ERC Starting Grant 2014
Research proposal [Part B2]

Section a: State-of-the-art and objectives

This proposal will explore three manifestations of structure – orderings (specifically, whether a suitably-defined class is well-quasi-ordered or not), parameters (giving some indication of structural complexity), and enumeration (where structure facilitates counting) – and the interplay between them. In particular, the primary objective of this proposal is the resolution of three distinct but related conjectures, all of which rely on a common increased understanding of structure. The motivation for this project comes from the consequences that such a structure theory would have in the study of graph algorithms.

The term *combinatorial object* can potentially be used to refer to a wide variety of objects, including tournaments, posets, matroids, codes, designs, and so on. The two primary types of object in this project are graphs and permutations, and the interaction between these. Nevertheless, it is expected that theory developed for graphs and permutations could be extended to other structures.

1 Orderings on objects, downsets, and infinite antichains

Many algorithmic problems on graphs ask whether a given graph G possesses certain substructures, such as a large set of vertices with no edges between them, or some obstruction to it being drawn in the plane in such a way that no pair of edges cross (i.e. is the graph planar?). For such questions, the ability to compare two graphs is key, and this motivates the study of partial (or quasi) orderings on them. Three examples of such orderings are the 'minor' relation, the subgraph ordering, and induced subgraphs.

Robertson and Seymour's celebrated Graph Minor Theorem [38] proves that in any infinite collection of graphs there exists one which is a minor of another. An *infinite antichain* in an ordered collection is an infinite set for which no pair of elements are comparable. Thus, the Graph Minor Theorem states that there are no infinite antichains of graphs when they are ordered by the minor relation, and so it is said to be a *well-quasi-order*. The consequences of both the statement of this result and the structure theory underlying it are extensive, particularly in the study of computational complexity and fixed parameter tractability (see, for example, [19, 25]).

Often, collections of graphs that satisfy specified properties form *downsets* within some ordering. That is, whenever a graph is in a downset, then so too are all the graphs that are 'less than' it in the ordering. A convenient way to describe such downsets is by the unique set of *minimal forbidden graphs*. For example, in the minor ordering the planar graphs are a downset defined by forbidding two graphs: the complete graph on 5 vertices, and the complete bipartite graph with 3 vertices in each part. That is, no planar graph contains either of these two graphs as a minor, and every graph which is not planar *must* contain at least one of them as a minor. A consequence of the Graph Minor Theorem is that every downset in the minor ordering has a *finite* collection of minimal forbidden graphs, and this is a key component in the algorithmic consequences mentioned earlier.

On the other hand, infinite antichains do exist in the subgraph and induced subgraph orderings – see Figure 1 – but this does not mean that every downset must contain one. In fact, Ding [24] showed that if a downset in the subgraph order does not have infinite intersection with either of the two antichains illustrated in Figure 1, then that downset is well-quasi-ordered.

The situation for the induced subgraph ordering is much less straightforward. We will refer to downsets in this ordering as *graph classes*, though we note the term 'hereditary property' is used widely. To determine whether a graph class is well-quasi-ordered or not, there are an arbitrarily large number of infinite antichains that may need to be considered. However, of the three graph orderings – minors, subgraphs, and induced subgraphs – the induced subgraph ordering is the only one that can handle

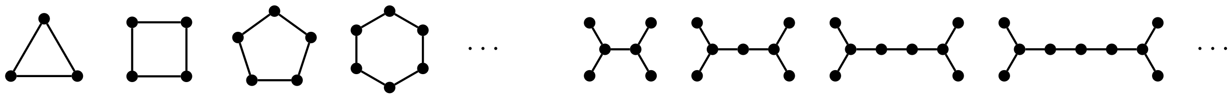


Figure 1: The first four graphs in two infinite antichains of graphs. Both are infinite antichains in the subgraph and induced subgraph orderings: no vertices or edges can be deleted from a larger graph to form a smaller one in either case.

collections of graphs which have a high density of edges. In order to derive algorithmic and other consequences in such collections, methods are needed to determine whether a graph class is well-quasi-ordered or not. Put precisely, we seek answers to instances of the following question:

Question 1.1. *Given a graph class defined by its minimal forbidden graphs, is it well-quasi-ordered?*

Since the set of minimal forbidden graphs of a class must form an antichain, for the induced subgraph ordering (and unlike downsets in the graph minor ordering) there exist graph classes which are defined by an infinite set of minimal forbidden elements (for example, every cycle is a minimal forbidden element of the graph class consisting of all trees). Algorithmically, this can be problematic: when seeking to develop algorithms which take the set of minimal forbidden graphs of a class as input, the finiteness of this set is important.

To prove that a class is well-quasi-ordered, a typical approach is to give a structural characterisation of the objects in the class which is amenable to one of the results from a standard 'toolkit'. Two of the most frequently used methods in this toolkit are Higman's lemma on ordering by divisibility [28], and modifications of Nash-Williams' 'minimal bad sequence' argument [34], but another approach is to use the stronger notion of *better quasi ordering*, also originally due to Nash-Williams [35] in establishing a short proof of Kruskal's Tree Theorem [31].

On the other hand, the development of general techniques to construct infinite antichains are more recent. The PI is on the cutting edge of this area, having described a generic construction for infinite antichains of permutations in [6], and having demonstrated how these methods can be applied to graph theory in [3].

In a sense, Question 1.1 can be viewed as an indicator for whether a given graph class has well-behaved structure (meaning it is well-quasi-ordered), or is intractably complicated. A recent survey by Cherlin [21] considers variants of Question 1.1 for several objects, and specifically asking whether there really is a 'dichotomy' between well-quasi-ordering and not, or whether this interface is subtler.

