Infinite Antichains of Combinatorial Structures
Case for Support

Robert Brignall

Previous Track Record

Robert Brignall (PI) is a Lecturer in Combinatorics at The Open University. After being an undergraduate at the University of Cambridge, he received his PhD in 2007 from the University of St Andrews. From 2007–2010, he was a Research Fellow at the Heilbronn Institute for Mathematical Research – a joint venture between the University of Bristol and the Government Communications Headquarters (GCHQ). During this appointment, the PI worked for half of his time on his own research, and the other half on research directed by the Heilbronn Institute. Although the latter research cannot be published, and despite finishing his PhD less than 4 years ago, he still has 9 papers published and 2 submitted for publication.

Since starting his research career, he has been centrally involved in a programme of research to advance the structure theory of permutation classes. He has ongoing international collaborations with researchers in the USA and New Zealand. Within the UK he continues to work with combinatorialists at the University of Bristol (where in 2009 he co-organised a workshop on graph theory), and at the University of St Andrews. His standing in the international permutation patterns community is demonstrated by his membership of the organising committees for the 2010 and 2011 Permutation Patterns Conferences (in Dartmouth College and California Polytechnic University, respectively) and of the local committee for the planned 2012 meeting in Edinburgh, and his co-editorship of the proceedings for the 2010 conference.

The PI has also always sought to place the study of permutation patterns in a "bigger picture": primarily this involves seeking parallels and connections with other combinatorial structures (most notably graphs), but it also means finding connections with other areas of mathematics, for example model theory (through the study of relational structures).

The research in this project builds on the PI's highly-developed knowledge and intuition of the structure of infinite antichains of permutations. Progress is expected to be made within the study of permutation classes, but the primary aim is to use this existing knowledge to build a clearer unified picture of the theory of infinite antichains for combinatorial structures.

The PI anticipates that this proposal will provide a firm foundation on which his research career can continue to develop: there is a longer-term programme of internationally important research here. This will enable the PI to maintain existing collaborations and establish new ones, and to create an environment at the Open University that can attract future funding and research students.

The Open University has a long-standing history in combinatorial research, made highly visible through the Winter Combinatorics Meeting which has run at the university for the past 12 years. In 2011, the PI was involved in the organisation of this meeting, and this is anticipated to continue in the future. The combinatorics group, counting 13 members from postgraduates to emeritus professors, provides the ideal environment for this project. Individuals have a diverse range of interests, including (pertinent to this proposal) various aspects of graph theory, and the connections between model theory and infinite designs.

Collaborators

In addition to the appointment of an RA to this project, the proposal includes support to enable collaboration with two other researchers:

Vincent Vatter is an Assistant Professor at the University of Florida, USA. His research interests relate very strongly to the PI's – he has 29 papers in combinatorics, of which 5 are with the PI. This ongoing collaboration makes him a natural addition to this project, and the research that has already come from this partnership lends evidence to support the anticipated success of the current project.

Vadim Lozin is an Associate Professor at DIMAP, University of Warwick. With over 80 publications, in recent years he has devoted considerable attention to the study of infinite antichains and well-quasi-order in graphs, and also the connections with permutations. Thus his expertise complements the PI's, and Lozin and the PI have together already identified the potential for cross-fertilisation between the study of permutations and graphs.

Relevant Work

Progress in the structural theory of permutation patterns in the last 7 years has led to numerous breakthroughs. One such has been in the study of *simple permutations* — in a series of three papers the PI, in collaboration with Huczynska and Vatter [8, 9] and with Ruškuc and Vatter [10], established three wide-ranging results: first, a Ramsey-type result for simple permutations themselves; next, an application of this structural result to describe a decision procedure to determine whether a finitely based permutation class contains only finitely many simple permutations; finally, within such permutation classes, a general framework for a wide variety of enumerative problems. These three papers, together with the PI's survey article [6], firmly established the important role simple permutations play in a general structure theory, and consequently all these papers have already been cited numerous times by several authors.

