_q\}$ is the set of parallel classes of the design $\mathcal{E}_i$. The order in which these parallel classes of each $\mathcal{E}_i$ are listed is arbitrary. For each j, the set of blocks $\mathcal{Q}^*_j = \bigcup_{i=1}^{q} \mathcal{Q}_{j,i}$ forms a parallel

$\mathcal{D}_1$	$\mathcal{D}_2$	$\ldots$	$\mathcal{D}_q$
$\mathcal{P}_{1,1}$	$\mathcal{P}_{1,2}$	$\ldots$	$\mathcal{P}_{1,q}$
$\mathcal{P}_{2,1}$	$\mathcal{P}_{2,2}$	$\ldots$	$\mathcal{P}_{2,q}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\mathcal{P}_{j,1}$	$\mathcal{P}_{j,2}$	$\ldots$	$\mathcal{P}_{j,q}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$

$\mathcal{D}_i$ is on the point set V_i and has parallel classes $\mathcal{P}_{j,i}$ for $1 \leq j \leq \left[{n-1 \atop d-1}\right]_q$.

The blocks of the parallel classes $\mathcal{P}_{j,i}$ for $1 \leq i \leq q$ are the points of the transversal design $\mathcal{T}_j$, and the parallel classes themselves form the groups of this design.

Figure 1(a). The parallel classes of the designs $\mathcal{D}_i$.

$\mathcal{P}_{j,1}$	$\mathcal{P}_{j,2}$	$\ldots$	$\mathcal{P}_{j,q}$
$B_{1,j,1}$	$B_{1,j,2}$	$\ldots$	$B_{1,j,q}$
$B_{2,j,1}$	$B_{2,j,2}$	$\ldots$	$B_{2,j,q}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$B_{k,j,1}$	$B_{k,j,2}$	$\ldots$	$B_{k,j,q}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$

$\mathcal{P}_{j,i}$ is on the point set V_i and has blocks $B_{k,j,i}$ for $1 \leq k \leq q^{n-d}$.

Two possible parallel classes in $\mathcal{T}_j$ are formed by the rows and by the diagonals of this array; these are $\mathcal{P}^*_{1,j}, \mathcal{P}^*_{2,j}$, etc.

Figure 1(b). The transversal design $\mathcal{T}_j$.

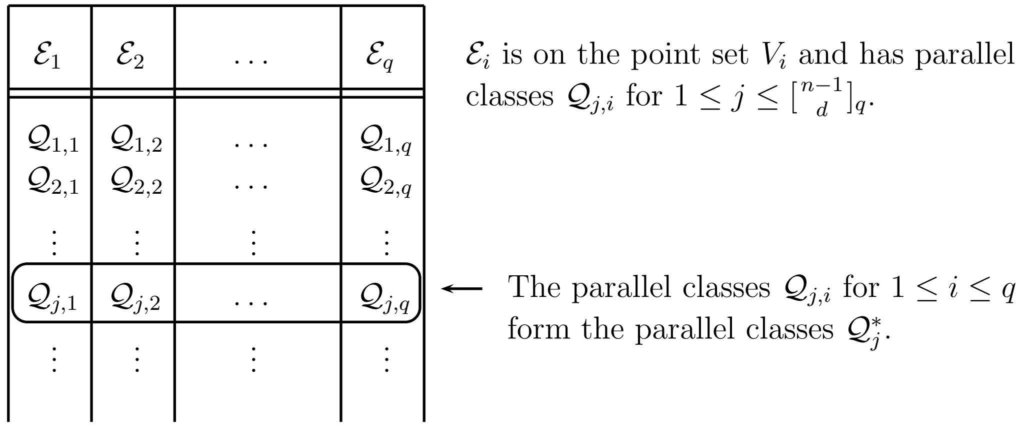

Figure 2. The parallel classes of the designs $\mathcal{E}_i$.

class of $q \cdot q^{n-d-1} = q^{n-d}$ blocks of length q^d on the point set V. Figure 2 illustrates this part of the construction. Let $\mathcal{B}_b$ be the collection of all the blocks from all the designs $\mathcal{E}_i$, $1 \le i \le q$, so that $\mathcal{B}_b = \bigcup_j \mathcal{Q}^*_j$ where j runs from 1 to $\left[\begin{smallmatrix} n-1 \\ d \end{smallmatrix}\right]_q$. Then $\mathcal{B}_b$ comprises $\left[\begin{smallmatrix} n-1 \\ d \end{smallmatrix}\right]_q$ parallel classes, each containing q^{n-d} blocks of length q^d on the point set V.

