A Small Basis for Four-Line Configurations in Steiner Triple Systems

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Dedicated to the memory of Gemma Holly Griggs.

Abstract

Formulae for the numbers of two, three, and four-line configurations in a Steiner triple system of order v, STS(v), are given. While the formulae for two and three-line configurations depend only on v, the same is true for only 5 of the 16 four-line configurations. For the other 11 and fixed v, the number of occurrences of any one of them, in particular the Pasch configuration, determines the number of occurrences of all the others.

1 Introduction

A Steiner triple system of order v (STS(v)) is a pair $(V, \mathcal{B})$ comprising a base set V of cardinality v and a family $\mathcal{B}$ of 3-element subsets of V which collectively have the property that every 2-element subset of V occurs precisely

once. The 3-element subsets of V which form $\mathcal{B}$ are called *blocks* or *lines*. An *n-line configuration*, $n \geq 1$, is simply any n lines of an STS(v). This paper is principally concerned with the case where $n = 4$.

Recently, there has been considerable interest in the study of Steiner triple systems and configurations. One particular aspect of this work is concerned with the decomposition of STS(v) into two, three, and four-line configurations and the articles [3, 5, 6] contain many results, as well as unsolved problems, in this area. Another part investigates avoidance problems: the construction of STS(v) containing no copies of a particular configuration [1, 4]. In this paper we begin a further strand of the work; a study of how the numbers of occurrences of each configuration in a particular STS(v) are interrelated. In some ways this is a more fundamental question than decomposition or avoidance yet it seems to have been neglected. This may be because it was not suspected that any interesting results could be obtained. However this is far from the case as will be shown. We will refer to a configuration C as being *constant* or *variable*. By saying that C is constant we mean that for each admissible value of $v \geq 13$ it occurs the same number of times in every STS(v). By saying that C is variable we mean that for some value of $v \geq 13$ there are at least two nonisomorphic STS(v)s containing different numbers of the configuration. We denote the number of occurrences of a configuration C in an STS(v) by $c(v)$ or simply by c when no confusion is likely. Thus, referring to the four-line configurations, $c_1, c_2, \ldots, c_{16}$ denote the number of $C_1, C_2, \ldots, C_{16}$ in an STS(v).

In order to count four-line configurations we shall first count smaller configurations. These results are easy to determine and we include them only for completeness and because we make use of them subsequently. All the configurations are constant and proofs of the formulae are left as exercises for the reader. There is only one one-line configuration, a single line. In any STS(v) there are $v(v-1)/6$ one-line configurations. There are 2 two-line configurations: a pair of parallel blocks denoted by A_1, and a pair of intersecting blocks denoted by A_2.

$$\begin{aligned} a_1 &= v(v-1)(v-3)(v-7)/72. \\ a_2 &= v(v-1)(v-3)/8. \end{aligned}$$

There are 5 three-line configurations and these are shown in Figure 1 where they are denoted by $B_1, B_2, \ldots, B_5$.

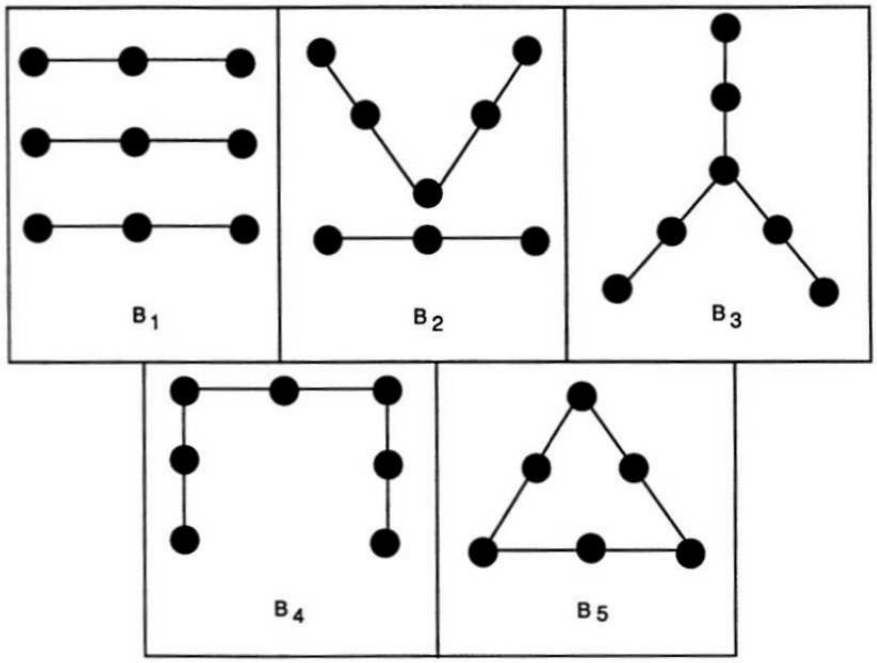

Fig. 1. The Three Line Configurations.