Regardless of the answer to this, however, instances where Question 1.1 can be answered positively typically provide a structural characterisation that has far-reaching consequences. The algorithmic implications of the Graph Minor Theorem are a case in point: the well-quasi-orderability of graph minors is important, but the underlying polynomial time algorithm to answer an instance of 'is a graph H a minor of another graph G?' needs the structural results.

Resolutions of Question 1.1 for other objects

In addition to the Graph Minor Theorem, there are two results of interest where analogues of Question 1.1 have been fully answered for other orderings or objects. The first is the result due to Ding [24] for the subgraph ordering, which was mentioned earlier.

The second is more recent: Atminas, Lozin and Moshkov [17] resolved the equivalent to Question 1.1 for downsets of words over a finite alphabet, equipped with the 'factor order' (so-called *factorial languages*). Importantly, they show that in order to determine whether a downset in this ordering is well-quasi-ordered, one need only consider the infinite antichains which exhibit a 'periodicity' in their structure, where the length of period that needs to be checked is bounded by a function of the size of the minimal forbidden factors of the downset.

While the factor order for words is a substantially less complicated ordering than the induced subgraph ordering, this result provides the first real evidence to support a widely-held belief that the only infinite antichains of graphs which would be needed to answer Question 1.1 should exhibit a similar 'periodicity' in their structure.

Infinite antichains of permutations

A relative newcomer to structural combinatorics and well-quasi-ordering are permutations equipped with the "containment" ordering. In this combinatorial (rather than algebraic) context, a permutation π of length n is viewed as a picture comprising the n points $(i, \pi(i))$, $(i = 1, 2, \ldots, n)$ in the plane. A permutation is then *contained* in another if points can be deleted from the picture of the larger permutation, and the axes rescaled in order to produce the picture of the smaller permutation.

This ordering forms a partial order on the set of all permutations, and it is strongly related to the induced subgraph ordering. Associated with each permutation π is a well-defined *permutation graph* G_π, and in this translation, if π contains σ, then G_σ is an induced subgraph of G_π. Note that the reverse implication is not always true, as the mapping from permutations to graphs is typically many-to-one.

Downsets of permutations in the containment ordering are called *permutation classes*, and by the connection with permutation graphs, every permutation class corresponds directly to a graph class (necessarily consisting entirely of permutation graphs). Moreover, if a permutation class is well-quasi-ordered, then so too is the corresponding graph class. This connection is particularly important, because the ability to visualise permutations graphically has led to fast progress on questions concerning structure and well-quasi-ordering (see, for example, [6] and [10, 39]). Until October 2013, the PI was funded by the UK's Engineering and Physical Sciences Research Council (EPSRC, grant number EP/J006130/1), and a major component in this project was to exploit this connection [3, 8, 9].

The graphical perspective for permutations has meant that the construction of infinite antichains of permutations has also progressed further than comparative studies in other objects, most significantly by the PI in [6]. In the translation from permutations to graphs, an infinite permutation antichain does not always become an infinite graph antichain, but with care the translation can often be made to work, as recently demonstrated in [3].

Four examples of permutation antichains are indicated in Figure 2. It is striking that these three examples all exhibit regularity in how they are constructed: to create a longer permutation in each antichain from those shown, add one or more extra 'oscillation' or 'spiral'. The two pairs of circled points in each picture are called *anchors*, and these are crucial to prevent embeddings.

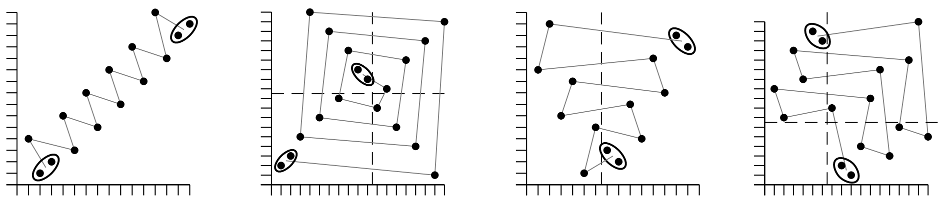


Figure 2: Four typical elements from four different infinite permutation antichains. The grey lines connecting the points indicate the process by which the permutations are constructed in each case.

Coloured objects and well-quasi-ordering

In 1972 Pouzet [37] proposed the following strengthening of well-quasi-ordering: for $n \in \mathbb{N}$, a graph class is *n-well-quasi ordered* if it contains no n-coloured infinite antichain: that is, an infinite set of graphs where the vertices have been coloured using n colours, and the ordering refined so that in embedding one graph as an induced subgraph of another, the colours of the vertices must match.

One might assume that increasing the number of colours that can be used would give greater freedom to construct coloured infinite antichains, but Pouzet conjectured that only the step from 1 to 2 colours makes any difference.

Conjecture 1 (Pouzet [37], see also Fraïsse [26])**.** *A graph class is 2-well-quasi-ordered if and only if it is n-well-quasi-ordered for all* $n \geq 2$*.*

Whereas there are (uncoloured) well-quasi-ordered graph classes defined by an infinite list of minimal forbidden elements, 2-well-quasi-ordered graph classes are always defined by a *finite* list of minimal forbidden elements – note that this is exactly what is required when seeking algorithmic ramifications of Question 1.1, and it enables (for example) the development of nonconstructive polynomial time recognition procedures. The resolution of Conjecture 1, therefore, is a highly desirable step forward.

If true, Conjecture 1 would also lend more evidence to the belief that the only infinite antichains that need to be considered must have some form of well-behaved construction. Heurisitically, one can mimic the role of 'anchors' for antichains by assigning the 'end vertices' of each antichain element a different colour from the rest of the antichain, so two colours will always suffice.

