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Rigid Steiner triple systems obtained from projective triple systems

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Abstract

It was shown by Babai in 1980 that almost all Steiner triple systems are rigid; that is, their only automorphism is the identity permutation. Those Steiner triple systems with the largest automorphism groups are the projective systems of orders $2^n - 1$. In this paper we show that each such projective system may be transformed to a rigid Steiner triple system by at most $n$ Pasch trades whenever $n \geq 4$.

Running head: Rigid STS

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1 Introduction

A Steiner triple system of order \(v\), \(\text{STS}(v)\), is an ordered pair \((V, \mathcal{B})\) where \(V\) is a \(v\)-element set (the points) and \(\mathcal{B}\) is a set of triples from \(V\) (the blocks), such that each pair from \(V\) appears in precisely one block. The necessary and sufficient condition for the existence of an \(\text{STS}(v)\) is that \(v \equiv 1 \text{ or } 3 \pmod{6}\) [9]; such values of \(v\) are called admissible. We often omit set brackets and commas from triples of points so that \(\{x, y, z\}\) may be written as \(xyz\) when no confusion is likely, and pairs (or \(n\)-tuples) may be treated similarly. An automorphism of an \(\text{STS}(v) = (V, \mathcal{B})\) is a permutation on the points of \(V\) that preserves the set of blocks \(\mathcal{B}\). An \(\text{STS}(v)\) is said to be rigid if its only automorphism is the identity permutation. Alternative terms to “rigid” are “automorphism-free” and “asymmetric”.

It was shown by Lindner and Rosa [10] that a rigid \(\text{STS}(v)\) exists for admissible \(v\) if and only if \(v \geq 15\). Subsequently, Babai [1] proved that at most \(\exp((2-\epsilon)\log v)\) distinct \(\text{STS}(v)\)s have an automorphism group of order greater than 1. Since (for admissible \(v\)) the number of distinct \(\text{STS}(v)\)s is \(\exp((2-\epsilon)\log v)\) [2, 12], it follows that the proportion of rigid \(\text{STS}(v)\)s tends to 1 as \(v \to \infty\). Speaking colloquially, almost all Steiner triple systems are rigid.

The most symmetric \(\text{STS}(v)\)s are the projective systems; these exist when \(v\) is of the form \(2^n - 1\). The projective \(\text{STS}(v)\) of order \(v = 2^n - 1\) may be represented on the points of \(\mathbb{Z}_3^2 \setminus \{0\}\) by taking the block set to comprise all triples of points \(xyz\) such that \(x \oplus y \oplus z = 0\) in \(\mathbb{Z}_3^2\). Here and subsequently we use \(\oplus\) to denote addition of points in \(\mathbb{Z}_3^2\). We will identify the integer \(2^{n-1}a_{n-1} + 2^{n-2}a_{n-2} + \ldots + 2a_1 + a_0\) with the point \((a_{n-1}, a_{n-2}, \ldots, a_1, a_0) \in \mathbb{Z}_3^n \setminus \{0\}\) so that, for example, \(12 \oplus 5 = 9\) because 12 is identified with 1100, 5 with 0101 and 9 with 1001 = 1100 \(\oplus\) 0101.

(In the vector representation, leading zeros are suppressed so that, for example, 101 and 0101 are both identified with 5, and it is not necessary to specify \(n\) when using \(\oplus\).) We use the symbol \(S_n\) to denote the projective \(\text{STS}(2^n - 1)\). It is well-known that \(S_n\) has automorphism group \(\text{PSL}(n, 2)\) of order \(2^{n-1} \prod_{i=2}^n (2^i - 1)\) [7, page 41].

It is an interesting question how close the most and the least symmetric systems can be to one another. In this paper we investigate how far \(S_n\) is from a rigid system of the same order. The systems \(S_2\) and \(S_3\) are, up to isomorphism, the unique Steiner triple systems of orders 3 and 7 (the latter being generally known as the Fano plane), so there are no rigid systems of these orders.

If \(T_1\) and \(T_2\) are disjoint sets of triples from a common point set \(V\) that cover the same pairs of points, then the pair \(\mathcal{T} = \{T_1, T_2\}\) is called a trade pair, and \(T_1\) and \(T_2\) are called tradeable configurations. If an \(\text{STS}(v)\) contains a copy of \(T_1\), then that copy may be replaced by the corresponding copy of \(T_2\) to give another \(\text{STS}(v)\). This operation is called a \(\mathcal{T}\)-trade. The set of points covered by \(T_1\) and \(T_2\) is called the foundation of the trade, and the number of blocks in each \(T_i\) \((i = 1, 2)\) is called the volume of the trade. A Pasch configuration or quadrilateral or 4-cycle \(P(a, b, c, d, e, f)\) is a set of four triples on six distinct points having the form \(\{abc, ade, bdf, cef\}\). The opposite Pasch configuration is...
Let \( \mathcal{P}(a, b, c, d, e, f) = P(f, b, c, d, e, a) \), and this covers the same pairs with a disjoint set of triples. If \( P_1 \) and \( P_2 \) are opposite Pasch configurations then \( \mathcal{P} = \{ P_1, P_2 \} \) is a trade pair and the corresponding replacement operation is called a Pasch trade. This is the smallest possible trade in an STS(\( v \)), both by foundation and volume.