It is easily determined that

$$\begin{aligned}
b_1 &= v(v-1)(v-3)(v-7)(v^2-19v+96)/1296,\\
b_2 &= v(v-1)(v-3)(v-7)(v-9)/48,\\
b_3 &= v(v-1)(v-3)(v-5)/48,\\
b_4 &= 9a_1 = v(v-1)(v-3)(v-7)/8,\\
b_5 &= 4a_2/3 = v(v-1)(v-3)/6.
\end{aligned}$$

2 Four-line configurations

There are 16 four-line configurations and these are shown in Figure 2 where they are denoted by $C_1, C_2, \ldots, C_{16}$. It happens that the constant four-line configurations are C_4, C_7, C_8, C_{11} and C_{15}, and that all the others are variable. We deal with the constant ones first.

The C_4 configurations can be counted by adjoining an extra line to B_2. The extra line passes through the common vertex but not through any of the

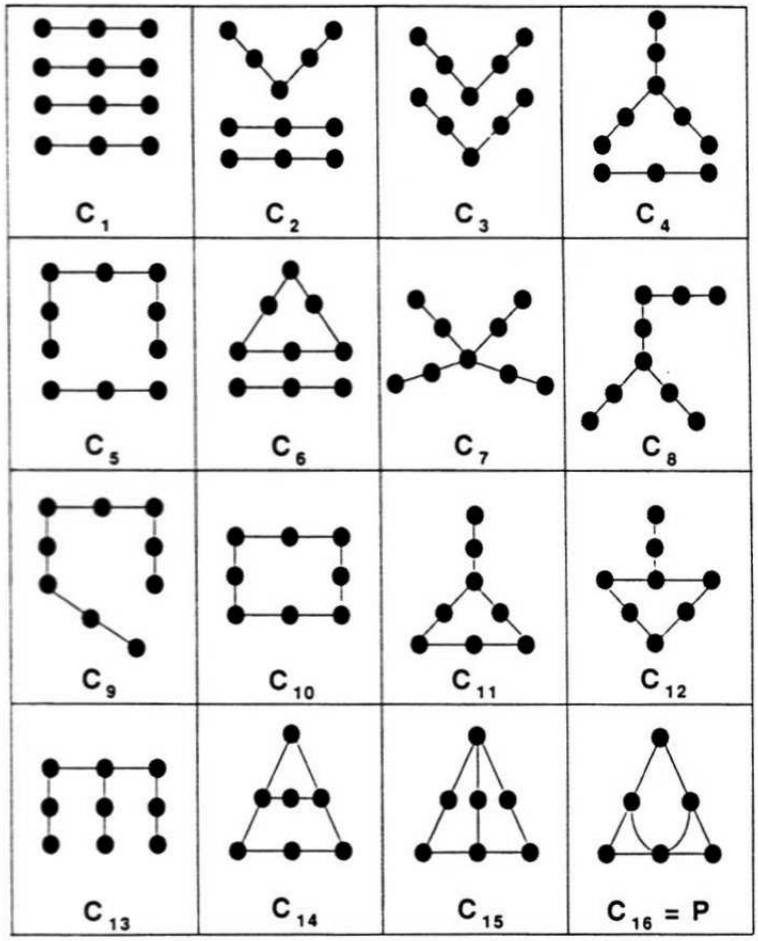

Fig. 2. The Four Line Configurations.

three points on the single line component, thus five line choices are excluded. The resulting C_4 configuration can be arrived at in three ways. Hence

$$c_4 = b_2 \left[\frac{v-1}{2} - 5 \right] /3 = v(v-1)(v-3)(v-7)(v-9)(v-11)/288.$$

For C_7 we choose the common point and then select any quadruple of distinct lines through that point. This gives

$$\begin{aligned} c_7 &= v \left[\left(\frac{v-1}{2} \right) \left(\frac{v-1}{2} - 1 \right) \left(\frac{v-1}{2} - 2 \right) \left(\frac{v-1}{2} - 3 \right) /24 \right] \\ &= v(v-1)(v-3)(v-5)(v-7)/384. \end{aligned}$$