Objective 1. *Resolve Pouzet's Conjecture (Conjecture 1).*

2 Graph parameters

Formally, a *graph parameter* is any function that takes as input a graph, and outputs some number. The parameters of interest here are those that give some measurement of structure. A famous example, through its role in the proof of the Graph Minor Theorem, is 'tree width'. Specifically, Robertson and Seymour distinguish between the case where all the graphs in a downset in the minor ordering have tree width bounded by some k, and the case where the tree width is unbounded. Notably, it is the proof of a structure theorem to handle this latter case that commands much of the work of the whole Graph Minor Theorem. For a short introduction to the key steps in the Graph Minor Theorem, see Lovász [32].

A more recent parameter is *clique width*: for a graph G, its clique width, $\mathrm{cw}(G)$, is the smallest k for which G can be constructed from the set of labels $\{1, 2, \ldots, k\}$ using the following four operations:

- add a new vertex with a label from $\{1, 2, \ldots, k\}$,
- take the disjoint union, $G \cup H$, of two already-constructed labelled graphs G and H,
- add edges between all vertices labelled i and all vertices labelled j (for $i \neq j$), and
- relabel all vertices labeled i by j.

Clique width can be viewed as a generalisation of tree width, insofar as a graph with bounded tree width also has bounded clique width. A major motivation behind its study is due to Courcelle, Makowsky and Rotics [22], who showed that a large number of graph algorithms which are NP-hard in general can be solved in linear time for graphs with clique width at most some fixed k.

Similar – and indeed, stronger – algorithmic results exist for graphs of bounded tree width. However, as with the minor ordering on graphs, tree-width is sensitive to the density of edges of a graph, and this makes it incompatible with the induced subgraph ordering. On the other hand, the induced subgraph ordering respects clique-width: if H is an induced subgraph of a graph G, then $\mathrm{cw}(H) \leq \mathrm{cw}(G)$. As a result, we may define the *clique width of a graph class* to be the smallest k for which every graph in the class has clique width at most k.

The connection between clique width and well-quasi-ordering is very recent. Daligault, Rao and Thomassé [23] suggest that clique width may be used as a tool towards resolving Conjecture 1 in a way that mirrors the role of tree width for the Graph Minor Theorem. They conjecture that every 2-well-quasi-ordered graph class must have bounded clique width, which would mean that Conjecture 1 would be resolved by considering only graph classes with bounded clique width, which provides some level of structural control.

In fact, they ask whether the following stronger statement is true, which by [22] would have important ramifications for the study of well-quasi-ordering in graph classes. We state it here as a conjecture.

Conjecture 2. *Every well-quasi-ordered graph class has bounded clique-width.*

In the four years since the publication of [23], several graph classes have been found to support this belief, see [2] and [30, 33]. Moreover, the recent cross-fertilisation of results between the study of permutations and graphs (see the next section) is beginning to indicate what type of structure might be forced to appear in graph classes with unbounded clique width, making the resolution of this conjecture feasible.

Objective 2. *Resolve Conjecture 2.*

As well as underpinning an approach to the proof of Pouzet's conjecture, the techniques developed for this objective would change the landscape of well-quasi-orderability for the induced subgraph ordering, by connecting it directly to the algorithmic process by which graphs are built using the clique width operations.

For comparison, recall that much of the proof of the Graph Minor Theorem is devoted to developing a structure theory for downsets in the minor ordering which have unbounded tree width. Reassuringly, a positive resolution of Conjecture 2 would mean that classes with unbounded clique width are immediately not well-quasi-ordered. Thus, some instances of Question 1.1 would be answered without the need for an analogue of Robertson and Seymour's intricate structure theory.

3 Enumeration

The third component in this proposal is to investigate a connection between the combinatorial study of permutations and graph parameters.

There are effectively two ways to answer the question 'In a permutation class, how many permutations are there of each length n?': asymptotically (where the *growth rate* gives an approximate measure of the size – and hence structural complexity – of the class), and precisely, by means of an exact formula. The recent advances in structure theory, and indeed well-quasi-ordering, have been both motivated by and extensively applied in both camps. Notable recent examples include Albert and the PI [13], Albert, Ruškuc and Vatter [15], Bevan (a PhD student of the PI) [18] and Vatter [39]. Similar work exists for graphs and other combinatorial objects, but the work is not as far-reaching – see Bollobás [20] for a survey.

The *generating function* for a permutation class is a formal power series in which the coefficient of x^n is equal to the number of permutations of length n. It is often of interest to characterise generating functions for permutation classes as rational, algebraic, holonomic, etc, as this can be an indicator of a certain level of structural complexity in the class. At the end where structure is very well-behaved, we find the following strong property: if a permutation class and *all* of its subclasses have rational generating functions then the class is said to be *strongly rational.* It turns out that all 'small' permutation classes are strongly rational [15]. Here, small means those classes with growth rate less than an algebraic number $\lambda = 2.20557\ldots$, which is also the point where permutation classes fail to be well-quasi-ordered for the first time.

When a permutation class is not well-quasi-ordered, then it cannot be strongly rational because it has uncountably many subclasses (but there are only countably many rational generating functions available). However, it can be *broadly rational*, meaning that it and all subclasses defined by finitely many minimal forbidden permutations have rational generating functions.

A recent preprint of the PI [8] indicates a connection between the enumeration of permutation classes and a graph parameter called *linear clique width.* This is a restricted version of clique width, in which one constructs graphs using the same operations, except that the operation of taking the disjoint union of two previously-constructed graphs is not permitted.

Building on evidence in [8] on linear clique width in subclasses of cographs, and results concerning strong rationality in the analogous permutation class due to Albert, Atkinson, and Vatter [12], the PI and coauthors made the following conjecture.