It is sometimes impossible to transform one given STS(\( v \)) to another by any sequence of Pasch trades. For 79 of the 80 nonisomorphic STS(15)s, it is possible to transform any one system to an isomorphic copy of another by a sequence of Pasch trades [4]. It was shown in [5] that, by allowing more general \( k \)-cycle trades, all 80 STS(15)s are connected. More recently, it was shown in [8] that the same is true for STS(19)s. However, the existence of perfect Steiner triple systems [6, 3] establishes that such transformations are not possible for all admissible orders.

Our main result is that, for \( n \geq 4 \), the projective system \( S_n \) of order \( 2^n - 1 \) may be converted to a rigid system of the same order by Pasch trades, and that \( n \) block-disjoint Pasch trades suffice. So, if the distance between systems is measured by Pasch trades, \( S_n \) is distance at most \( n \) from a rigid system whenever \( n \geq 4 \).

If the distance is measured by blocks then \( S_n \) is distance at most \( 4n \) from a rigid system whenever \( n \geq 4 \).

2 Preliminaries

An STS(\( 2^n - 1 \)) with point set \( A_n = \{ 1, 2, \ldots, 2^n - 1 \} \), may be extended to an STS(\( 2^{n+1} - 1 \)) with point set \( A_{n+1} \) by adjoining new blocks. Put \( B_n = \{ 2^n, 2^n + 1, \ldots, 2^{n+1} - 1 \} \), so that \( A_n \cup B_n = A_{n+1} \) and take the new blocks to be all triples of the form \( xyz \), where \( x \in A_n, y, z \in B_n \) and \( x \oplus y \oplus z = 0 \). If this construction is applied to the projective system \( S_n \), then \( S_{n+1} \) is the result. We will apply the construction recursively, starting with a rigid STS(15) which may be obtained from the projective system of order 15 (that is, \( S_4 \)) by applying four block-disjoint Pasch trades. At each stage of the recursion we will apply one further block-disjoint Pasch trade and show that the resulting system is rigid. Thus for \( n \geq 4 \) we obtain a rigid STS(\( 2^n - 1 \)), which we denote by \( S_n^* \), and which is \( n \) Pasch trades distant from \( S_n \).

The rigid STS(15), \( S_4^* \), with which we start the recursion is given in Table 1. In fact it is system number 23 in the standard listing of [11]. It follows from results in [5] that the minimum number of Pasch trades required to convert \( S_4 \) to a rigid STS(15) is 4. The blocks which result from the trades are indicated by asterisks. More generally, a block \( xyz \) of an STS(\( 2^n - 1 \)) with point set \( A_n \) will be called a projective block (pr-block for short) if \( x \oplus y \oplus z = 0 \), otherwise it will be called a non-projective block (npr-block for short). Thus in Table 1, the asterisked blocks are the npr-blocks and all remaining blocks are pr-blocks. The number of Pasch configurations containing the point \( x \) in \( S_n^* \) will be denoted by \( p_n(x) \), and \( p_4(x) \) is also tabulated in Table 1. Altogether there are 18 Pasch configurations in \( S_4^* \).
First we apply the recursive construction as described above, adding new points to form an STS(2^n+1). We take the Pasch configuration P* = P(6, 2, 3, 4, 5, 1) which is the only Pasch configuration in which all four blocks lie in one and only one Pasch configuration. Second, the block \{1, 14, 15\} is the only pr-block containing the point 1. In general for \(n \geq 4\), we will call the triple \(D_n = \{1, 2^n - 2, 2^n - 1\}\) the distinguished triple of order \(n\), so \{1, 14, 15\} is the distinguished triple of order 4, \(D_4\).

As previously indicated, the construction of \(S^*_n\) from \(S^*_n\) is in two stages. First we apply the recursive construction as described above, adding new points and new projective blocks to form an STS(2^{n+1} - 1), denoted by \(T_{n+1}\). Then we apply a Pasch trade involving the distinguished triple of order \(n\) and three of the new triples. We take the Pasch configuration \(P_{n+1} = P(1, 2^n - 2, 2^n - 1, 2^n + 1, 2^n, 2^{n+1} - 1)\) in \(T_{n+1}\) and trade it for the opposite Pasch configuration, \(\overline{T}_{n+1}\). If \(\{a, b, c\}\) is one of the four blocks of \(S^*_n\) lying in \(\overline{T}_{n+1}\), then either all the points \(a, b, c\) are in \(B_n\), or one is in \(B_n\) and two are in \(A_n\). On the other hand, if this block is not in \(\overline{T}_{n+1}\), then either all the points \(a, b, c\) are in \(A_n\), or one is in \(A_n\) and two are in \(B_n\).

It is easy to show that \(S^*_n\) is rigid. Examining Table 1, the only point lying in 11 Pasch configurations is the point 11, so any automorphism must fix this point. The points 8 and 12 are the unique points lying in 8 Pasch configurations, so they are either fixed or transposed, and consideration of the block \{4, 8, 12\} establishes that 4 is a fixed point. Then 9 and 13 are either fixed or transposed, so consideration of \{9, 13, 14\} gives 14 fixed. But then \{4, 11, 15\} gives 15 fixed, \{4, 10, 14\} gives 10 fixed and \{5, 11, 14\} gives 5 fixed. From \{1, 14, 15\} we deduce that 1 is fixed and from \{1, 2, 4\}, 2 must be fixed. It is now trivial to show that all the remaining points are fixed by any automorphism.