Next put $\mathcal{B} = \mathcal{B}_a \cup \mathcal{B}_b$. We will prove that $\mathcal{B}$ forms an $RA(d, n, q)$ design. By construction, $\mathcal{B}$ is resolved into parallel classes each containing q^{n-d} blocks of length q^d on the point set V of cardinality q^n. Then, allowing possible multiplicity, we have

$$|\mathcal{B}| = |\mathcal{B}_a| + |\mathcal{B}_b| = q^{n-d}\left(q^{n-d}\begin{bmatrix} n-1 \\ d-1 \end{bmatrix}_q + \begin{bmatrix} n-1 \\ d \end{bmatrix}_q\right) = q^{n-d}\begin{bmatrix} n \\ d \end{bmatrix}_q.$$

Hence $|\mathcal{B}|$ is the number of blocks in an $RA(d, n, q)$ design.

To prove that the blocks of $\mathcal{B}$ form an $RA(d, n, q)$ design on the point set V, it remains only to show that every pair of distinct points $x, y \in V$ appears $\left[\begin{smallmatrix} n-1 \\ d-1 \end{smallmatrix}\right]_q$ times. So take any two distinct points $x_i \in V_i$ and $y_h \in V_h$. Suppose initially that $i \neq h$. The point x_i appears in precisely one block of the j^{th} parallel class $\mathcal{P}_{j,i}$ of $\mathcal{D}_i$, say $B_{k_x,j,i}$. Similarly y_h appears in precisely one block of the j^{th} parallel class $\mathcal{P}_{j,h}$ of $\mathcal{D}_h$, say $B_{k_y,j,h}$. The blocks $B_{k_x,j,i}$ and $B_{k_y,j,h}$ appear together in precisely one block of the transversal design $\mathcal{T}_j$. Hence the points x_i and y_h appear together in precisely one block from the parallel classes $\mathcal{P}^*_{l,j}$, $1 \le l \le q^{n-d}$. This applies to each j in the range

$1 \leq j \leq [{n-1 \atop d-1}]_q$ and consequently the pair $x_i y_h$, $i \neq h$, appears in precisely $[{n-1 \atop d-1}]_q$ blocks of $\mathcal{B}_a$.

Finally consider the case $i = h$ by taking two points x_i and y_i where $x \neq y$. The pair $x_i y_i$ does not appear in any $\mathcal{D}_j$ with $j \neq i$, and it appears in precisely $[{n-2 \atop d-2}]_q$ blocks of $\mathcal{D}_i$. Each point of an $RTD(q, q^{n-d})$ appears in q^{n-d} blocks of the transversal design. Consequently each of the $[{n-2 \atop d-2}]_q$ blocks containing the pair $x_i y_i$ appears as a subset of q^{n-d} blocks of $\mathcal{B}_a$. Hence the pair $x_i y_i$ appears in precisely $q^{n-d}[{n-2 \atop d-2}]_q$ blocks of $\mathcal{B}_a$. Furthermore, the pair $x_i y_i$ appears in precisely $[{n-2 \atop d-1}]_q$ blocks of $\mathcal{E}_i$ and so it appears in precisely $[{n-2 \atop d-1}]_q$ blocks of $\mathcal{B}_b$. But

$$q^{n-d} \begin{bmatrix} n-2 \\ d-2 \end{bmatrix}_q + \begin{bmatrix} n-2 \\ d-1 \end{bmatrix}_q = \begin{bmatrix} n-1 \\ d-1 \end{bmatrix}_q,$$

and so the pair $x_i y_i$ appears in $[{n-1 \atop d-1}]_q$ blocks of $\mathcal{B}$.

It follows that the blocks of $\mathcal{B}$ form an $RA(d, n, q)$ design on the point set V. □

It was noted in Construction 2.2 that for each i, $1 \leq i \leq q$, the ordering of the parallel classes $\mathcal{P}_{j,i}$ for $1 \leq j \leq [{n-1 \atop d-1}]_q$ is arbitrary. If the ordering for $i = 1$ is fixed, then the number of ways of entering the parallel classes of the remaining designs $\mathcal{D}_i$ $(i > 1)$ into the table shown in Figure 1(a) is $\big(([{n-1 \atop d-1}]_q)!\big)^{q-1}$. We will refer to these arrangements as *alignments* of the parallel classes $\mathcal{P}_{j,i}$. Similarly, with the ordering of the parallel classes $\mathcal{Q}_{j,1}$ for $1 \leq j \leq [{n-1 \atop d}]_q$ fixed, the number of ways of entering the parallel classes of the remaining designs $\mathcal{E}_i$ $(i > 1)$ into the table shown in Figure 2 is $\big(([{n-1 \atop d}]_q)!\big)^{q-1}$ and we will refer to these arrangements as *alignments* of the parallel classes $\mathcal{Q}_{j,i}$. In each case, a subsequent re-ordering of the rows of the table will be regarded as giving rise to the same alignment. Thus alignments are determined by the entries in each row, irrespective of the order of the rows. If the designs $\mathcal{D}_i$ are all semi-simple then the $\big(([{n-1 \atop d-1}]_q)!\big)^{q-1}$ alignments of the parallel classes $\mathcal{P}_{j,i}$ are all distinct; a similar observation applies to the designs $\mathcal{E}_i$ and their parallel classes $\mathcal{Q}_{j,i}$.