Conjecture 3 (Brignall, Korpelainen and Vatter [8])**.** *A permutation class is broadly rational if and only if the corresponding class of permutation graphs has bounded linear clique-width. Moreover, the permutation class is strongly rational if and only if the graph class has bounded linear clique-width and is well-quasi-ordered.*

For the class of cographs (which are well-quasi-ordered but have unbounded linear clique-width [27]), this conjecture is known to hold amongst all its subclasses: i.e. a subclass of cographs has bounded linear clique width if and only if the corresponding permutation class is strongly rational (see [8]).

Further evidence for this conjecture comes from the class of bipartite permutation graphs, which, significantly and unlike cographs, are not well-quasi-ordered. Lozin [33] showed that this class (which is equivalent to '321-avoiding permutations') is a minimal class with unbounded clique width. Very recently Albert, Ruškuc and Vatter [16] have shown that every subclass has bounded linear clique width (and thus the bipartite permutation graphs are in fact minimal with unbounded *linear* clique width). Additionally, [16] also shows that while the 321-avoiding permutations have a nonrational generating function, all of its subclasses are broadly rational.

Objective 3. *Resolve Conjecture 3.*

A positive resolution of Conjecture 3 would represent a very powerful tool for studying both linear clique-width and permutation class enumeration. Its proof would provide a generalisation of several different techniques for enumerating permutation classes [10, 11, 14, 15], while the result on its own would enable direct translation between graphs and permutations. For example, the many permutation classes which are known to have nonrational generating functions would give a much faster route than [2] and [33] to demonstrate that a graph class has unbounded linear clique width.

Section b: Methodology

The proposed route to the resolution of these three conjectures is to develop and apply our structural understanding of graph classes and permutation classes. As demonstrated, these conjectures appear to be intricately connected, so a unified approach is required, that begins by adding the insight and techniques from the study of permutations to the current graph theoretic methods. We begin with a brief introduction to two key structural techniques.

Modular decomposition

The modular decomposition is a much-used technique (both for graphs and permutations) to carry out the vital process of describing larger objects in terms of smaller ones. The objects which are 'indivisible' in this decomposition are particularly important: these determine the well-quasi-ordering of every class, the clique width of every graph class, and for permutations their enumeration is a common route to the enumeration of permutation classes. The PI has worked with and developed these techniques continually since the start of his research career – see [5] for a survey in the permutation context.

Grid classes, letter graphs and k-uniform graphs

Another approach to describing structure in a class is to show that the vertices of every graph (or the entries of every permutation) in the class can be partitioned in a controlled way. For example, one might show that every partition is either a clique or an independent set, and the edges between partitions are restricted. For graphs, two notions have emerged recently in work towards understanding well-quasi-ordering, namely *letter graphs* [36] and *k-uniform graphs* [29].

For permutations, the term *grid class* now describes a suite of methods where the graphical representation of a permutation is divided into cells. Like the notions above for graphs, this is important for well-quasi-ordering [6], but it also enables both asymptotic [18] and exact [1] enumeration.

Work Package 0: Coloured antichains and graph parameters

The key aim of this '0th' work package is to resolve the following conjecture, which underpins both Objective 1 and Objective 2.

Conjecture 4 (Daligault, Rao and Thomassé [23])**.** *Every 2-well-quasi-ordered graph class has bounded clique width.*

While the PI strongly believes this conjecture to be true, it is worth noting that a counterexample — namely a class of unbounded clique-width which is 2-well-quasi-ordered — would be a significant result to aid our understanding of the interplay between clique width and well-quasi-ordering. For example, Daligault et al. [23] note that if such a counterexample were to exist, then there would exist a counterexample with the following strong property:

Proposition 5 ([23])**.** *If there exists a counterexample to Conjecture 4, then there exists a counterexample class $\mathcal{C}$ for which every strict subclass of $\mathcal{C}$ has* bounded *clique width.*

To resolve this conjecture, therefore, requires an increased understanding of both the freedom of structure that must exist in a class of unbounded clique-width, and the construction of 2-coloured infinite antichains.

Unbounded clique width

Activity 0.1. *Discover more minimal classes of unbounded clique width.*

As well as informing our understanding of unbounded clique width, this activity also provides the opportunity to find a counterexample to Conjecture 4, if it is in fact false. The PI has recently worked with Vadim Lozin, a leading expert in unbounded clique-width, to establish that the class of 'split permutation graphs' is a minimal class of unbounded clique width [2]. The connection with permutation classes here provided the crucial insight, and it is expected that this connection will reveal more such minimal graph classes in due course.

Note that any class which has unbounded clique width necessarily has unbounded linear clique width. It appears that many of the known minimal classes of unbounded clique width are also minimal classes of unbounded *linear* clique width, so this activity has ramifications towards Objective 3, too.

Activity 0.2. *Identify key structural features which imply that a graph class has unbounded clique width.*

As new minimal classes are found in Activity 0.1, it will be important to analyse these classes in order to discover structural features that can be exploited to construct 2-coloured infinite antichains.

The current belief is that unbounded clique-width implies that the class has a structure which is arbitrarily complicated in two 'directions': roughly, this means that there is no way to partition the vertices into a bounded number of parts, nor is there a way to partition the vertices so that each part has a bounded number of vertices within it. Progress on the understanding of unbounded linear clique width will also be important here.

Building 2-coloured infinite antichains

The second part of Work Package 0 has two related goals, which by their nature will largely run concurrently.

Activity 0.3. *Develop new techniques for constructing 2-coloured infinite antichains.*

Activity 0.4. *Find graph parameters that 'measure' the ability to produce 2-coloured infinite antichains, and which tie in with clique width.*

The idea behind Activity 0.4 is to take a general construction from Activity 0.3, and encode this as a parameter. If it can be shown that this parameter must be unbounded whenever clique width is unbounded, then Conjecture 4 will be resolved.