The PI's unique expertise in the study of infinite antichains of permutations dates back to his first year as a PhD student, from which came the results of [5]. In [4], the PI developed a more general construction for "fundamental" infinite antichains of permutations than has previously existed for any combinatorial structure, as well as describing a major advance in showing when permutation classes do not contain infinite antichains.

Other contributions to the study of permutation classes include applying the developing structure theory of permutation classes to enumerative problems, as exemplified by [7] and research carried out during a total of eight weeks spent on three research visits to the University of Otago in Dunedin, New Zealand [1–3].

In the broader context of combinatorial structures (and, in particular, relational structures), the PI together with Ruškuc and Vatter [11] used the expertise and intuition gained in the study of simple permutations to prove results across a wide range of combinatorial structures. This work demonstrates not only the PI's growing expertise in crossing the boundaries between different structures, but also the enormous potential for research in this wider context.

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Description of Proposed Research

Background

Three celebrated results in combinatorics show that in certain quasi-orders of combinatorial structures, infinite antichains do not exist – Higman's Theorem on ordering by divisibility [23], Kruskal's Tree Theorem [27] and Robertson and Seymour's Graph Minor Theorem [33].

For many other structures and orders, however, infinite antichains do exist, and in some cases they exist in abundance. Driven by the widespread impact of results such as the Graph Minor Theorem, the study of how and when infinite antichains occur has drawn continual attention since Higman [23] effectively founded the area in the 1950s.

In addition to "traditional" combinatorial structures such as graphs and tournaments, a relative newcomer to this study of infinite antichains are permutations equipped with the "containment" ordering. Despite the systematic study starting less than 10 years ago, the emergence of graphical methods to analyse permutations has led to fast progress, most recently and significantly by the PI [4]. To date, progress for individual combinatorial structures has only rarely been used to make connections with others. Now that the study of permutations has advanced so far, there is significant potential for cross-fertilisation.

This proposal seeks to take a unified view to explore the phenomenon and structure of infinite antichains of general combinatorial structures by drawing on the connections between the various objects, and study how these quasi-orders differ from arbitrary ones.

The rest of this background is organised as follows. First, we give an overview of well-quasi-order for combinatorial structures, and the motivation behind this study. Next, we narrow our view slightly to "relational structures", a level of generality that enables comparisons to be made between several prominent objects. We discuss the differences between quasi-orders of combinatorial structures and arbitrary ones, and describe some of the theory of infinite antichains that exists in this latter context. Finally, we introduce the permutation containment ordering and survey the recent antichain constructions.

Orderings on Combinatorial Structures

The term *combinatorial structure* can potentially be used to refer to a wide variety of objects: graphs, tournaments, posets, permutations, matroids, codes, designs, and so on. For each type of structure, there are typically a variety of *quasi-orders*: reflexive, transitive binary relations, such as the subgraph ordering for graphs. An *antichain* in such an ordering is a set of objects for which no pair of elements are comparable. For example, in the subgraph ordering this means no graph in an antichain is the subgraph of another.

The major results of [23, 27, 33] show that particular orders contain no infinite antichains: they are *well-quasi-ordered* (wqo, for short). For orderings which are not wqo, a lot of attention has been devoted to instances of the following general question:

Question 2.1. *Given some finite description of a downset in an ordering on some combinatorial structure, is it possible to decide whether the downset is wqo?*

A set D is a *downset* of a quasi-order if $a \in D$ and $b \leq a$ implies $b \in D$. These are prominent because many natural sets of objects (e.g. planar graphs) are actually downsets. Showing that some downset is wqo indicates a "regularity" in its structure, typically leading to enumerative and algorithmic results [14, 15].

Question 2.1 for the subgraph ordering is effectively answered: Ding [19] showed that a downset in this ordering is wqo if and only if it has finite intersection with the set of cycles and the "split-end" graphs – see Figure 1. However, for most structures and orderings, only partial results exist.

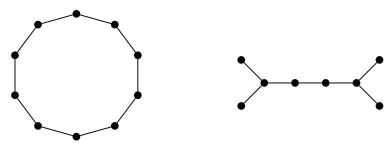

Figure 1: Typical elements of the antichain of cycles (left) and the "split end" antichain (right) under both the subgraph and induced subgraph orderings.