As in the case of Construction 2.1, we wish to use the construction recursively, so we make the following observations in the form of a lemma using the terminology of Construction 2.2.

Lemma 2.2 *If each $\mathcal{D}_i$ and $\mathcal{E}_i$ is semi-simple then the resulting $RA(d, n, q)$ design is semi-simple. If each $\mathcal{D}_i$ and $\mathcal{E}_i$ is simple then the resulting $RA(d, n, q)$*

design is simple. In either case, varying any of the ingredients $\mathcal{D}_i$, $\mathcal{E}_i$ or $\mathcal{T}_j$, or varying the alignments of the parallel classes $\mathcal{P}_{j,i}$, or the alignments of the parallel classes $\mathcal{Q}_{j,i}$ results in distinct designs.

Proof Suppose that each $\mathcal{D}_i$ and $\mathcal{E}_i$ is semi-simple. There is no repetition of parallel classes lying in $\mathcal{B}_b$ because each of the designs $\mathcal{E}_i$ is semi-simple and each has a different point set. Furthermore, each of the blocks of $\mathcal{B}_b$ contains only points x_i with a common suffix i, whereas each of the blocks in $\mathcal{B}_a$ contains points with all suffices $i = 1, 2, \ldots, q$. Hence no block of $\mathcal{B}_b$ appears in $\mathcal{B}_a$. Suppose that two parallel classes of $\mathcal{B}_a$ are identical, that is $\mathcal{P}^*_{l,j} = \mathcal{P}^*_{t,s}$. If $j \neq s$ then these two parallel classes originate from different $RTD(q, q^{n-d})$ designs which have different groups and hence different parallel classes, resulting in a contradiction. So $j = s$ and the two parallel classes originate from the same $RTD(q, q^{n-d})$. If $l \neq t$ then the blocks of the two classes will be different, so we must have $(l, j) = (t, s)$. Thus none of the parallel classes of $\mathcal{B}_a$ is repeated and so the resulting design $(V, \mathcal{B})$ is semi-simple.

Suppose next that each $\mathcal{D}_i$ and $\mathcal{E}_i$ is simple. There is no repetition of blocks lying in $\mathcal{B}_b$ because each of the designs $\mathcal{E}_i$ is simple and each has a different point set. As previously, no block of $\mathcal{B}_b$ appears in $\mathcal{B}_a$. Suppose that two blocks of $\mathcal{B}_a$ are identical. That is, $\bigcup_{i=1}^{q} B_{k_i,j,i} = \bigcup_{i=1}^{q} B_{m_i,l,i}$. Since $B_{k,j,i}$ is a block on the point set V_i, it follows that $B_{k_i,j,i} = B_{m_i,l,i}$ for each value i. But for each i both $B_{k_i,j,i}$ and $B_{m_i,l,i}$ are blocks of the simple design $\mathcal{D}_i$, so they come from the same parallel class $\mathcal{P}_{j,i}$ (so that $j = l$) and consequently $\{B_{k_1,j,1}, B_{k_2,j,2}, \ldots, B_{k_q,j,q}\}$ and $\{B_{m_1,l,1}, B_{m_2,l,2}, \ldots, B_{m_q,l,q}\}$ are identical blocks of the transversal design $\mathcal{T}_j$. Hence $\bigcup_{i=1}^{q} B_{k_i,j,i}$ and $\bigcup_{i=1}^{q} B_{m_i,l,i}$ are produced at one and the same point in the construction and this block is not repeated. It follows that $\mathcal{B}$ has no repeated blocks.

Now consider two designs, $\mathcal{R}$ and $\mathcal{R}'$, resulting from the construction, the first formed from $\mathcal{D}_i$, $\mathcal{E}_i$ and $\mathcal{T}_j$, and the second from $\mathcal{D}'_i$, $\mathcal{E}'_i$ and $\mathcal{T}'_j$ where $1 \leq i \leq q$ and $1 \leq j \leq \left[{n-1 \atop d-1}\right]_q$. We assume that each $\mathcal{D}_i$, $\mathcal{E}_i$, $\mathcal{D}'_i$ and $\mathcal{E}'_i$ is semi-simple. (This is weaker than assuming they are simple.) If X is any structure in $\mathcal{R}$, then we will denote by X' the corresponding structure in $\mathcal{R}'$.

Choose any i such that $1 \leq i \leq q$. In each of $\mathcal{R}$ and $\mathcal{R}'$, delete all the blocks that contain points from $V \setminus V_i$. The remaining blocks are those of $\mathcal{E}_i$ (respectively, $\mathcal{E}'_i$) grouped into the parallel classes $\mathcal{Q}_{j,i}$ ($\mathcal{Q}'_{j,i}$). Hence if $\mathcal{E}_i$ and $\mathcal{E}'_i$ are different resolved designs, then $\mathcal{R} \neq \mathcal{R}'$.