Recent unpublished work by the PI has identified a general and abstract technique to build *3-coloured* infinite antichains. Roughly, infinite antichain elements are made by taking intial segments from an infinite sequence of vertices, and colouring the first and last vertices of the segment with two different colours. The third colour is used solely for the convenience of being able to distinguish between the two 'ends' of the segment for each antichain element; it should be possible to reduce the construction to two colours later.

Associated with this construction is a new parameter called *clone-width*, which gives a precise measure of how many antichain elements can be created using this construction within a given graph. Thus, a class containing graphs with arbitrarily large clone-width must contain an infinite 3-coloured antichain of the form described.

This construction and associated parameter is an encouraging first step, but Activity 0.3 requires that this method can be made to tie in with clique width: if it could be shown that unbounded clone-width implied unbounded clique-width, then Work Package 0 would be complete. It may be, in fact, that the current construction is too abstract, so further investigation here is certainly required.

Work Package 1: Bounded clique-width and 2-well-quasi-ordering (Objective 1)

The key aim of this work package is to resolve the following special case of Conjecture 1:

Conjecture 6. *For a graph class $\mathcal{C}$ of bounded clique width, if $\mathcal{C}$ is 2-well-quasi-ordered then it is n-well-quasi-ordered for all $n \geq 2$.*

Combined with a positive resolution to Conjecture 4 from Work Package 0 above, a proof of this conjecture would complete the proof of Pouzet's conjecture, and thus Objective 1.

Daligault et al [23] describe an approach to prove Pouzet's conjecture that follows this pattern. They start with a graph parameter called *node-label controlled (NLC)-width*, which is equivalent to clique width in the sense that one parameter is at most twice the other. From this, they define graph classes $NLC_k^{\mathcal{F}}$ which have NLC-width k, but where the ability to relabel vertices is restricted to those relabelling functions inside some family $\mathcal{F}$. Crucially, they prove that the nature of $\mathcal{F}$ completely determines whether the graph class $NLC_k^{\mathcal{F}}$ contains a 2-coloured infinite antichain, or whether it is n-well-quasi-ordered for all n.

They then conjecture that every 2-well-quasi-ordered graph class must be contained within one of these 'good' $NLC_k^{\mathcal{F}}$ classes, and are consequently n-well-quasi-ordered.

This general approach seems plausible, but we note that the only 2-coloured infinite antichain required in their result requires the presence of arbitrarily long paths. Meanwhile, the graph antichain constructions exhibited in [3] show that there are several ways in which a graph class can contain an infinite antichain but not arbitrarily long paths, so the 'absence of long paths' may not be enough on its own.

Activity 1.1. *Investigate how graph classes without long paths relate to the $NLC_k^{\mathcal{F}}$ construction.*

The idea here is to ascertain whether it is possible for an n-well-quasi-ordered graph class to avoid long paths, but not be wholly contained in any of the 'good' $NLC_k^{\mathcal{F}}$ classes. If this is possible, then a refinement to the construction will be needed, and in this case it is likely that other 2-coloured infinite antichains will need to be considered.

The outcome from Activity 1.1 will largely determine the approach taken thereafter to complete Objective 1. It is clear that the structure of graph classes without long paths needs to be better understood, and if further 2-coloured infinite antichains are found to be important, these will also need to be excluded. The general follow-up activity, therefore, is:

Activity 1.2. *Investigate generalisations of the $NLC_k^{\mathcal{F}}$ parameters which might be used to show that graph classes with bounded clique width and which do not contain any known obstructions to 2-well-quasi-ordering are n-well-quasi-ordered.*

Work Package 2: From 2-colouring to 1-colouring – completing Objective 2

Assuming positive resolution of Conjecture 4, the remaining task in Objective 2 is to describe how 2-coloured infinite antichains might be converted into monochromatic antichains, in such a way as to continue to respect unbounded clique width.

Activity 2.1. *Convert constructions for 2-coloured antichains into constructions for monochromatic antichains.*

A priori, any 2-coloured antichain construction is likely to use one colour for the 'end' vertices (the anchors), and another colour for all the other vertices. This activity, therefore, seeks to find a way to replace the coloured end vertices with 'gadgets' which preserve the inability to embed one antichain element inside another, and which do not introduce any of the minimal forbidden graphs that define the class. The PI has some experience of choosing exotic 'anchors' in this way, in the study of permutation containment [4].

Activity 2.2. *Find a graph parameter which ties in with unbounded clique width and the monochromatic infinite antichain construction.*

This final activity is the 1-coloured antichain equivalent to Activity 0.4, and its resolution would complete the proof of Conjecture 2. It may, in fact, transpire that the same parameter can be used here as was used in the 2-coloured case, with little or no modification required to allow for the 'gadgets' introduced in Activity 2.1.

Work Package 3: Encodings and graph parameters – resolving Objective 3

The final conjecture in this proposal takes a slightly different stance, but it shares a common requirement to understand classes which have unbounded (linear) clique width. As noted earlier, it appears that classes which are minimal with unbounded clique width are often also minimal with unbounded linear clique width, so an important activity in this work package is the following.

Activity 3.1. *Establish when a permutation graph class has bounded clique width but unbounded linear clique width.*

Two examples are known at present: First, the graph class consisting of all induced subgraphs of arbitrarily large binary trees has clique width 3, but unbounded linear clique width. Second, as illustrated by the PI in [8], the class of 'quasi-threshold' graphs (a subclass of cographs) have clique width 2, but unbounded linear clique width.

Both illustrate a similar phenomenon, that linear clique width cannot construct trees which are simultaneously arbitrarily tall and arbitrarily wide. For the class of binary trees the graphs themselves are of this form, while for quasi-threshold graphs it can be found in the modular decomposition trees. However, binary trees are not permutation graphs, so could it be that quasi-threshold graphs and the class of their complements are the only minimal classes of permutation graphs that witness Activity 3.1?