Downsets are often described by their set of *minimal forbidden elements*: elements q of the quasi-order such that $q \notin D$, but $b \in D$ for every $b < q$. One common interpretation of Question 2.1 is to consider downsets whose set of minimal forbidden elements is finite.

Relational Structures

Many objects can be described as *relational structures*: a set of points (or vertices) with one or more *relations*. For example, a graph is a set of vertices with a binary symmetric relation (the edges). The study of relational structures provides links with model theory, where wqo has also been studied intensely – see Fraïssé's book [21]. For our purposes, viewing different objects as relational structures can provide the "right" level of generality to build connections between them.

There is a natural *induced substructure* ordering on relational structures that unifies many important quasi-orders on specific objects. As well as the induced subtournament order [17], we identify two in particular: permutation containment (see later), and the induced subgraph ordering. For the latter, downsets are called *hereditary properties*, and the study of wqo remains an active area of research (see e.g. [18–20]). One aim of this proposal is to translate the wqo results of individual structures to this more general setting.

Quasi-Orders and Fundamental Antichains

We may ask Question 2.1 for abstract quasi-orders with no underlying combinatorial structure. To be wqo, these must also be *well-founded* – i.e. contain no infinite descending chain. The lack of structure in such orders enables the creation of pathological infinite antichains, in stark contrast to those of combinatorial structures which seem to be better behaved. This proposal aims to identify specific features that explain this distinction.

Some general results do exist, however. For Question 2.1 we can restrict our view to infinite antichains that are in some sense "elementary". Several authors have adopted different notions: minimal [17, 22], fundamental, trim and maximal [28] and canonical [20]. We discuss only two here, but there is a clear requirement to consolidate and relate these concepts. Following Gustedt [22], an infinite antichain A is *minimal* if it is minimal under the following order on infinite antichains: $A \preceq B$ if for every $b \in B$ there exists $a \in A$ with $a \leq b$. Importantly:

Proposition 2.2 (Gustedt [22], following [30]). *Let Q be a well-founded quasi-order. For each infinite antichain A in Q, there exists a minimal antichain B with $B \preceq A$.*

Thus minimal antichains satisfy this "elementary" requirement, but often it is easier to consider the weaker notion of *fundamental* antichains [28], which in practice often have neater descriptions.

Proposition 2.3 (Cherlin and Latka [17]). *For a well-founded quasi-order Q and integer k, there exists a finite set Λ_k of fundamental antichains such that any downset of Q, defined by at most k minimal forbidden elements, is wqo if and only if it has only finite intersection with every antichain in Λ_k.*

The strength of this result lies in the evidence it provides that the antichains of combinatorial structures are "nice": Λ_1 is known for tournaments [17] and permutations [13] – see Figure 2 for the permutation case. For graphs, Ding's result [19] shows that for the subgraph ordering Λ_i is known for all i (it consists of the two antichains in Figure 1), while not even Λ_1 is known for the induced subgraph ordering.

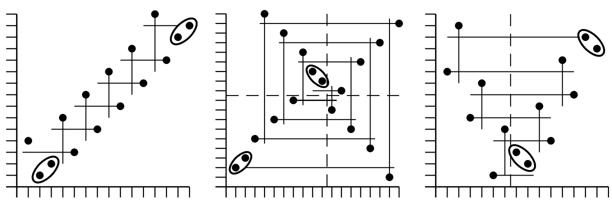

Figure 2: Typical elements of the three fundamental permutation antichains in Λ_1 (up to symmetry). From left to right: the increasing oscillating antichain, Widdershins and the antichain V.