<table>
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<th>7</th>
<th>8</th>
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<th>12</th>
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<td>9</td>
<td>7</td>
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<td>8</td>
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Table 1. The rigid STS(15), \(S^*_4\).
3 Main result

In this section we prove that $S_n^*$ is rigid for $n \geq 5$. The main step in our proof is Lemma 3.4, where we establish that any automorphism $\phi$ of $S_n^*$ fixes the Pasch configuration $P^* = P(6, 2, 3, 4, 5, 1)$. In order to do this, in Lemmas 3.1, 3.2 and 3.3, we determine some lower bounds on the number of Pasch configurations containing each block of $S_n^*$. Note that if a Pasch configuration contains three pr-blocks and a fourth block $S$, containing each block of Lemma 3.1. Suppose that $i \geq 4$ and that $B = xyz$ is an npr-block of $S_i^*$ which appears in exactly $k$ Pasch configurations in $S_i^*$. Then

(i) $B$ appears in at least $k - 2$ Pasch configurations in $S_{i+1}^*$;

(ii) if $i \geq 5$ and $B \neq \{1, 2^{i-1} - 2, 2^{i-1} + 1\}, \{1, 2^{i-1} - 1, 2^{i-1}\}, \{2^{i-1} - 2, 2^{i-1} - 1, 2^{i-1}\}$ or $\{2^{i-1}, 2^{i-1} + 1, 2^{i-1}\}$ then $B$ appears in at least $k$ Pasch configurations in $S_{i+1}^*$;

(iii) if $i \geq 5$ then $B$ appears in at least $k - 2$ Pasch configurations in $S_j^*$ for $j \geq i + 1$ (at least $k$ if $B$ is not one of the four npr-blocks identified in (ii));

(iv) if $i = 4$ then $B$ appears in exactly $k$ Pasch configurations in $S_j^*$ for $j \geq 4$.

Proof, (i) The distinguished triple $D_i = \{1, 2^i - 2, 2^i - 1\}$ of order $i$ is the only block of $S_i^*$ which is not a block of $S_{i+1}^*$. So $B$ can only appear in fewer Pasch configurations in $S_{i+1}^*$ if in $S_i^*$ it lies in Pasch configurations with $D_i$. But two distinct blocks can lie together in at most two Pasch configurations, so $B$ must appear in at least $k - 2$ Pasch configurations in $S_{i+1}^*$.

(ii) Now suppose that $i \geq 5$ and that $B$ and $D_i$ lie together in a Pasch configuration $Q$ in $S_i^*$. Without loss of generality, the block $B = xyz$ cannot lie with $D_i$ in a Pasch configuration unless $x \in \{1, 2^i - 2, 2^i - 1\}$. But there are no npr-blocks of $S_i^*$ containing the point $2^i - 2$. The only npr-blocks of $S_i^*$ that contain the point $2^i - 1$ are $\{2^{i-1} - 2, 2^{i-1} - 1, 2^{i-1}\}$ and $\{2^{i-1} - 1, 2^{i-1} + 1, 2^{i-1}\}$. The only remaining possibility is that $x = 1$ and $Q$ has blocks $B, D_i, \{2^i - 2, y, w\}, \{2^i - 1, z, w\}$. Since there are no npr-blocks in $S_i^*$ containing the point $2^i - 2$, $\{2^i - 2, y, w\}$ must be a pr-block and therefore $y = (2^i - 2) \oplus w$. If $\{2^i - 1, z, w\}$ were also a pr-block then $Q$ would comprise three pr-blocks and one npr-block, which is impossible. Thus $\{2^i - 1, z, w\}$ must be an npr-block, but the only npr-blocks containing the point $2^i - 1$ are $\{2^{i-1} - 2, 2^{i-1} - 1, 2^{i-1}\}$ and $\{2^{i-1} - 1, 2^{i-1} + 1, 2^{i-1}\}$. Hence $\{z, w\} = \{2^{i-1} - 2, 2^{i-1} - 1\}$. Examining the resulting four possibilities for the ordered pair $(z, w)$, and computing $y$ in each case, there are just two possibilities for $B$ when $x = 1$, namely $\{1, 2^{i-1} - 2, 2^{i-1} + 1\}$ and $\{1, 2^{i-1} - 1, 2^{i-1}\}$. (In fact, the four blocks $B$ identified in this paragraph each lie in two Pasch configurations with $D_i$, but we do not use this result.)
(iii) By applying (i) and (ii) it follows that if $i \geq 5$ then $B$ lies in at least $k - 2$ Pasch configurations in $S^*_j$ for $j \geq i + 1$ (at least $k$ if $B$ is not one of the four npr-blocks identified in (ii)).

(iv) In the case $i = 4$, note that $D_4$ appears in no Pasch configurations in $S^*_4$. Hence $B$ does not lie in any Pasch configurations with $D_4$ in $S^*_4$, so $B$ appears in at least $k$ Pasch configurations in $S^*_5$ and also in $S^*_i$ for $i > 5$. Suppose $B$ lies in an additional Pasch configuration $Q$ in $S^*_5$. At least one of the other blocks must be an npr-block. If this is an npr-block in $S^*_1$, then at least 5 points of $Q$ lie in $A_4$, and in this case the remaining two blocks of $Q$ must each contain two points from $A_4$ and one from $B_4$. But the only such blocks are three of the npr-blocks from $P_5$, namely $\{1, 14, 17\}$, $\{1, 15, 16\}$ and $\{14, 15, 31\}$. Hence $Q$ must contain an npr-block not in $S^*_1$ and consequently $B$ must contain one of the points 1, 14 and 15. From Table 1, there are six npr-blocks in $S^*_1$ containing the point 1, two containing the point 14 and none containing the point 15. Pairing each of these eight npr-blocks in turn with an intersecting block from $P_5$ gives 16 pairs. For each such pair there are two possibilities for the formation of a Pasch configuration, giving a total of 32 cases to be considered. In each of these cases one of the two additional blocks contains a point from $A_4$ and the other contains a point from $B_4$, and so these additional blocks cannot intersect one another and no Pasch configuration is formed. Thus if $B$ is an npr-block which appears in exactly $k$ Pasch configurations in $S^*_5$, then it appears in exactly $k$ Pasch configurations in $S^*_j$. It only remains to prove that it lies in no additional Pasch configurations in $S^*_j$ for $j \geq 6$.