To count C_8 configurations we start with B_4 and adjoin a new line to either one of the two common points. In each case the number of choices for the new line is $((v-1)/2)-4=(v-9)/2$. But each C_8 configuration arises in two ways, so

$$c_8 = 2b_4(v-9)/4 = v(v-1)(v-3)(v-7)(v-9)/16.$$

In the case of C_{11} we can form the configuration by adding a new line through any one of the three common points of B_5. In each case there are $((v-1)/2)-3=(v-7)/2$ choices for the new line. Hence

$$c_{11} = 3b_5(v-7)/2 = v(v-1)(v-3)(v-7)/4.$$

For C_{15} we start with the B_5 configuration and add a line through a common point and the opposite noncommon point. This can be done in three ways but each resulting C_{15} configuration can itself be obtained in three ways. Therefore

$$c_{15} = b_5 = v(v-1)(v-3)/6.$$

We must now deal with the nonconstant configurations. The configuration C_{16} is usually known as the Pasch configuration or quadrilateral. We will use P and p rather than C_{16} and c_{16} to denote it and its number of occurrences in an STS(v). We count C_{14} configurations by taking a B_5 configuration, choosing two of the three noncommon points and adding the line which joins them. This results in either a C_{14} or a P configuration. When P results from such a choice, all three choices of the two noncommon points give rise to the same P. Moreover, any of the four lines can be deleted from a P to form a B_5. Thus each P is counted 12 times. Furthermore, each C_{14} can be obtained from two distinct B_5 configurations and so each C_{14} is counted twice. Hence

$$2c_{14} + 12p = 3b_5,$$

and so

$$c_{14} = \frac{v(v-1)(v-3)}{4} - 6p.$$

It is convenient to count C_{12} configurations next. To do this we take a B_5 configuration and choose one of the three noncommon points. We then add any line through this point which does not pass through the common vertex of the B_5. The resulting configuration is either a C_{12} or a C_{14} or

a P configuration. There are three choices for the noncommon point and $(v-1)/2-2$ choices for the new line. Each C_{12} arises in only one way. However, when a C_{14} arises then the choice of the other noncommon point on the new line will give rise to the same C_{14} and, moreover, each C_{14} can be obtained from two distinct B_5 configurations. When a P arises then the choice of either of the other two points on the new line will give the same P and, in addition, each P can be obtained from four distinct B_5 configurations. Therefore

$$c_{12} + 4c_{14} + 12p = 3b_5(v-5)/2.$$

Hence

$$c_{12} = \frac{v(v-1)(v-3)(v-9)}{4} + 12p.$$

Next we count C_{13} configurations. To do this we first select a B_4, take a new point and add the line joining this point to the noncommon point on the line with two common points. The resulting configuration is either a C_{13} or a C_{12}. There are $(v-7)$ choices for the new point. Each C_{13} arises in six ways and each C_{12} arises in two ways. Hence

$$6c_{13} + 2c_{12} = b_4(v-7)$$

and so

$$c_{13} = \frac{v(v-1)(v-3)(v^2-18v+85)}{48} - 4p.$$

To count C_{10} configurations we again select a B_4. We consider the two lines of the B_4 which each contain only one common point. On each of these lines we select one of the two noncommon points and add the line joining them. This can be done in four ways. The resulting configuration is either a C_{10} or a C_{14}. Each C_{10} can be obtained from four distinct B_4 configurations and each C_{14} from two distinct B_4 configurations. Hence

$$4c_{10} + 2c_{14} = 4b_4.$$

This gives

$$c_{10} = \frac{v(v-1)(v-3)(v-8)}{8} + 3p.$$

We count C_9 configurations using B_4 again. We select a B_4 and any new point. We then add the line joining the new point to any one of the four noncommon points on the two lines of the B_4 which each contain only one

common point. There are $(v-7)$ choices for the new point and four choices for the new line. The resulting configuration is one of $C_9, C_{10}, C_{11}, C_{12}$. Each C_9 arises in four ways from a B_4 and a new point. When a C_{10} arises then two of the four line-choices coincide; moreover, each C_{10} arises in four ways from a B_4 and a new point. Each C_{11} and C_{12} arise in two ways from a B_4 and an extra point. Hence