Permutation Classes

Writing permutations in one-line notation, we say that σ is *contained* in π if there exists a subsequence of π that is *order isomorphic* to σ, i.e. the pairwise comparisons of the subsequence of π are the same as the corresponding comparisons in σ. In this ordering, downsets are called *permutation classes*, and if B is the set of minimal forbidden elements of a class $\mathcal{C}$, we write $\mathcal{C} = \text{Av}(B)$. The study of permutation classes dates back to Knuth [25], who showed that the set of permutations which can be sorted by passing the elements through a stack is precisely Av(231). In addition to sorting machines, connections exist with the computation of Schubert varieties and aspects of bioinformatics, such as genome rearrangement (for example by reversals or transpositions).

Infinite antichains of permutations have been known to exist since the 1970s [32], but the direct consideration of Question 2.1 in this context started more recently [13, 28, 29]. Antichains have also been central to the recent structure theory that has begun to emerge, being used directly by Vatter to find the smallest growth rate where there exist uncountably many permutation classes [34], and a point above which every real number is the growth rate of a permutation class [35].

Grid Classes and Antichain Constructions

The most promising progress towards Question 2.1 for permutation classes comes from the study of grid classes. A *grid class* $\mathrm{Grid}(\mathcal{M})$ is defined by a matrix $\mathcal{M}$ whose entries are permutation classes (allowing the empty class $\emptyset$). The permutations in $\mathrm{Grid}(\mathcal{M})$ are those that, pictorially, can be divided into cells so the points in each cell represent a permutation of the class in the corresponding cell of $\mathcal{M}$. Note that grid classes are themselves permutation classes.

The connection between grid classes and wqo was first identified by Murphy and Vatter [29], who solved Question 2.1 for *monotone* grid classes (where every non-empty cell of the matrix is either Av(21) or Av(12)). In [4], the PI proves two important advances. The first unifies and generalises the results of [12] and [29]:

Theorem 2.4. *For a matrix $\mathcal{M}$ whose cells are permutation classes containing only finitely many simple permutations,* $\mathrm{Grid}(\mathcal{M})$ *is wqo if the graph is acyclic, and each component has at most one non-monotone cell.*

We build the *graph* of a gridding matrix $\mathcal{M}$ by taking a vertex for every non-empty cell, and placing an edge between two cells that share a row or column, with all cells in-between being empty. Infinite antichains that "wind round" the cells in a cycle (as in the Widdershins antichain in Figure 2) have been known since [29]. In [4] the PI presents a new way to build infinite antichains that includes and simplifies the earlier constructions:

Theorem 2.5. *Let $\mathcal{M}$ be a gridding matrix. Then* $\mathrm{Grid}(\mathcal{M})$ *is not wqo if in the graph of $\mathcal{M}$ there exists a cycle, or a component containing two or more non-monotone-griddable cells.*

Following [24], a *non-monotone-griddable* cell is a permutation class which is not the subclass of a monotone grid class. The antichain construction starts with an infinite *grid pin sequence* $p_1, p_2, \ldots$, extending a concept used by the PI in [5, 8, 10]. Roughly, it builds a permutation by sequentially placing points in a grid so that each point relates in a unique way to its predecessors. See Figure 3 for an illustration, and [4] for the formal definition.

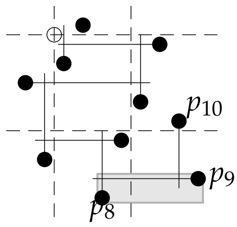

Figure 3: A grid pin sequence on the 3×3 grid.

From these infinite sequences, all but finitely many elements of every known fundamental permutation can be built. First, take permutations corresponding to initial segments $p_1, \ldots, p_n$ for varying n, and apply an *anchoring* construction to the first and last points: either "inflate" them to 12 or 21, or "tie" them together. These permutations are incomparable because a shorter prefix can only embed in a longer one as a contiguous sequence, but the anchoring prevents this. The target antichain is then obtained by performing "grid symmetries": permutations and reflections of the points in rows or columns of the gridding.

Academic Impact

This research will both advance and unify several strands of research in individual structures. Consequently it is expected to interest researchers within all of these strands, and more generally in model theory. This proposal represents the start of a long-term programme, the impact of which is expected to continue well beyond the end of this project. To maximise this impact, a survey article will be published at the start of the project, new research will be submitted to quality mathematical journals (with preprints published on the arXiv), and results will be presented at seminars and major international conferences.