So, suppose that $j \geq 6$ and that $B$ lies in an additional Pasch configuration $Q$ in $S^*_j$, but not in $S^*_{j-1}$. As above, at least one of the other blocks must be an npr-block. If this is an npr-block in $S^*_{j-1}$, then at least 5 points of $Q$ lie in $A_{j-1}$, and in this case the remaining two blocks of $Q$ must each contain two points from $A_{j-1}$ and one from $B_{j-1}$. But the only such blocks are the npr-blocks $\{1, 2^{j-1} - 2, 2^{j-1} + 1\}$, $\{1, 2^{j-1} - 1, 2^{j-1}\}$ and $\{2^{j-1} - 2, 2^{j-1} - 1, 2^{j-1}\}$. Hence $Q$ must contain an npr-block not in $S^*_{j-1}$. The only possible intersection that $B$ can have with such a block is the point 1 so, without loss of generality, we may take $x = 1$ and the two blocks as $1yz$ (which implies that neither $y$ nor $z$ is 14 or 15) and either $\{1, 2^{j-1} - 2, 2^{j-1} + 1\}$ or $\{1, 2^{j-1} - 1, 2^{j-1}\}$. Note there are no npr-blocks containing any pairs from $\{y, z\} \times \{2^{j-1} - 2, 2^{j-1} - 1, 2^{j-1}, 2^{j-1} + 1\}$. Hence, in either case, the remaining two blocks of the Pasch configuration are pr-blocks. By adding the six entries in these two blocks in each case, it can be seen that $y \oplus z = 2^{j-1} - 1$. But this contradicts the fact that $y, z \in A_4$. Hence $B$ does not appear in any additional Pasch configurations in $S^*_j$ for $j \geq 6$.

**Lemma 3.2.** For $i \geq 5$, all the blocks of the Pasch configuration $P_i$, lie in at least 4 Pasch configurations in $S^*_j$, and hence in at least 2 Pasch configurations in $S^*_j$ for $j \geq i + 1$. 

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Proof. The blocks of $P_i$ lie in an STS(7) subsystem of $T_i$, with the additional point $2i - 2$ and the three additional blocks $\{2^{i-1} - 2, 2^{i-1}, 2i - 2\}$, $\{2^{i-1} - 1, 2^{i-1} + 1, 2i - 2\}$ and $\{1, 2^i - 2, 2^i - 1\}$. When $P_i$ is traded for $\mathcal{P}_i$, the blocks of $\mathcal{P}_i$ continue to form an STS(7) subsystem with the same three additional blocks. Since every block of an STS(7) lies in 4 Pasch configurations within that STS(7), it follows that all the blocks of $\mathcal{P}_i$ lie in at least 4 Pasch configurations in $S_i^*$. □

Lemma 3.3. For $i \geq 5$ every pr-block of $S_i^*$ lies in at least 12 Pasch configurations in $S_i^*$.

Proof. In $S_i^*$ there are 6 npr-blocks through the point 1, so in $S_i^*$ ($i \geq 4$) there are $6 + 2(i - 4) = 2i - 2$ npr-blocks through the point 1. In $S_i^*$ there are 2 npr-blocks through the point 14, none through the point 15 and at most 4 npr-blocks through every other point, so in $S_i^*$ ($i \geq 4$) there are at most 4 npr-blocks through any other point other than 1.

Let $xyz$ be any pr-block of $S_i^*$, where $i \geq 6$. First we estimate the number of other pr-blocks containing $x$. There are $(2^i - 2)/2 = 2^{i-1} - 1$ blocks of $S_i^*$ that contain $x$. If $x = 1$, $2i - 2$ of these are npr-blocks, otherwise at most 4 of these are npr-blocks. So the number of pr-blocks through $x$ other than $xyz$ is $2^{i-1} - 2i$ if $x = 1$ and at least $2^{i-1} - 6$ otherwise. Potentially, each of these other pr-blocks $xvw$ may be paired with $xyz$ to give two Pasch configurations. A Pasch configuration will certainly result if the two blocks generated by the pairs $yw$ and $zw$ are pr-blocks (and likewise for the pairs $yw$ and $zv$). But if $y \neq 1$ at most 8 pairs through $y$ lie in an npr-block, and similarly for $z$. So, for $i \geq 6$, there are at least $2(2^{i-1} - 2i) - 16 = 2^i - 4i - 16 \geq 24$ Pasch configurations containing the block $xyz$.

Finally, in the case $i = 5$, direct computation establishes that for each pr-block of $S_i^*$, the minimum number of Pasch configurations in $S_i^*$ containing it is 12. □

Lemma 3.4. For $i \geq 4$ $P^* = P(6, 2, 3, 4, 5, 1)$ is the only Pasch configuration in $S_i^*$ that has all four of its blocks lying in exactly one Pasch configuration. Consequently, any automorphism of $S_i^*$ maps this Pasch configuration to itself.