$$4c_9 + 8c_{10} + 2c_{11} + 2c_{12} = 4b_4(v-7).$$

This gives

$$c_9 = \frac{v(v-1)(v-3)(v-9)^2}{8} - 12p.$$

To count C_6 configurations we return to B_5. Firstly we select a B_5, then we choose two new points and add the line joining them. The two new points can be selected in $(v-6)(v-7)/2$ ways. The resulting configuration is one of C_6, C_{11}, C_{12}. Each C_6 arises in three ways from such a choice but each C_{11} and each C_{12} arise only in one way. Therefore

$$3c_6 + c_{11} + c_{12} = b_5(v-6)(v-7)/2.$$

Hence

$$c_6 = \frac{v(v-1)(v-3)(v-9)(v-10)}{36} - 4p.$$

To count C_5 configurations we first select a B_4, then choose two new points and add the line joining them. The two new points can be chosen in $(v-7)(v-8)/2$ ways. The resulting configuration is one of C_5, C_8, C_9, C_{13}. Each C_5 arises from two choices of the two additional points. Each C_8 and each C_9 arises from two choices of the two additional points. Each C_{13} arises from three choices of the two additional points. Therefore

$$3c_5 + 2c_8 + 2c_9 + 3c_{13} = b_4(v-7)(v-8)/2.$$

This gives

$$c_5 = \frac{v(v-1)(v-3)(v-9)(v^2-20v+103)}{48} + 12p.$$

We count C_3 configurations by selecting a B_2 and any new point. We join the new point to any one of the three points on the single-line component of the B_2. The point can be selected in $(v-8)$ ways and the line selected in

three ways. The resulting configuration is one of C_3, C_8, C_9. Each C_3 arises in eight ways, each C_8 in one way and each C_9 in two ways. Hence

$$8c_3 + c_8 + 2c_9 = 3b_2(v-8).$$

This gives

$$c_3 = \frac{v(v-1)(v-3)(v-9)^2(v-11)}{128} + 3p.$$

We count C_2 configurations again using B_2. This time we select two new points and add the line joining them. The new points can be chosen in $(v-8)(v-9)/2$ ways. The resulting configuration is one of C_2, C_3, C_4, C_5. Each C_2 arises from six selections of a B_2 plus two points. Each C_3 arises four times, each C_4 arises three times and each C_5 arises twice. Hence

$$6c_2 + 4c_3 + 3c_4 + 2c_5 = b_2(v-8)(v-9)/2.$$

This gives

$$c_2 = \frac{v(v-1)(v-3)(v-9)(v-10)(v^2-22v+129)}{576} - 6p.$$

Finally we count C_1 configurations. We select a B_1, choose two new points and add the line joining them. The two new points can be chosen in $(v-9)(v-10)/2$ ways. The resulting configuration is either a C_1 or a C_2. Each C_1 configuration arises in 12 ways from a B_1 and two extra points. Each C_2 arises in two ways. Hence

$$12c_1 + 2c_2 = b_1(v-9)(v-10)/2.$$

This gives

$$c_1 = \frac{v(v-1)(v-3)(v-9)(v-10)(v-13)(v^2-22v+141)}{31104} + p.$$

We summarize the results of this section below. The total number of four-line configurations in an STS(v) is $\binom{v(v-1)/6}{4}$ and the dedicated reader can check that the entries below sum to this total.

$$\begin{aligned}
c_1 &= v(v-1)(v-3)(v-9)(v-10)(v-13)(v^2-22v+141)/31104 + p \\
c_2 &= v(v-1)(v-3)(v-9)(v-10)(v^2-22v+129)/576 - 6p \\
c_3 &= v(v-1)(v-3)(v-9)^2(v-11)/128 + 3p
\end{aligned}$$

$$
\begin{aligned}
c_4 &= v(v-1)(v-3)(v-7)(v-9)(v-11)/288 \\
c_5 &= v(v-1)(v-3)(v-9)(v^2-20v+103)/48+12p \\
c_6 &= v(v-1)(v-3)(v-9)(v-10)/36-4p \\
c_7 &= v(v-1)(v-3)(v-5)(v-7)/384 \\
c_8 &= v(v-1)(v-3)(v-7)(v-9)/16 \\
c_9 &= v(v-1)(v-3)(v-9)^2/8-12p \\
c_{10} &= v(v-1)(v-3)(v-8)/8+3p \\
c_{11} &= v(v-1)(v-3)(v-7)/4 \\
c_{12} &= v(v-1)(v-3)(v-9)/4+12p \\
c_{13} &= v(v-1)(v-3)(v^2-18v+85)/48-4p \\
c_{14} &= v(v-1)(v-3)/4-6p \\
c_{15} &= v(v-1)(v-3)/6 \\
c_{16} &= p
\end{aligned}
$$