Now suppose that $\mathcal{E}_i = \mathcal{E}'_i$ for $1 \leq i \leq q$. In each of $\mathcal{R}$ and $\mathcal{R}'$, delete all the blocks that contain points from more than one set V_i. The remaining blocks are those of $\mathcal{B}_b$ ($\mathcal{B}'_b$) grouped into the parallel classes $\mathcal{Q}^*_j$ ($\mathcal{Q}^{*\prime}_j$). From a single $\mathcal{Q}^*_j$, the parallel classes $\mathcal{Q}_{j,1}, \mathcal{Q}_{j,2}, \ldots, \mathcal{Q}_{j,q}$ may be recovered to give one of the rows of the table in Figure 2 (of course, the value of j cannot be recovered). By repeating this process, each row of the table may be obtained, although the ordering of the rows is not determined. Nevertheless, this is sufficient to give the alignment of the parallel classes of the designs $\mathcal{E}_i$. The same operation may be applied to the blocks of $\mathcal{B}'_b$ to give the alignment of the parallel classes of the designs $\mathcal{E}'_i$. Hence, if these alignments are different, then $\mathcal{R} \neq \mathcal{R}'$.

Subsequently we assume that $\mathcal{E}_i = \mathcal{E}'_i$ for $1 \leq i \leq q$ and that the parallel classes $\mathcal{Q}_{j,i}$ are aligned identically in $\mathcal{R}$ and $\mathcal{R}'$.

In each of $\mathcal{R}$ and $\mathcal{R}'$, delete all the blocks of $\mathcal{B}_b$ ($= \mathcal{B}'_b$) and then choose any i such that $1 \leq i \leq q$. In all the remaining blocks (those of $\mathcal{B}_a$ ($\mathcal{B}'_a$)) delete all points except those in V_i. The resulting blocks form q^{n-d} copies of each parallel class of $\mathcal{D}_i$ ($\mathcal{D}'_i$). Removing the duplicate copies gives the blocks of $\mathcal{D}_i$ ($\mathcal{D}'_i$) grouped into the parallel classes $\mathcal{P}_{j,i}$ ($\mathcal{P}'_{j,i}$). Hence if $\mathcal{D}_i$ and $\mathcal{D}'_i$ are different resolved designs, then $\mathcal{R} \neq \mathcal{R}'$.

Now suppose that $\mathcal{D}_i = \mathcal{D}'_i$ for $1 \leq i \leq q$. Again delete the blocks of $\mathcal{B}_b$ from $\mathcal{R}$ and $\mathcal{R}'$, so that only the blocks of $\mathcal{B}_a$ ($\mathcal{B}'_a$) remain; these are grouped into the parallel classes $\mathcal{P}^*_{l,j}$ ($\mathcal{P}^{*\prime}_{l,j}$). From a single $\mathcal{P}^*_{l,j}$, the parallel classes $\mathcal{P}_{j,1}, \mathcal{P}_{j,2}, \ldots, \mathcal{P}_{j,q}$ may be recovered to give one of the rows of the table in Figure 1(a). By repeating this process, each row of the table may be obtained, although the ordering of the rows is not determined. Nevertheless, this is sufficient to give the alignment of the parallel classes of the designs $\mathcal{D}_i$. The same operation may be applied to the blocks of $\mathcal{B}'_a$ to give the alignment of the parallel classes of the designs $\mathcal{D}'_i$. Hence, if these alignments are different, then $\mathcal{R} \neq \mathcal{R}'$.

Subsequently we assume that $\mathcal{D}_i = \mathcal{D}'_i$ for $1 \leq i \leq q$ and that the parallel classes $\mathcal{P}_{j,i}$ are aligned identically in $\mathcal{R}$ and $\mathcal{R}'$.

As above, from each $\mathcal{P}^*_{l,j}$, the parallel classes $\mathcal{P}_{j,1}, \mathcal{P}_{j,2}, \ldots, \mathcal{P}_{j,q}$ may be recovered, and these form the groups of both $\mathcal{T}_j$ in $\mathcal{R}$ and $\mathcal{T}'_j$ in $\mathcal{R}'$ (of course, the value of j cannot be recovered). If all the classes $\mathcal{P}^*_{t,s}$ are examined, and the blocks of those which give rise to a common set of parallel classes $\mathcal{P}_{j,1}, \mathcal{P}_{j,2}, \ldots, \mathcal{P}_{j,q}$ are recorded, then noting that no parallel class is repeated, the blocks of $\mathcal{T}_j$ may be unambiguously obtained along with their resolution into parallel classes. Likewise for $\mathcal{T}'_j$. If these blocks differ, or if the resolutions differ, then $\mathcal{R} \neq \mathcal{R}'$. This completes the proof of the lemma. $\square$

Denote by $NRTD(k,n)$ the number of distinct resolved transversal designs $RTD(k,n)$. Lemma 2.2 gives the following inequality.

$$\begin{aligned} NS^*RA(d,n,q) \;\; &\geq \;\; \left(\left(\left[{n-1 \atop d-1}\right]_q\right)! \left(\left[{n-1 \atop d}\right]_q\right)!\right)^{q-1} \times (NS^*RA(d-1,n-1,q))^q \\ &\times (NS^*RA(d,n-1,q))^q \times \left(NRTD(q,q^{n-d})\right)^{\left[{n-1 \atop d-1}\right]_q}. \quad (2) \end{aligned}$$

The inequality remains valid if S^* is replaced throughout by S. In both cases, the number of isomorphism classes may be estimated by dividing by $(q^n)!$.