Proof. By computation, this is true for $i = 4$. Moreover, all four blocks of $P^*$ are npr-blocks, so they continue to lie in exactly 1 Pasch configuration in $S_i^*$ for $i \geq 5$. Apart from the four npr-blocks of $P^*$, every other npr-block in $S_i^*$ lies in at least 2 Pasch configurations in $S_i^*$ and hence also in $S_i^*$ for $i \geq 5$. All npr-blocks of $S_i^*$ other than those already present in $S_i^*$ arise from $\mathcal{P}_i$ for some $j \geq 4$, and so these appear in at least 2 Pasch configurations in $S_i^*$. Finally, for $i \geq 5$, all pr-blocks of $S_i^*$ appear in at least 12 Pasch configurations in $S_i^*$.

Theorem 3.1. For $n \geq 4$, the Steiner triple system $S_n^*$ is rigid.
Proof. Let $\phi$ be an automorphism of $S_n^*$. When $n = 4$ we have already shown that $\phi$ is the identity permutation, so now assume $n \geq 5$. Consider $P^* = P(6, 2, 3, 4, 5, 1)$. This has blocks 124, 135, 236, 456 and it must be fixed by $\phi$; in other words, $\phi$ is an extension of an automorphism of this Pasch configuration. A Pasch configuration has an automorphism group of order 24. Note that $n^*$ contains all the pairs 16, 25 and 34 do not appear in the blocks of $P^*$, so $\phi(1)$ forces $\phi(6)$, and $\phi(2)$ forces $\phi(5)$. Table 2 lists all the automorphisms of $P^*$ in two formats. For example, the entry labelled $\phi_1$ indicates that $\phi_1$ is the permutation

\[
\begin{pmatrix}
1 & 2 & 3 & 4 & 5 & 6 \\
1 & 3 & 2 & 5 & 4 & 6
\end{pmatrix} = (1)(2, 3)(4, 5)(6).
\]

Our aim is to show that only $\phi_0$ extends to an automorphism of $S_n^*$. Since $\phi_{21}^2 = \phi_{22}^2 = \phi_3$, showing that $\phi_3$ cannot be extended will suffice to do the same for $\phi_{21}$ and $\phi_{22}$. Similarly, elimination of $\phi_{20}$ (respectively, $\phi_{23}$) eliminates $\phi_{10}$ and $\phi_{14}$ (respectively, $\phi_7$ and $\phi_{16}$). We also have $\phi_8^2 = \phi_5$, $\phi_{12}^2 = \phi_6$, $\phi_{17}^2 = \phi_9$, $\phi_{18}^2 = \phi_{11}$. So we need only consider $\phi_i$ for $i = 1, 2, 3, 4, 5, 6, 9, 11, 13, 15, 19, 20, 23$. In each case we assume that $\phi_i$ extends to an automorphism of $S_n^*$ and derive a contradiction. Note that $S_n^*$ contains all the blocks listed in Table 1, apart from the block \{1, 14, 15\}.

For $i = 1, 2$, $\phi_i$ fixes the block \{1, 6, 12\}, and maps \{2, 5, 7\} to \{3, 4, 13\} and vice-versa. Hence $\phi_i$ fixes the point 12 and transposes 7 and 13. But then $\phi_i$ maps the block \{7, 12, 11\} to \{13, 12, 10\} and vice-versa, so it transposes the points 10 and 11. But consideration of the block \{10, 11, 2\} then shows that $\phi_i$ fixes the point 2, a contradiction.

For $i = 4, 19$, $\phi_i$ fixes the block \{3, 4, 13\}, and maps \{2, 5, 7\} to \{1, 6, 12\} and vice-versa. Hence $\phi_i$ fixes the point 13 and transposes 7 and 12. But then it must map the block \{13, 12, 10\} to \{13, 7, 1\}, which implies that $\phi_i(10) = 1$, a contradiction.

For $i = 9, 13$, $\phi_i$ fixes the block \{2, 5, 7\}, and maps \{1, 6, 12\} to \{3, 4, 13\} and vice-versa. Hence $\phi_i$ fixes the point 7 and transposes 12 and 13. But then it

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<td>$\phi_5$</td>
<td>246135</td>
<td>(1, 2, 4)(3, 6, 5)</td>
<td>$\phi_2$</td>
<td>541632</td>
<td>(1, 5, 3)(2, 4, 6)</td>
</tr>
<tr>
<td>$\phi_6$</td>
<td>264315</td>
<td>(1, 2, 6, 5)(3, 4)</td>
<td>$\phi_3$</td>
<td>563942</td>
<td>(1, 2)(5, 6)(3, 4)</td>
</tr>
<tr>
<td>$\phi_7$</td>
<td>312564</td>
<td>(1, 3, 2)(4, 5, 6)</td>
<td>$\phi_4$</td>
<td>624351</td>
<td>(1, 6)(2, 3, 4)(5)</td>
</tr>
<tr>
<td>$\phi_8$</td>
<td>321654</td>
<td>(1, 3, 2)(4, 6, 5)</td>
<td>$\phi_5$</td>
<td>635241</td>
<td>(1, 6, 2)(3, 5, 4)</td>
</tr>
<tr>
<td>$\phi_9$</td>
<td>356124</td>
<td>(1, 3, 6, 4)(2, 5)</td>
<td>$\phi_6$</td>
<td>642531</td>
<td>(1, 6, 2)(4, 5, 3)</td>
</tr>
<tr>
<td>$\phi_{10}$</td>
<td>365214</td>
<td>(1, 3, 5)(2, 6, 4)</td>
<td>$\phi_7$</td>
<td>653421</td>
<td>(1, 6, 2)(5, 3)(4)</td>
</tr>
</tbody>
</table>

Table 2. Automorphisms of $P^* = P(1, 2, 4, 3, 5, 6)$. 


must map the block \( \{7, 12, 11\} \) to \( \{7, 13, 1\} \), which implies that \( \phi_i(11) = 1 \), a contradiction.