A structure theory for infinite antichains will enable further progress in the structural study of downsets of specific combinatorial objects. From this, enumerative results typically follow, particularly in asymptotics – for example growth rates of permutation classes, or speeds of hereditary properties [15]. As seen by the impact of the Graph Minor Theorem, results in wqo theory are also important in algorithmics – see [14] for a survey. For example, establishing that some downset is wqo typically facilitates a polynomial-time algorithm for the *membership problem*: is an object a member of the downset?

Being newly appointed at the Open University, this project will assist the PI to integrate with the existing combinatorics research group, through running informal reading groups and the local presentation of results.

Starting with the completion of the survey article and continuing throughout the project, close collaboration is expected with Vincent Vatter (University of Florida), particularly on Objectives 1, 3 and 4 (see below). Vatter and the PI have a successful history of collaboration, and the intersection of research interests makes him an essential addition to this proposal.

Objectives 2 and 3 provide an ideal opportunity to build upon informal discussions and collaborate with Vadim Lozin (University of Warwick), with the close geographical proximity enabling frequent visits. Lozin's interests in wqo complement the PI's own, and being already fully aware of the potential for cross-fertilisation he is the ideal researcher to add breadth to and engage fully with the project.

Research Hypothesis and Objectives

With a view to Question 2.1, this project aims to unify and advance the theory of infinite antichains for combinatorial structures. The evidence provided by known constructions and results for fundamental antichains has led the PI to develop the following hypothesis, which underpins the proposed research:

Main Hypothesis. Fundamental antichains of combinatorial structures have a *spine*: an infinite structure defined by a sequence of points $p_1, p_2, \ldots$, where each p_i is unique in the way it relates to its predecessors $p_1, \ldots, p_{i-1}$. This spine defines all but possibly finitely many elements of the antichain by taking finite prefixes, and then anchoring the first and last points.

The main evidence supporting this hypothesis is the PI's description of all known permutation antichains; further connections are discussed later. If this or a similar hypothesis were true, this would represent a major step towards a general answer to Question 2.1. Never before has there been a sufficiently general construction to allow such a hypothesis to be considered, and consequently the work outlined in this proposal differs substantially from any previous programmes of research into wqo. It is the recent major advances in permutation antichain constructions that make this project so timely. Such progress has not been possible before, but for the first time the tools are now available to begin this line of enquiry.

Working towards this hypothesis, we identify four objectives that comprise this proposal:

Objective 1. *Determine the relationship between infinite antichain constructions and grid pin sequences.*

Objective 2. *Advance the theory of wqo in specific structures, and particularly graphs, by directly converting antichain constructions for permutations.*

Objective 3. *Through cross-fertilisation between structures, advance the general theory of wqo for relational structures.*

Objective 4. *Identify properties of infinite antichains in orderings of combinatorial structures that distinguish them from those in arbitrary quasi-orders.*

Programme and Methodology

The first task is for the PI and Vatter to complete a survey article on infinite antichains of combinatorial structures, setting the unified viewpoint that the rest of the research will take. With this completed within the first month of the proposal, it will form a crucial part of the induction material for the RA, as well as a solid platform for the ensuing research.

Objective 1

Following the induction programme, the PI and RA will work together closely on Objective 1 to enable the RA to start developing intuition on the key concepts underlying the later research.

Although the use of grid pin sequences has been shown to generalise all existing antichain constructions, it is possible that these sequences offer too much freedom – Theorem 2.5 uses essentially only one family of these, so a first step is to consider others and establish a general proof technique to state when infinite antichains can be built. Another key concept here are the anchoring techniques: given the spine of a fundamental antichain, do there exist other anchoring methods?

In the other direction, the following important question will feed into Objective 4.

Question 2.6. *Do there exist fundamental antichains of permutations which cannot be constructed from grid pin sequences?*

If such antichains do exist, is this because the Main Hypothesis is wrong, or are grid pin sequences simply not the right construction? Finding different constructions would give invaluable insight and open up new strands of research. On the other hand, if the answer to this question is "no" this would be an extremely far-reaching result, giving strong evidence in support of the Main Hypothesis.