In connection with applying (2) recursively, note that an $RA(n,n,q)$ design consists of a single block, so that $NS^*RA(n,n,q) = 1$. Also, an $RA(1,n,q)$ design is a resolved $2-(q^n,q,1)$ design, which is a resolved Steiner system on q^n points with block length q, and $NS^*RA(1,n,q)$ is just the number of such systems, all of which are simple. Use can also be made of the fact that for $1 \leq d \leq n-1$, $AG_d(n,q)$ has full automorphism group $A\Gamma L(n,q)$, and so $NSRA(d,n,q) \geq (q^n)!/|A\Gamma L(n,q)|$. Furthermore, if $q = p^s$, where p is prime, then $|A\Gamma L(n,q)| = sq^{\frac{n(n+1)}{2}} \prod_{i=1}^{n}(q^i-1)$.

In Example 2.1, an estimate of the number of semi-simple $2-(16,4,7)$ designs ($S^*RA(2,4,2)$ designs) was obtained. This estimate was facilitated by the existence of the unique $2-(16,4,1)$ design. In the absence of such a coincidence, Lemma 2.2 and inequality (2) may be used to obtain an estimate. The next example shows how this may be done in this case.

Example 2.2 (The construction of $S^*RA(2,4,2)$ designs)

The ingredients required for each such design are:

(a) two $S^*RA(1,3,2)$ designs on different point sets; these are resolutions of $2-(8,2,1)$ designs (necessarily simple), that is to say 1-factorizations of the complete graph K_8,

(b) two $S^*RA(2,3,2)$ designs on different point sets; these are resolutions of semi-simple $2-(8,4,3)$ designs,

(c) seven resolved transversal designs $RTD(2,4)$ on different point sets.

Lemma 2.2 can be used to provide a lower bound on the number of distinct $S^*RA(2,4,2)$ designs and also on the number of isomorphism classes.

There are six nonisomorphic 1-factorizations of K_8, each with seven 1-factors. These give 6240 distinct 1-factorizations of K_8 and hence 6240 distinct $S^*RA(1,3,2)$ designs [2]. Up to isomorphism, there are four $2-(8,4,3)$ designs, but only one is resolvable; this is $AG_2(3,2)$ [2]. This is simple with a unique resolution and has full automorphism group $A\Gamma L(3,2)$ of order 1344; hence there are $8!/1344 = 30$ distinct $S^*RA(2,3,2)$ designs. An $RTD(2,4)$ is equivalent to a 1-factorization of the complete bipartite graph $K_{4,4}$, and there are precisely 24 distinct 1-factorizations of this graph [2]. Hence there are 24 resolved transversal designs $RTD(2,4)$. Applying inequality (2) gives the number of distinct $S^*RA(2,4,2)$ designs as at least

$$\begin{aligned}\left(\begin{bmatrix}3\\1\end{bmatrix}_2\right)!\left(\begin{bmatrix}3\\2\end{bmatrix}_2\right)! \times 6240^2 \times 30^2 \times 24^{\left[{3 \atop 1}\right]_2} &= (7!)^2 \times 6240^2 \times 30^2 \times 24^7 \\ = 4\,082\,737\,461\,092\,790\,829\,056\,000\,000 &\geq 4.082737 \times 10^{27}.\end{aligned}$$

Although this is not as good as the estimate of approximately 2.810906×10^{31} obtained for the same quantity by a different method in Example 2.1, this method works in general and does not rely on fortunate coincidences between the parameters. Furthermore, the designs produced are in fact simple, rather than just semi-simple, since the ingredients are simple. To obtain a lower bound for the number of isomorphism classes, since the design has 16 points, we simply divide the estimate by 16!, the largest possible size of an isomorphism class in this case.

3 Applications

In this section we examine some consequences of applying inequalities (1) and (2). The bounds we obtain rely on very crude estimates; many large terms are replaced by 1, and use is made of the inequalities $m^{\frac{m}{2}} \leq m! < m^m$ for $m \geq 2$. We also note that if $j \geq i \geq 1$ then $q^{j-i} \leq (q^j-1)/(q^i-1) < 2q^{j-i}$, so that $\left[{n \atop d}\right]_q \geq q^{d(n-d)}$ and $\left[{n \atop 1}\right]_q < 2q^{n-1} \leq q^n$. The bounds that could be obtained by more exact use of (1) and (2) in particular cases would be much greater than those recorded here.