For \( i = 3, 20, 23 \), \( \phi_i \) fixes the blocks \( \{1, 6, 12\} \), \( \{2, 5, 7\} \) and \( \{3, 4, 13\} \). So \( \phi_i \) fixes each of the points 7, 12 and 13. Hence \( \phi_i \) fixes the block \( \{7, 13, 1\} \), and consequently the point 1. The blocks \( \{7, 12, 11\} \) and \( \{12, 13, 10\} \) are fixed by \( \phi_i \), so the points 11 and 10 are also fixed. But then the block \( \{2, 10, 11\} \) establishes that \( \phi_i \) fixes the point 2. But points 1 and 2 are not both fixed in any of these cases, so we have a contradiction.

For \( i = 5, 6 \), consideration of the blocks \( \{1, 6, 12\} \), \( \{2, 5, 7\} \) and \( \{3, 4, 13\} \) gives \( \phi_i(12) = 7, \phi_i(7) = 13 \), so \( \phi_i \) maps the block \( \{7, 12, 11\} \) to \( \{13, 7, 1\} \), implying that \( \phi_i(11) = 1 \), a contradiction.

For \( i = 11, 15 \) consideration of the blocks \( \{1, 6, 12\} \), \( \{3, 4, 13\} \) and \( \{2, 5, 7\} \) gives \( \phi_i(12) = 13, \phi_i(13) = 7 \), so \( \phi_i \) maps the block \( \{12, 13, 10\} \) to \( \{13, 7, 1\} \), implying that \( \phi_i(10) = 1 \), a contradiction.

It now follows that any automorphism \( \phi \) of \( S^n_6 \) fixes each of the points 1, 2, \ldots, 6. Consideration of the following blocks in the order given proves that the points 7, 8, \ldots, 15 are also fixed by \( \phi \): \( \{1, 6, 12\}, \{2, 5, 7\}, \{3, 4, 13\}, \{3, 7, 14\}, \{4, 7, 9\}, \{4, 8, 12\}, \{4, 10, 14\}, \{5, 10, 15\} \) and \( \{5, 11, 14\} \). Note that the block \( D_4 = \{1, 14, 15\} \) of \( S^n_6 \) which is destroyed by a Pasch trade in creating \( S^n_6 \) has not been employed in this argument.

Now suppose that for some \( i \) with \( 4 \leq i < n \), all the points of \( A_i \) are known to be fixed by \( \phi \); this has just been proven for \( i = 4 \). We will prove that all the points of \( B_i \) are also fixed by \( \phi \). The system \( S^n_6 \) contains the Pasch configuration \( T_{i+1} \) and consequently the block \( \{1, 2^i - 1, 2^i\} \). Hence the point \( 2^i \) is fixed by \( \phi \). The block \( \{2, 2^i, 2^i + 2\} \) lies in \( S^n_6 \), so the point \( 2^i + 2 \) is also fixed by \( \phi \). Now consider all the triples of the form \( \{x, 2^i + 2, (2^i + 2) \oplus x\} \) for \( x \in A_i \). These are all blocks of \( S^n_6 \) and so all the points \( (2^i + 2) \oplus x \) are fixed points of \( \phi \). But these cover all points of \( B_i \setminus \{2^i + 2\} \). Hence all the points of \( B_i \) are fixed by \( \phi \).

Since \( A_i \cup B_i = A_{i+1} \), we deduce that all the points of \( A_{i+1} \) are fixed by \( \phi \). Then, by induction, \( \phi \) fixes all the points of \( A_n \), and so \( \phi \) is the identity permutation on \( A_n \), and \( S^n_6 \) is rigid.

\section{4 Concluding remarks}

In this section we investigate the scope for improvements to the result of Section 3. In the discussions below it is convenient to consider each system of order \( 2^n - 1 \) as having point set \( \mathbb{Z}_2^n \setminus \{0\} \).

\textbf{Theorem 4.1.} Suppose that \( S_n \) is converted to another STS, say \( S'_n \), of the same order by a trade \( \mathcal{T} \) whose foundation lies in a subspace \( V_d \subseteq \mathbb{Z}_2^n \) of dimension \( d \). If \( d < n \) then \( S'_n \) is not rigid.

\textbf{Proof.} If \( d < n - 1 \), add points to the subspace \( V_d \) to form a subspace \( V_{n-1} \) of dimension \( n - 1 \) that contains the foundation of \( \mathcal{T} \). Put \( W = \mathbb{Z}_2^n \setminus V_{n-1} \) so that
Theorem 4.2. Suppose that $n$ is an odd number. If $w^*$ is any fixed point of $W$ then all the points $w^* \oplus v$ for $v \in V_{n-1}$ must lie in $W$ for otherwise we have $w^* = (w^* \oplus v) \oplus v \in V_{n-1}$, a contradiction. Hence $W = \{w^* \oplus v : v \in V_{n-1}\}$.

First suppose that $xyz$ is a block of $S'_n$ with $x, y \in V_{n-1}$. If it is an npr-block then it arose in the trade and so $z \in V_{n-1}$, while if it is a pr-block then $z = x \oplus y \in V_{n-1}$. So a block with two points in $V_{n-1}$ has all three points in $V_{n-1}$.