Activity 3.2. *Show that classes of permutation graphs which are minimal with unbounded linear clique width correspond to permutation classes with nonrational generating functions.*

The resolution of this activity would complete one direction of the proof of Conjecture 3.

If the conclusion from Activity 3.1 is that the structure of modular decomposition trees determines whether a class with bounded clique width has unbounded linear clique width, then part of this activity will be complete: the permutation equivalent of the modular decomposition is a powerful tool for enumeration [7], and it should be possible to relate nonrational generating functions with arbitrarily tall and wide modular decomposition trees.

This, then, would leave permutation graph classes with unbounded clique width to consider, and here the outcomes from Activity 0.2 will play a part: the key is to take the 'forced' structure in such a graph class, and show that this translates to give a permutation class with a nonrational generation function.

Bounded linear clique width, and permutation encodings

For the remaining direction of Conjecture 3, it will be sufficient to show that a class of permutation graphs with bounded linear clique width gives a strongly- or broadly rational permutation class. This essentially requires the translation of linear clique width expressions for constructing graphs into encodings for permutations.

Activity 3.3. *Expand the toolkit for encoding permutation structure using regular languages, with a view to carrying out enumeration.*

The key technique to show that permutation classes are strongly rational is to show how all the permutations can be encoded by a *regular language*, and that this regular language is compatible with the restriction to subclasses (defined by finitely many minimal forbidden permutations). The state-of-the-art in this area is due to [15], and it relies heavily on understanding the structure of permutations which are indivisible under the modular decomposition.

When a permutation class is not well-quasi-ordered, then the question of broad rationality is more open. One example is due to Albert, Atkinson and Ruškuc [11], but the encoding they use is substantially different from that in [15] for strongly rational classes.

Activity 3.4. *Explore the structure of classes of permutation graphs with bounded linear clique width, and how this relates to permutation encodings.*

Activity 3.1 should provide a number of properties that permutation graph classes *must* possess if they are to have bounded linear clique width, for example a strong restriction on the structure of the modular decomposition trees in the class. This, in turn, should limit the ways in which linear clique width expressions can be used to build the graphs in the class, thereby making the process of translation to permutation encodings more tractable.

Risk management

The prior intention is to prove all the conjectures mentioned in this proposal. However, the activities towards these are based around building up a better understanding by looking at progressively more complicated examples, which should enable counterexamples to be found, if they exist. Moreover, the discovery of counterexamples will inevitably lead to a better understanding of the situation, and hence generate further lines of enquiry.

The other risk is that a counterexample or mere failure to resolve Conjecture 4 (Work Package 0) will affect the proposed route to prove Conjecture 1. It is worth noting, however, that this conjecture could be resolved by building on Activity 1.1 directly.

External Research Collaborators

The PI expects to build on existing collaborations with two key individuals from other institutions, whose input will greatly enhance the ability to deliver on the objectives.

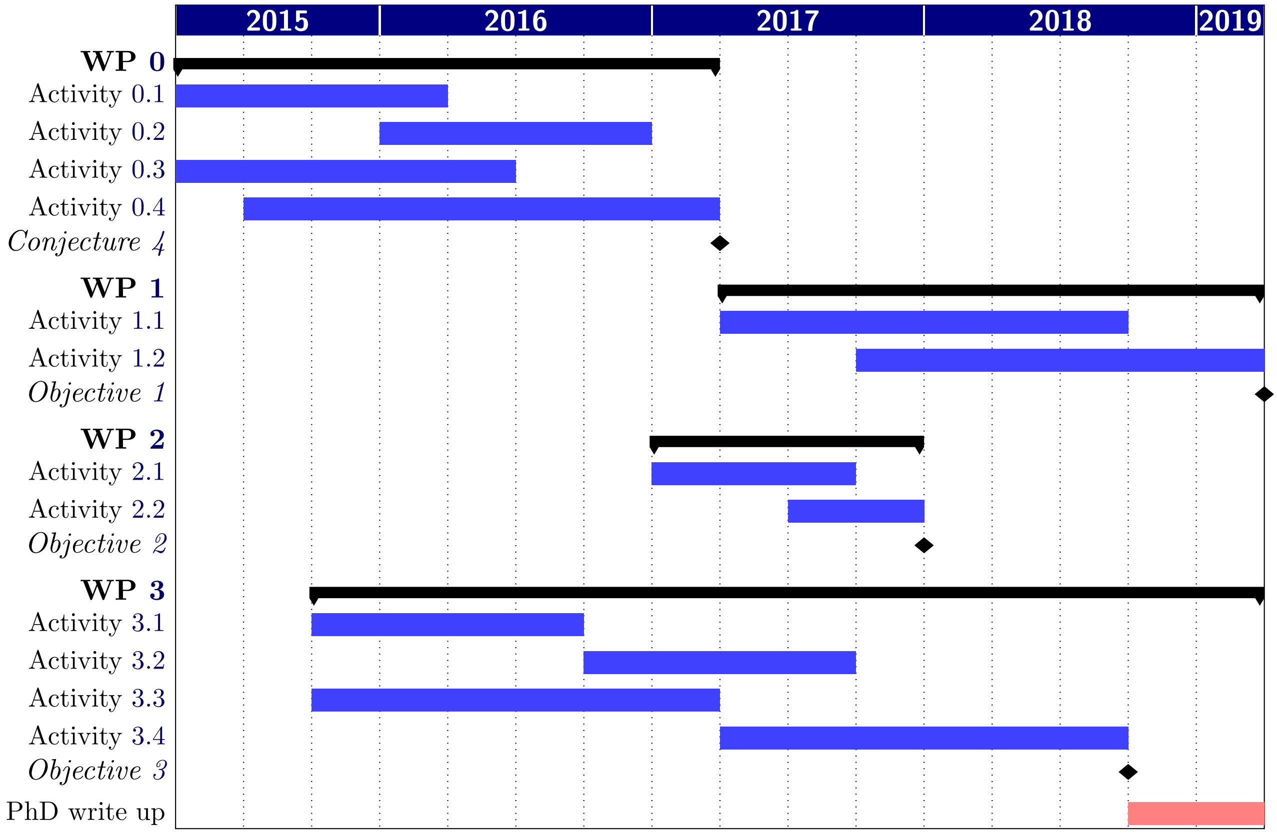
Vincent Vatter (University of Florida, USA) is a world-leader in the structural study of permutation classes, and particularly the applications of this to the study of well-quasi-ordering and enumeration. The PI has a long successful history of publishing with Vatter, with a total of 6 coauthored publications since 2007.