3 Concluding remarks

Perhaps the most surprising outcome of the above results is that the number of occurrences of any one of the variable four-line configurations, in particular the Pasch configuration, determines the number of occurrences of all of the others. The manner in which they are distributed within an STS(v) is completely irrelevant. This raises a number of very interesting questions. Anti-Pasch STS(v)s have been investigated [1, 4] as avoidance results. It now follows that these systems are extremal with respect to all other (variable) four-line configurations. At the opposite extreme will be systems which contain the maximum number of Pasch configurations, which we call maxi-Pasch STS(v). Elementary counting, or consideration of C_{14} given above, gives max(p)= $v(v-1)(v-3)/24$ but this can only be achieved in the projective spaces. i.e., $v = 2^n - 1$ [7]. The paper [7] also contains results on the maximum number of Pasch configurations in known STS(v) for other small admissible values of v. However these results are only "best so far" and construction of truly maxi-Pasch STS(v) for other values of $v \neq 2^n - 1$ would be of interest.

Secondly, the same questions, and techniques for answering them, can be applied to many other combinatorial structures. Restricting attention to designs with block size 3, we may first enquire about partial Steiner triple

systems PSTS(v). Clearly here no general results can apply but results for *maximal* PSTS(v) may very well exist and it would be interesting to discover how these compare with those for STS(v). Results for triple systems with general λ, TS(v, λ), would also be of interest.

Further, and of greater importance is the question: What is a basis for five-line configurations in a Steiner triple system? We conjecture that, in addition to v, the numbers of each of the two tightest configurations (on seven points) form a basis. These configurations are illustrated in Figure 3 and are commonly called the *mitre* and the *mia*. We do have some empirical evidence for the conjecture for which we thank Peter Danziger of the University of Toronto for computing help. The results obtained by counting the numbers of each five-line configuration in a "random" sample of STS(19)s were assembled in a matrix. Using a computer algebra package, the rank of this matrix was shown to be 3 as expected. [Added in Proof - This conjecture has now been confirmed and the explicit formulae determined.]

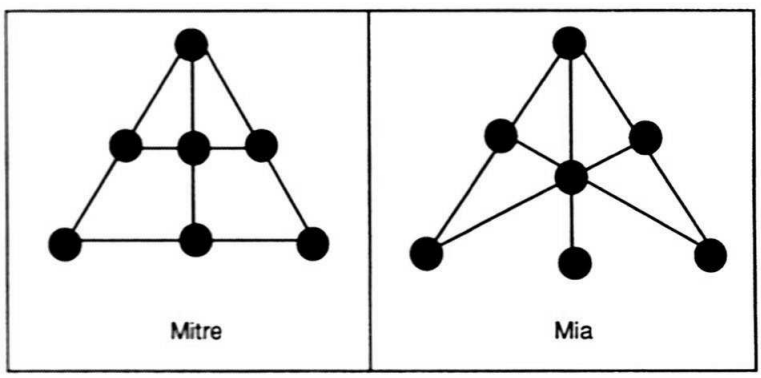

Fig. 3. Mitre and Mia.

Finally, we can enquire about the situation for general n-line configurations in a Steiner triple system. Observe that configurations consisting of n lines and $n+2$ points arise naturally in the results. The significance of such configurations has also been identified elsewhere [2] in relation to avoidance problems. It would be tempting to speculate that for each n, such configurations together with any constant configuration form a basis. However in general these do not encompass all the most economical configurations, e.g., deleting a line from a Fano plane gives a 6-line, 7-point configuration, and surely these configurations too must be of importance. However we can

recover a similar conjecture as follows. Following [2], define an *Erdös configuration* of order n, $n > 1$, in a Steiner triple system to be any configuration consisting of n lines and $n+2$ points, which contains no subconfiguration of m lines and $m+2$ points, $1 < m < n$. We then make the following conjecture.

A basis for n-line configurations in a Steiner triple system consists of any constant n-line configuration, e.g., the "star", n lines all intersecting in a common point, together with all Erdös configurations of order m, $1 < m \leq n$.

The result is trivially true for $n = 1, 2, 3$ and we have proved it for $n = 4$. For $n = 5$, if the results of the empirical investigations are indeed true, then the conjecture would follow since the number of mias in an STS(v) is three times the number of quadrilaterals. In general however the conjecture may be difficult to prove.

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