Second suppose that $xyz$ is a block of $S'_n$ with $x, y \in W$ and consequently $xyz$ is a pr-block. Then $z = x \oplus y = (w^* \oplus u) \oplus (w^* \oplus v)$ for some $u, v \in V_{n-1}$. This gives $z = u \oplus v \in V_{n-1}$, and so a block with two points in $W$ has its third point in $V_{n-1}$.

Now choose a fixed point $v^* \neq 0 \in V_{n-1}$ and define a mapping $\phi$ on $Z_2^n$ by

$$\phi(z) = \begin{cases} 
z & \text{if } z \in V_{n-1}, \\
z \oplus v^* & \text{if } z \in W.
\end{cases}$$

If $xyz$ is a block of $S'_n$ with all three points in $V_{n-1}$ then $\phi(xyz) = xyz$. On the other hand, if $xyz$ is a block of $S'_n$ with $x, y \in W$ then $z \in V_{n-1}$ and so $\phi(x, y, z) = \{(x \oplus v^*), (y \oplus v^*), z\}$ and this is a block of $S'_n$ because $(x \oplus v^*) \oplus (y \oplus v^*) = x \oplus y = z$. Hence $\phi$ is a non-trivial automorphism of $S'_n$, which is therefore not rigid.

**Corollary 4.1.** Suppose that $S_n$ is converted to another STS, say $S'_n$, of the same order by a trade $T$ consisting of $p$ Pasch trades. If $p < \frac{n}{3}$ then $S'_n$ is not rigid.

**Proof.** The points of a Pasch configuration generate a subspace of dimension 3. So, if $p < \frac{n}{3}$, then the foundation of $T$ lies in a subspace of dimension $d < n$, and the result follows.

Theorem 3.1 shows that for $n \geq 4$, we can convert the projective system $S_n$ to a rigid system using $n$ Pasch trades. Corollary 4.1 proves that any such conversion requires at least $n/3$ Pasch trades. Our next result shows that an incremental approach has some limitations if we wish to improve on Theorem 3.1. First we define a projective extension.

Suppose that $U_n$ is an STS$(2^n - 1)$ on the point set $Z_2^n \setminus \{0\}$. Then $U_n$ may be embedded in an STS$(2^{n+1} - 1)$, say $U_{n+1}$, by the addition of a new coordinate so that a point $x \in Z_2^n \setminus \{0\}$ generates two new points $0x, 1x \in Z_2^{n+1} \setminus \{0\}$, and we also add a further new point $100 \cdots 0$. Each block $\{a, b, c\}$ of $U_n$ generates the block $\{0a, 0b, 0c\}$ of $U_{n+1}$, and the remaining blocks of $U_{n+1}$ are all the triples of the form $\{0x, 1y, 1z\}$ where $x \oplus y \oplus z = 0$ in $Z_2^n$ and $y \neq z$. The system $U_{n+1}$ will be called the projective extension of $U_n$. The process may be repeated to give successive projective extensions $U_{n+1}, U_{n+2}, \ldots$ of $U_n$.

**Theorem 4.2.** Suppose that $R_n$ is a rigid STS$(2^n - 1)$ on the point set $Z_2^n \setminus \{0\}$ obtained from $S_n$ by some sequence of block-disjoint Pasch trades. If $T_{n+2}$ is the projective extension of $R_n$ to a system of order $2^{n+2} - 1$, then $T_{n+2}$ is not rigid.
it cannot be converted to a rigid system by a single further block-disjoint Pasch trade.

Proof. A block \{a, b, c\} of \(R_n\) generates a block \{00a, 00b, 00c\} of \(T_{n+2}\). We will call blocks of this form type 0 blocks. The remaining blocks, which are all pr-blocks, are of four further types:

- **type 1**: \{00x, 01y, 01z\},
- **type 2**: \{00x, 10y, 10z\},
- **type 3**: \{00x, 11y, 11z\},
- **type 4**: \{01x, 10y, 11z\},

where, in each case, \(x \oplus y \oplus z = 0\).

Using this classification of blocks, we may classify the Pasch configurations of \(T_{n+2}\). If a Pasch configuration has an 00x point then it must have two blocks containing this point, so the four blocks have the block types 0000, 0111, 0222, 0333, 1111, 1244, 1344, 2244, or 3333. If the Pasch configuration has no 00x point then the only possibility is 4444. So there are 11 types of Pasch configuration to consider. The type 0000 Pasch configurations correspond to those present in \(R_n\).

We only consider trading a block-disjoint Pasch configuration, that is to say one that does not include any blocks resulting from trades already made in generating \(R_n\) from \(S_n\). Thus any type 0 block involved in such a trade will be a pr-block. We will denote the resulting system obtained from \(T_{n+2}\) by \(\overline{T}_{n+2}\).

First we argue that \(T_{n+2}\) itself is not rigid. We may consider that \(T_{n+2}\) is formed directly from \(S_{n+2}\) by a sequence of Pasch trades all of whose points have their first two coordinates 0. All the points of these trades lie in the subspace of dimension \(n+1\) given by the equation \(\xi_{n+2} = 0\), where \(\xi_{n+2}\) denotes the first coordinate of a point. So, by Theorem 4.1, \(T_{n+2}\) is not rigid.