It is useful to have an estimate for $NRTD(q, q^{n-d})$. So, let $\mathcal{T}$ be a resolved transversal design $TD(q,m)$, having groups G_i, $1 \leq i \leq q$, where $G_i = \{g_{i,j} : 1 \leq j \leq m\}$. For $1 \leq j \leq m$, let B_j be the block of $\mathcal{T}$ containing the points $g_{1,j}$ and $g_{2,1}$. Each block B_j lies in a distinct parallel class $\mathcal{P}_j$. Suppose that

π_2, π_2' are permutations of $\{g_{2,2}, g_{2,3}, \ldots, g_{2,m}\}$ and that for $i \geq 3$, π_i, π_i' are permutations of G_i. Put $\pi = \prod_{i=2}^{q} \pi_i$ and $\pi' = \prod_{i=2}^{q} \pi_i'$. Note that both $\pi(B_j)$ and $\pi'(B_j)$ contain the pair $\{g_{1,j}, g_{2,1}\}$, and that $\pi(\mathcal{P}_j)$ $(\pi'(\mathcal{P}_j))$ is the only parallel class in $\pi(\mathcal{T})$ (respectively, $\pi'(\mathcal{T})$) that covers this pair.

Now suppose that for some $i \geq 3$, $\pi_i \neq \pi_i'$. Then there exists $g_{i,k}$ such that $\pi_i(g_{i,k}) \neq \pi_i'(g_{i,k})$. The point $g_{i,k}$ must occur in one of the blocks B_j, say B_{j_k}. Then $\pi(B_{j_k}) \neq \pi'(B_{j_k})$, so that $\pi(\mathcal{T}) \neq \pi'(\mathcal{T})$.

Next suppose that $\pi_i = \pi_i'$ for $i \geq 3$, but $\pi_2 \neq \pi_2'$. Then there exists $g_{2,k}$ $(k \geq 2)$ such that $\pi_2(g_{2,k}) \neq \pi_2'(g_{2,k})$. But then if B is the block of $\mathcal{P}_1$ containing $g_{2,k}$, $\pi(B) \neq \pi'(B)$, and consequently $\pi(\mathcal{P}_1) \neq \pi'(\mathcal{P}_1)$, so that $\pi(\mathcal{T}) \neq \pi'(\mathcal{T})$.

It follows that each of the $(m-1)!(m!)^{q-2}$ permutations π gives a distinct copy of $\mathcal{T}$. Hence $NRTD(q,m) \geq (m-1)!(m!)^{q-2} = (m!)^{q-1}/m$. In particular,

$$NRTD(q, q^{n-d}) \geq \frac{((q^{n-d})!)^{q-1}}{q^{n-d}}. \tag{3}$$

In [4] it is proved that for $2 \leq d \leq n-2$, the number of nonisomorphic $SP(d,n,q)$ designs is at least

$$\frac{(q-1)\left(\begin{bmatrix} n \\ d \end{bmatrix}_q\right)!}{(q^{n+1}-1)|P\Gamma L(n,q)||A\Gamma L(n,q)|}. \tag{4}$$

We show that the bound given by our inequalities is better than this for $n \geq 7$.

If $1 < d < n-1$, applying inequality (1) to $NSP(d, n-1, q)$ we obtain $NSP(d, n-1, q) \geq (\begin{bmatrix} n-1 \\ d \end{bmatrix}_q)!$. Also, using inequality (2), $NSRA(d,n,q) \geq \left(\left(\begin{bmatrix} n-1 \\ d-1 \end{bmatrix}_q\right)! \left(\begin{bmatrix} n-1 \\ d \end{bmatrix}_q\right)!\right)^{q-1}$. Thus for $2 \leq d \leq n-2$, inequality (1) gives

$$\begin{aligned} NSP(d,n,q) &\geq \left(\begin{bmatrix} n \\ d \end{bmatrix}_q\right)! \left(\left(\begin{bmatrix} n-1 \\ d-1 \end{bmatrix}_q\right)!\right)^{q-1} \left(\left(\begin{bmatrix} n-1 \\ d \end{bmatrix}_q\right)!\right)^{q} \\ &\geq \left(\begin{bmatrix} n \\ d \end{bmatrix}_q\right)! \left(\left(q^{(d-1)(n-d)}\right)!\right)^{q-1} \left(\left(q^{d(n-1-d)}\right)!\right)^{q}. \end{aligned} \tag{5}$$

Now assume that $n \geq 7$ and that $2 \leq d \leq n-2$. Put $f(d) = d(n-1-d)$. Then $f(d)$ has a local maximum at $d = (n-1)/2$, which lies in the interval