Vadim Lozin (University of Warwick, UK) has unique expertise in developing methods relating to the well-quasi-ordering and clique width of graph classes. Although Lozin and the PI have only started collaborating relatively recently, this collaboration has already led to the production of 3 preprints.

In addition to these two collaborators, **Michael Albert** (University of Otago, New Zealand) and the PIs PhD supervisor **Nik Ruškuc** (University of St Andrews, UK) may also be involved, particularly by providing expertise in the encoding of permutations for enumeration, relevant to Objective 3. The PI has previously written 6 articles with Albert, and 3 with Ruškuc. Finally, it is the PI's intention to use this project to establish new collaborations, for example with the authors of [23].

Gantt Chart

The following Gantt Chart gives an approximate indication of the expected duration for each Activity and Work Package. Details concerning the distribution of workload to the research team can be found in Section c.



Section c: Resources (including project costs)

Cost Category			**Total in Euro**
Direct costs	**Personnel**	PI	187 211
		Senior Staff	0
		Postdocs	230 567
		Students	64 574
		Other	0
	i. Total Direct costs for Personnel (in Euro)		482 352
	Travel		45 029
	Equipment		0
	Other goods and services	Consumables	0
		Publications (including Open Access fees), etc.	5 376
		Other: audit fee	1 572
		conference fees	2 150
		recruitment fees	1 344
	ii. Total other Direct costs (in Euro)		55 471
A – Total Direct Costs (i + ii) (in Euro)			537 823
B – Indirect Costs (overheads) 25% of Direct Costs (in Euro)			134 456
C1 – subcontracting Costs (no overheads) (in Euro)			0
C2 – Other Direct Costs with no overheads (in Euro)			0
Total Estimated Eligible Costs (A + B + C) (in Euro)			672 279
Total Requested EU Contribution (in Euro)			672 279

% of working time the PI will dedicate to the project over the period of the grant	**60%**

Costs in the above table are based on a Full Economic Costing for the project. This is based on a duration of 48 months starting on 1st April 2015.

The research team will comprise the PI (committing 60% of his working time to the project), one postdoctoral research associate (PDRA) for the whole period, and a postgraduate research student for 42 months, to begin on 1st October 2015.

Postdoctoral Research Associate

The PDRA will contribute primarily to the study of graph parameters, and their relationship with well-quasi-ordering. Within Work Package 0, they will focus primarily on Activities 0.1 and 0.2. Later, they will work on the activities within Work Package 1. This allocation of tasks to the PDRA also allows for some flexibility, should there be any need to reappoint the position after, say, 2 years.

PhD Student

The PhD student will contribute towards the work in Work Package 3, and particularly Activities 3.3 and 3.4. As demonstrated by the success of the PI's current PhD student, the study of structure and enumeration of permutations is an ideal area to train a new researcher. In particular, the wealth of new methods that have developed in recent years allows room for the student to undertake more routine follow-up research at first, and tackling deeper questions later.

Collaborators

No salary costs are included for any collaborators.

Travel

Approx €4 000 per person per year is requested for the PI and PDRA. As well as providing funding for conferences, this will enable collaboration with Lozin (University of Warwick), a two-week research visit to work with Vatter (University of Florida) each spring, and funding for other collaborations as they emerge.

Approx €1 300 per year is requested for the PhD student to attend conferences and, latterly, to forge collaborations. If necessary, this can be augmented by the fund for the PI and PDRA.

Approx €2 000 per year is requested to enable Vatter (University of Florida) to visit the Open University for two weeks each summer.

Open Access

All articles will be submitted to Green, Gold or Diamond Open Access journals. Additionally, preprints will be uploaded to the ArXiv, and made available on the PIs website. £4 000 (approx €4 800) has been requested to enable some articles to be submitted to Gold journals.

PI References

[1] Albert, M. H., Atkinson, M. D., and Brignall, R. The enumeration of three pattern classes using monotone grid classes. *Electron. J. Combin. 19(3)*, 3 (2012), P20, 34 pp. (electronic).

[2] Atminas, A., Brignall, R., Korpelainen, N., Lozin, V., and Stacho, J. A minimal class of bipartite graphs of unbounded clique-width. Preprint.

[3] Atminas, A., Brignall, R., Korpelainen, N., and Lozin, V.and Vatter, V. Well-quasi-order for permutation graphs omitting a path and a clique. Preprint.

[4] Brignall, R. Wreath products of permutation classes. *Electron. J. Combin. 14* (2007), Research paper 46, 15 pp. (electronic).

[5] Brignall, R. A survey of simple permutations. In *Permutation Patterns*, S. Linton, N. Ruškuc, and V. Vatter, Eds., vol. 376 of *London Math. Soc. Lecture Note Ser.* Cambridge Univ. Press, Cambridge, 2010, pp. 41–65.

[6] Brignall, R. Grid classes and partial well order. *J. Combin. Theory Ser. A 119*, 1 (2012), 99–116.