Objective 2

Anticipating the recruitment of an RA with a background in graph theory, Objective 2 commences soon after the start so that their expertise can be brought to bear at an early stage. Working at first with the PI and throughout in collaboration with Lozin, the RA will gradually assume control of this objective to choose the lines of enquiry.

Due to the intensity of research into wqo for graphs, the following conversion from permutations is a clear starting point: For a permutation π on n points, define a graph with n vertices (labelled $1, \ldots, n$ during the construction), with $i \sim j$ in the graph if and only if either $i < j$ and $\pi(i) > \pi(j)$, or $j < i$ and $\pi(j) > \pi(i)$ – these are precisely the *permutation graphs*. The conversion preserves containment: σ contained in π implies G_σ is an induced subgraph of G_π, but the reverse is not quite true (see [16]). One simple task is to describe the conditions under which this translation preserves antichains. Typically, they do map across – for example, the increasing oscillating antichain (Figure 2) maps to the split end antichain (Figure 1).

Some immediate consequences arise from this conversion of antichains. In 1992, Ding [19] conjectured that the hereditary property consisting of all permutation graphs which do not contain paths or their complements on 5 or more vertices is wqo, but the "Widdershins" antichain in Figure 2 can be converted to provide a counterexample.

Objective 2 also has a more involved component: to apply new and existing techniques for specific structures to build upon the foothold that this conversion provides. One aim here will be to find Λ_1 for the induced subgraph ordering, and Lozin's and the RA's expertise are vital to add to the PI's own experience.

Objective 3.

With Objective 2 advancing, the investigation team will be developing significant knowledge about how infinite antichains are built in different structures. Taking the expertise and intuition of the PI and the RA, together with Vatter (during his visit in Summer 2012) and Lozin, this phase of the project will begin by cross-fertilising between structures, including feeding back into the work for permutations. Building this theory for individual structures puts us in a strong position to work towards a more general theory of wqo for relational structures. A primary aim here is to describe a "generic" construction technique for fundamental antichains.

Objective 4 and the Main Hypothesis

Drawing together the results from the first three objectives, Objective 4 seeks to answer a deep question about the underlying properties of quasi-orders of combinatorial structures, thus building evidence for or against the Main Hypothesis. This objective will involve both the RA and the PI, and Vatter during the proposed visit to Florida in late Autumn 2012. We mention here two conjectures that represent the type of research to be considered; other similar conjectures exist.

Conjecture 2.7 (Murphy [28])**.** *Let A be a fundamental antichain of permutations. Then there exist at most finitely many lengths n such that A has two or more permutations of length n.*

Despite being stated for permutations, we will consider this conjecture more generally. It is not true for arbitrary quasi-orders: in the poset whose Hasse diagram is a rooted ternary tree, there exists a fundamental antichain that has two elements at every level except the lowest.

The second conjecture, first made in 1972, is both relevant to Objective 4 and strongly related to the Main Hypothesis, and again can be applied more generally. For graphs G, H whose vertices are coloured with a palette of n colours, we write $G \leq_n H$ if G is embeddable as a vertex-colour-preserving induced subgraph of H. A (monochromatic) hereditary property of graphs $\mathcal{G}$ is *n-wqo* if the set consisting of all n-colourings of graphs from $\mathcal{G}$ is wqo when viewed as a downset in the $\leq_n$-ordering.

Conjecture 2.8 (Pouzet [31]; see also [21, 26])**.** *Let $\mathcal{G}$ be a hereditary property of graphs. Then $\mathcal{G}$ is 2-wqo iff $\mathcal{G}$ is n-wqo for all $n \geq 2$.*

From the spine of a monochromatic fundamental antichain, there is an anchoring technique to build a 2-coloured antichain: assign a second colour to the first and last points of each prefix. If our Main Hypothesis is true, this conjecture is essentially reduced to showing that having more than two colours does not enable any further anchoring methods.

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