$[2, n-3]$. Furthermore, $f(2) = f(n-3) = 2n-6$. So, if $2 \le d \le n-3$ then $d(n-1-d) \ge 2n-6 \ge n+1$. If $d = n-2$ then $d(n-1-d) = n-2$. Replacing d by $d-1$ also gives that if $3 \le d \le n-2$ then $(d-1)(n-d) \ge 2n-6 \ge n+1$, and if $d = 2$ then $(d-1)(n-d) = n-2$. In particular, either $d(n-1-d) \ge n+1$ or $(d-1)(n-d) \ge n+1$. Hence, from (5), we have

$$NSP(d,n,q) \ge \begin{cases} ({\textstyle\left[{n \atop d}\right]_q})!((q^{n+1})!)^{2q-1} & \text{if } 3 \le d \le n-3, \\ ({\textstyle\left[{n \atop d}\right]_q})!((q^{n+1})!)^{q-1}((q^{n-2})!)^{q} & \text{if } d = n-2, \\ ({\textstyle\left[{n \atop d}\right]_q})!((q^{n+1})!)^{q}((q^{n-2})!)^{q-1} & \text{if } d = 2. \end{cases}$$

The number of nonisomorphic $SP(d,n,q)$ designs is therefore at least

$$\begin{aligned} NSP(d,n,q)/({\textstyle\left[{n+1 \atop 1}\right]_q})! &\ge NSP(d,n,q)/(q^{n+1})! \\ &\ge ({\textstyle\left[{n \atop d}\right]_q})!((q^{n-2})!)^{q}, \end{aligned}$$

which exceeds the estimate (4) by a factor greater than $((q^{n-2})!)^q$.

In [1] it is proved that for $q \ge 3$ and $2 \le d \le n-1$, the number of nonisomorphic simple resolvable $A(d,n,q)$ designs is at least

$$\frac{((q^{n-d})!)^{\left[{n-1 \atop d-1}\right]_q} \left(\left[{n-1 \atop d-1}\right]_q \right)!}{q!(q^{n-1} + q^{n-2} + \ldots + q + 1)|A\Gamma L(n-1,q)|^q}. \tag{6}$$

We show that the bound given by our inequalities is substantially greater than this for resolved designs when $q \ge 3$ and $2 \le d \le n-2$. Note that the latter inequality entails $n \ge 4$.

Applying inequality (2) gives

$$\begin{aligned} NSRA(d,n,q) &\ge \left(\left(\left[{n-1 \atop d-1}\right]_q \right)! \left(\left[{n-1 \atop d}\right]_q \right)! \right)^{q-1} \times (NSRA(d-1,n-1,q))^q \\ &\quad \times \left(NRTD(q, q^{n-d}) \right)^{\left[{n-1 \atop d-1}\right]_q}. \end{aligned}$$

The number of nonisomorphic $SRA(d,n,q)$ designs, N, is at least this number divided by $(q^n)!$. As noted earlier, $NSRA(d-1,n-1,q) \ge (q^{n-1})!/|A\Gamma L(n-1,q)|$, and combining this with our earlier estimate for $NRTD(q, q^{n-d})$ gives

$$\begin{aligned} N &\ge \left(\left(\left[{n-1 \atop d-1}\right]_q \right)! \left(\left[{n-1 \atop d}\right]_q \right)! \right)^{q-1} \left(\frac{(q^{n-1})!}{|A\Gamma L(n-1,q)|} \right)^q \\ &\quad \times \left(\frac{((q^{n-d})!)^{q-1}}{q^{n-d}} \right)^{\left[{n-1 \atop d-1}\right]_q} /(q^n)!. \end{aligned} \tag{7}$$

We also have $(q^n)! < q^{q^n}((q^{n-1})!)^q$, so that

$$\frac{((q^{n-1})!)^q}{(q^n)!} \quad > \quad \frac{1}{q^{q^n}}. \tag{8}$$

If $m \geq 8$ then $(m-1)! > m^{m/2}$ (this can be proved by induction using $7! = 5040, 8^4 = 4096$ and $(\frac{m+1}{m})^m < 3$ for $m \geq 1$).

Since $q \geq 3$ and $n-d \geq 2$, $q^{n-d} \geq 9 > 8$, so that $(q^{n-d}-1)! > q^{(n-d)q^{n-d}/2}$. Hence

$$\begin{aligned}
\left(\frac{(q^{n-d})!}{q^{n-d}}\right)^{\left[{n-1 \atop d-1}\right]_q} &> q^{\left(\frac{n-d}{2}\right)q^{n-d}\left[{n-1 \atop d-1}\right]_q} \\
&\geq q^{q^{n-d}\left[{n-1 \atop d-1}\right]_q} \\
&> q^{q^{n-d}q^{(d-1)(n-d)}} \\
&= q^{q^{d(n-d)}}.
\end{aligned}$$

But $d(n-d)$ has minimum value $2(n-2)$ for $2 \leq d \leq n-2$, and $2(n-2) \geq n$ since $n \geq 4$. Thus

$$\left(\frac{(q^{n-d})!}{q^{n-d}}\right)^{\left[{n-1 \atop d-1}\right]_q} \quad > \quad q^{q^n}. \tag{9}$$

Then, using (7), (8) and (9) we get

$$N \quad \geq \quad \frac{\left(\left(\left[{n-1 \atop d-1}\right]_q\right)! \left(\left[{n-1 \atop d}\right]_q\right)!\right)^{q-1} ((q^{n-d})!)^{(q-2)\left[{n-1 \atop d-1}\right]_q}}{|A\Gamma L(n-1,q)|^q}. \tag{10}$$

Use of recursion in particular cases would improve our bound (10) still further.