[7] Brignall, R., Huczynska, S., and Vatter, V. Simple permutations and algebraic generating functions. *J. Combin. Theory Ser. A 115*, 3 (2008), 423–441.

[8] Brignall, R., Korpelainen, N., and Vatter, V. Linear clique-width for subclasses of cographs, with connections to permutations. Preprint.

[9] Brignall, R., Lozin, V., and Stacho, J. Bichain graphs: geometric model and universal graphs. Preprint.

Other References

[10] Albert, M. H., Atkinson, M. D., Bouvel, M., Ruškuc, N., and Vatter, V. Geometric grid classes of permutations. *Trans. Amer. Math. Soc. 365*, 11 (2013), 5859–5881.

[11] Albert, M. H., Atkinson, M. D., and Ruškuc, N. Regular closed sets of permutations. *Theoret. Comput. Sci. 306*, 1-3 (2003), 85–100.

[12] Albert, M. H., Atkinson, M. D., and Vatter, V. Subclasses of the separable permutations. *Bull. Lond. Math. Soc. 43* (2011), 859–870.

[13] Albert, M. H., and Brignall, R. Enumerating indices of Schubert varieties defined by inclusions. *J. Combin. Theory Ser. A 123*, 1 (2014), 154–168.

[14] Albert, M. H., Linton, S., and Ruškuc, N. The insertion encoding of permutations. *Electron. J. Combin. 12*, 1 (2005), Research paper 47, 31 pp.

[15] Albert, M. H., Ruškuc, N., and Vatter, V. Inflations of geometric grid classes of permutations. To appear in *Israel Journal of Mathematics.*

[16] Albert, M. H., Ruškuc, N., and Vatter, V. Rational generating functions for 321-avoiding subclasses. In preparation.

[17] Atminas, A., Lozin, V. V., and Moshkov, M. Deciding wqo for factorial languages. In *LATA* (2013), pp. 68–79.

[18] Bevan, D. Growth rates of permutation grid classes, tours on graphs, and the spectral radius. To appear in *Transactions of the American Mathematical Society.*

[19] Bienstock, D., and Langston, M. A. Algorithmic implications of the graph minor theorem. In *Network models*, vol. 7 of *Handbooks Oper. Res. Management Sci.* North-Holland, Amsterdam, 1995, pp. 481–502.

[20] Bollobás, B. Hereditary and monotone properties of combinatorial structures. In *Surveys in Combinatorics 2007*, A. Hilton and J. Talbot, Eds., no. 346 in London Mathematical Society

Lecture Note Series. Cambridge University Press, 2007, pp. 1–39.

[21] Cherlin, G. Forbidden substructures and combinatorial dichotomies: WQO and universality. *Discrete Math. 311*, 15 (2011), 1543–1584.

[22] Courcelle, B., Makowsky, J. A., and Rotics, U. Linear time solvable optimization problems on graphs of bounded clique-width. *Theory Comput. Syst. 33*, 2 (2000), 125–150.

[23] Daligault, J., Rao, M., and Thomassé, S. Well-quasi-order of relabel functions. *Order 27*, 3 (2010), 301–315.

[24] Ding, G. Subgraphs and well-quasi-ordering. *J. Graph Theory 16*, 5 (1992), 489–502.

[25] Downey, R. G., and Fellows, M. R. *Parameterized complexity.* Monographs in Computer Science. Springer-Verlag, New York, 1999.

[26] Fraïssé, R. *Theory of relations*, revised ed., vol. 145 of *Studies in Logic and the Foundations of Mathematics*. North-Holland Publishing Co., Amsterdam, 2000.

[27] Gurski, F., and Wanke, E. On the relationship between NLC-width and linear NLC-width. *Theoret. Comput. Sci. 347*, 1-2 (2005), 76–89.

[28] Higman, G. Ordering by divisibility in abstract algebras. *Proc. London Math. Soc. (3) 2* (1952), 326–336.

[29] Korpelainen, N., and Lozin, V. Bipartite induced subgraphs and well-quasi-ordering. *Discrete Mathematics 311* (2011), 1813–1822.

[30] Korpelainen, N., Lozin, V., and Mayhill, C. Split permutation graphs. *Graphs and Combinatorics* (2013), 1–14.

[31] Kruskal, J. B. Well-quasi-ordering, the Tree Theorem, and Vazsonyi's conjecture. *Trans. Amer. Math. Soc. 95* (1960), 210–225.

[32] Lovász, L. Graph minor theory. *Bull. Amer. Math. Soc. (N.S.) 43*, 1 (2006), 75–86 (electronic).

[33] Lozin, V. V. Minimal classes of graphs of unbounded clique-width. *Ann. Comb. 15*, 4 (2011), 707–722.

[34] Nash-Williams, C. S. J. A. On well-quasi-ordering finite trees. *Proc. Cambridge Philos. Soc. 59* (1963), 833–835.

[35] Nash-Williams, C. S. J. A. On well-quasi-ordering infinite trees. *Proc. Cambridge Philos. Soc. 61* (1965), 697–720.

[36] Petkovšek, M. Letter graphs and well-quasi-order by induced subgraphs. *Discrete Math. 244*, 1-3 (2002), 375–388. Algebraic and topological methods in graph theory (Lake Bled, 1999).

[37] Pouzet, M. Un bel ordre d'abritement et ses rapports avec les bornes d'une multirelation. *C. R. Acad. Sci. Paris Sér. A-B 274* (1972), A1677–A1680.

[38] Robertson, N., and Seymour, P. D. Graph minors. XX. Wagner's conjecture. *J. Combin. Theory Ser. B 92*, 2 (2004), 325–357.

[39] Vatter, V. Small permutation classes. *Proc. Lond. Math. Soc. (3) 103* (2011), 879–921.