Now consider the case when \(\overline{T}_{n+2}\) is obtained from \(T_{n+2}\) by trading a type 0000 Pasch configuration. Again, all the points of all the trades used to convert \(S_{n+2}\) to \(\overline{T}_{n+2}\) satisfy \(\xi_{n+2} = 0\), so \(\overline{T}_{n+2}\) is not rigid. The same argument applies when \(\overline{T}_{n+2}\) is obtained from \(T_{n+2}\) by trading a type 0111 or a type 1111 Pasch configuration. For types 0222 and 2222, the argument is essentially the same but with the equation \(\xi_{n+1} = 0\), where \(\xi_{n+1}\) denotes the second coordinate of a point. For types 0333 and 3333, the argument may be repeated but with the equation \(\xi_{n+2} \oplus \xi_{n+1} = 0\).

Consider next trading a type 1244 Pasch configuration \(P\). This has blocks of the form:

- \{00a, 01b, 01c\} \((a \oplus b \oplus c = 0)\),
- \{00a, 10d, 10e\} \((a \oplus d \oplus e = 0)\),
- \{01b, 10d, 11f\} \((b \oplus d \oplus f = 0)\),
- \{01c, 10e, 11f\} \((c \oplus e \oplus f = 0)\).

The traded Pasch configuration \(\overline{P}\) comprises the blocks:

- \{11f, 01b, 01c\},
- \{11f, 10d, 10e\},
- \{01b, 10d, 00a\},
- \{01c, 10e, 00a\}.
Define $\phi : 00x \rightarrow 00x, \ 11x \rightarrow 11x, \ 01x \rightarrow 10(x \oplus f), \ 10x \rightarrow 01(x \oplus f)$. This mapping stabilizes $P$, maps the image of $R_n$ in $T_{n+2}$ to itself, and maps all the remaining blocks (which are pr-blocks) amongst themselves. So $\phi$ is a non-trivial automorphism of $T_{n+2}$.

A similar argument works for Pasch types 1344 and 2344.

Finally consider trading a type 4444 Pasch configuration $P$. This has blocks of the form:

$$\{01a, 10b, 11c\} \quad (a \oplus b \oplus c = 0), \quad \{01a, 11d, 10e\} \quad (a \oplus d \oplus e = 0),$$

$$\{10b, 11d, 01f\} \quad (b \oplus d \oplus f = 0), \quad \{11c, 10e, 01f\} \quad (c \oplus e \oplus f = 0).$$

The traded Pasch configuration $P'$ comprises the blocks:

$$\{01f, 10b, 11c\}, \quad \{01f, 11d, 10e\},$$

$$\{10b, 11d, 01a\}, \quad \{11c, 10e, 01a\}.$$

Define $\phi : 00x \rightarrow 00x, \ 11x \rightarrow 11x, \ 01x \rightarrow 10(x \oplus c), \ 10x \rightarrow 01(x \oplus c)$. This mapping stabilizes $P$, maps the image of $R_n$ in $T_{n+2}$ to itself, and maps all the remaining blocks (which are pr-blocks) amongst themselves. So $\phi$ is a non-trivial automorphism of $T_{n+2}$.

It follows that we cannot apply a single (block-disjoint) Pasch trade to $T_{n+2}$ to get a new rigid system of order $2^{n+2} - 1$.

Despite Theorem 4.2, the following example shows that it is possible to do better than the result of Theorem 3.1.

**Example 4.1.** The projective system $S_5$ may be converted to a rigid system using three block-disjoint Pasch trades.

Take the point set to be $A_5$ and the three block-disjoint Pasch configurations $P_1 = P(1, 2, 3, 4, 5, 6), \ P_2 = P(1, 6, 7, 8, 9, 14), \ and \ P_3 = P(2, 9, 11, 16, 18, 25)$. Trading all three gives a new system, say $R_5$. For each $x \in A_5$, Table 3 gives the number, $q_5(x)$, of Pasch configurations in $R_5$ containing the point $x$.

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<th>$x$</th>
<th>1</th>
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<th>3</th>
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<td>146</td>
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<td>154</td>
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<td>177</td>
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<td>110</td>
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<td>175</td>
<td>142</td>
<td>146</td>
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<td>143</td>
<td>110</td>
<td>154</td>
<td>150</td>
<td>117</td>
<td>154</td>
</tr>
</tbody>
</table>

Table 3. The number of Pasch configurations in $R_5$ containing $x$.

It follows from Table 3 that any automorphism of $R_5$ must fix the points 1, 5, 6, 7, 8, 10, 11, 15, 17, and 27. It is then easy to argue that all remaining points
are fixed; for example the block \{1, 3, 5\} must be stabilized, and so 5 is a fixed point. By exhaustive computer search we have also shown that it is not possible to convert \( S_5 \) to a rigid system using just two Pasch trades. We hope to publish a more comprehensive study of Pasch trades on \( S_5 \) in a future paper.

Example 4.1 is not an isolated case. For specific values of \( n \) it is easy to find examples which convert \( S_n \) to a rigid system by employing fewer than \( n \) Pasch trades. Thus the question of finding, as a function of \( n \), the minimum number of Pasch trades necessary to convert \( S_n \) to a rigid system remains open. It seems likely that there is some constant \( c \) satisfying \( \frac{1}{3} \leq c \leq 1 \) such that the minimum number of Pasch trades required is asymptotic to \( cn \). Given a collection of \( k < n_0 \) Pasch trades which convert \( S_{n_0} \) to a rigid system, we speculate that a modest generalization of our construction, albeit with a more complicated proof, might facilitate a result that for \( n \geq n_0 \), \( n - (n_0 - k) \) Pasch trades suffice to convert \( S_n \) to a rigid system. But to obtain a result which is substantially better than our Theorem 3.1, that is to say one that improves the putative constant \( c \), a new construction is likely to be required.

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References