Finally in this section we give a specific example which builds on Example 2.2, thereby illustrating the power of recursive application of our inequalities.

Example 3.1 (The construction of $SRA(3,5,2)$ designs)

Applying inequality (2) gives

$$\begin{aligned}
NSRA(3,5,2) \quad \geq \quad & \left(\begin{bmatrix} 4 \\ 2 \end{bmatrix}_2\right)! \left(\begin{bmatrix} 4 \\ 3 \end{bmatrix}_2\right)! \times (NSRA(2,4,2))^2 \\
& \times (NSRA(3,4,2))^2 \times (NRTD(2,4))^{\left[{4 \atop 2}\right]_2}.
\end{aligned}$$

It was shown in Example 2.2 that $NSRA(2,4,2) \geq 4.082737 \times 10^{27}$ and in the course of that example it was noted that $NRTD(2,4) = 24$. The design $AG_3(4,2)$ has automorphism group $A\Gamma L(4,2)$ of order 322560. So there are at least $16!/322560$ distinct $SRA(3,4,2)$ designs, giving $NSRA(3,4,2) \geq 6.48648 \times 10^7$. Using these values gives $NSRA(3,5,2) \geq 1.923299 \times 10^{171}$. Dividing by 32! gives the number of nonisomorphic $SRA(3,5,2)$ designs as at least 7.3×10^{135}. This compares with the figure of 1.6×10^{27} given by the formula in [1] for the number of nonisomorphic (simple) resolvable $A(3,5,2)$ designs.

4 Concluding remarks

The estimates in the previous section are extremely crude and could be improved in several respects, but at the expense of considerably lengthening the paper. Here we restrict ourselves to a few specific comments.

Most of the estimates can be strengthened by inclusion of all the terms from inequalities (1) and (2), some of which we simply replaced by 1. The factorial estimations can be greatly strengthened by use of Stirling's Theorem: $m! = (2\pi m)^{\frac{1}{2}} m^m \exp(-m + \frac{\theta(m)}{12m})$, where $0 < \theta(m) < 1$. Our estimate (3) for $NRTD(q, q^{n-d})$ can also be improved in many cases because it relies on permutations applied to a single $RTD(q, q^{n-d})$, whereas in general there will be many nonisomorphic designs on which to base this argument.

For specific values of q, some of the estimates can be significantly strengthened. In particular, in the case $q = 2$, $NRTD(2,m)$ is just the number of 1-factorizations of $K_{m,m}$ which is the number of Latin squares of order m divided by $m!$, and consequently $NRTD(2,m)$ is bounded below by $(m!)^{2m-1}/m^{m^2}$ [2].

Finally, in moving from the number of distinct designs on m points to the number of isomorphism classes, we have always divided by $m!$; in some cases this factor may be excessive.

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References

[1] D. Clark, D. Jungnickel and V. D. Tonchev, Exponential bounds on the number of designs with affine parameters, J. Combin. Des., to appear.

[2] C. J. Colbourn and J. H. Dinitz (eds.), The Handbook of Combinatorial Designs (second edition), CRC Press, Boca Raton, 2007.

[3] D. Jungnickel, The number of designs with classical parameters grows exponentially, Geom. Dedicata, 16 (1984), 167–178.

[4] D. Jungnickel and V. D. Tonchev, The number of designs with geometric parameters grows exponentially, Des. Codes Cryptogr., 55 (2010), 131–140.

[5] W. M. Kantor, Automorphisms and isomorphisms of symmetric and affine designs, J. Algebraic Combin., 3 (1994), 307–338.

[6] W. Kocay and G. H. J. van Rees, Some nonisomorphic $(4t+4, 8t+6, 4t+3, 2t+2, 2t+1)$-BIBDs, Discrete Math., 92 (1991), 159–172.

[7] C. Lam, S. Lam and V. D. Tonchev, Bounds on the number of affine, symmetric, and Hadamard designs and matrices, J. Combin. Theory Ser. A, 92 (2000), 186–196.

[8] C. Lam and V. D. Tonchev, A new bound on the number of designs with classical affine parameters, Des. Codes Cryptogr., 27 (2002), 111–117.

[9] R. A. Mathon, K. T. Phelps and A. Rosa, Small Steiner triple systems and their properties, Ars Combin., 15 (1983), 3–110.

[10] R. C. Mullin, Resolvable designs and geometroids, Util. Math., 5 (1974), 137–149.

[11] G. H. J. van Rees, A new family of BIBDs and non-embeddable $(16, 24, 9, 6, 3)$-designs, Discrete Math., 77 (1989), 357–365.