The Möbius function of permutations with an indecomposable lower bound

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Abstract

We show that the Möbius function of an interval in a permutation poset where the lower bound is sum (resp. skew) indecomposable depends solely on the sum (resp. skew) indecomposable permutations contained in the upper bound, and that this can simplify the calculation of the Möbius sum. For increasing oscillations, we give a recursion for the Möbius sum which only involves evaluating simple inequalities.

1 Introduction

Let σ and π be permutations of natural numbers, written in one-line notation, with $\sigma = \sigma_1\sigma_2\ldots\sigma_m$, and $\pi = \pi_1\pi_2\ldots\pi_n$. We say that σ is *contained* in π if there is a sequence $1 \leq i_1 < i_2 < \ldots < i_m \leq n$ such that for any $r, s \in \{1, \ldots, m\}$, $\pi_{i_r} < \pi_{i_s}$ if and only if $\sigma_r < \sigma_s$. We say that π *avoids* σ if π does not contain σ. The set of all permutations is a poset under the partial order given by containment.

An interval $[\sigma, \pi]$ in a poset is the sub-poset defined as $\{\tau : \sigma \leq \tau \leq \pi\}$. The Möbius function $\mu[\sigma, \pi]$ is defined on an interval of a poset as follows: for $\sigma \not\leq \pi$, $\mu[\sigma, \pi] = 0$; for all λ, $\mu[\lambda, \lambda] = 1$; and for $\sigma < \pi$,

$$\mu[\sigma, \pi] = -\sum_{\sigma \leq \lambda < \pi} \mu[\sigma, \lambda]. \qquad (1)$$

Our motivation for this paper is to find a *contributing set* $\mathfrak{C}_{\sigma,\pi}$ that is significantly smaller than the poset interval $[\sigma, \pi]$, and a $\{0, \pm 1\}$ weighting function

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$W(\sigma, \lambda, \pi)$ such that

$$\mu[\sigma, \pi] = \sum_{\lambda \in \mathfrak{C}_{\sigma,\pi}} -\mu[\sigma, \lambda] W(\sigma, \lambda, \pi). \tag{2}$$

Plainly, in Equation 2, we could set $\mathfrak{C}_{\sigma,\pi} = \{\lambda : \sigma \leq \lambda < \pi\}$, and have $W(\sigma, \lambda, \pi) = 1$, which is equivalent to Equation 1. Our aim is to find a definition of $\mathfrak{C}_{\sigma,\pi}$ such that the number of elements in $\mathfrak{C}_{\sigma,\pi}$ is less than the number of elements in the poset interval. For increasing oscillations, we will show that the elements of $\mathfrak{C}_{\sigma,\pi}$ can be determined using simple inequalities, and that as a consequence $\mu[\sigma, \pi]$ can be determined using inequalities.

The study of the Möbius function in the permutation poset was introduced by Wilf [9]. The first result in this area was by Sagan and Vatter [4], who determined the Möbius function on intervals of layered permutations. Steingrímsson and Tenner [8] gave a large class of pairs of permutations (σ, π) where $\mu[\sigma, \pi] = 0$, as well as determining the Möbius function where σ occurs exactly once in π, and σ and π satisfy certain other conditions. Burstein, Jelínek, Jelínková and Steingrímsson [1] determined the Möbius function for sum/skew decomposable permutations and hence for separable permutations. Their result includes a pair of recursions for decomposable permutations that underpins the first part of this paper. McNamara and Steingrímsson [3] investigated the topology of intervals in the permutation poset, and found a single recurrence equivalent to the recursions in [1].

Many results for $\mu[\sigma, \pi]$ have been obtained by considering ways in which σ can be found in π, which we call an *embedding* of σ in π. The main technique has been to restrict the ways in which σ can be embedded, and these restrictions give us *normal embeddings*. The specific definition of normal embedding changes from paper to paper in order to match the structures being considered.

Smith [5] determined the Möbius function on the interval $[1, \pi]$ for all permutations π with a single descent. In [6], Smith found a two-term formula for the Möbius function. The first term counts the number of normal embeddings, and the second term is a rather complicated double sum. In [7], Smith determined the Möbius function $\mu[\sigma, \pi]$ in terms of the normal embeddings of σ into π when σ and π have the same number of descents.

One problem with normal embeddings is that in cases when the absolute value of the Möbius function exceeds the number of possible embeddings, then either non-unitary weights or some additional error-correction factor is required.

In this paper we show that the Möbius function on intervals with a sum indecomposable lower bound depends on the sum indecomposable permutations contained in the upper bound. We provide a weighting function that determines which sum indecomposable permutations contribute to the Möbius sum. We then consider increasing oscillations. For these permutations, we show how we can find all of the permutations that contribute to the Möbius sum by applying simple numeric inequalities, which leads to a fast polynomial algorithm for determining the Möbius function.

We start with some essential definitions and notation in Section 2, then in Section 3 we provide a number of preliminary lemmas. We conclude this section with a theorem that gives the Möbius value where the lower bound is a sum indecomposable permutation. In Section 4 we consider the Möbius function for increasing oscillations where the lower bound of the interval is sum indecomposable. We finish with some concluding remarks in Section 5.

2 Definitions and notation

When discussing the Möbius function, $-\sum_{\sigma \leq \lambda < \pi} \mu[\sigma, \lambda]$, we will frequently be examining the value of $\mu[\sigma, \lambda]$ for a specific value of λ. We say that this is the *contribution* that λ makes to the sum. If we have a set of permutations $S \subseteq [\sigma, \pi]$ such that $\sum \lambda \in S \mu[\sigma, \lambda] = 0$, then we say that the set S makes *no net contribution* to the sum.

A *direct sum* of two permutations α and β of lengths m and n respectively is the permutation $\alpha_1, \ldots, \alpha_m, \beta_1 + m, \ldots, \beta_n + m$. We write a direct sum as $\alpha \oplus \beta$. A *skew sum*, $\alpha \ominus \beta$ is defined in the obvious way.

Let α be a permutation, and r a positive integer. Then $r\alpha$ is $\alpha \oplus \alpha \oplus \ldots \oplus \alpha \oplus \alpha$, with r occurrences of α.

The *direct interleave* of two permutations α and β is formed by taking the direct sum $\alpha \oplus \beta$, and then exchanging the value of the largest point from α with the value of the smallest point from β. We can also view this as increasing the largest point from α by 1, and simultaneously decreasing the smallest point from β by 1. We write an interleave as $\alpha \otimes \beta$. For example, $321 \otimes 213 = 421536$.

For completeness, we also define a *skew interleave*, $\alpha \oslash \beta$, which is formed by taking the skew sum $\alpha \ominus \beta$, and then exchanging the smallest point from α with the largest point from β.

The interleave operations, $\otimes$ and $\oslash$, are not associative, as $1 \otimes 1 \otimes 1$ could represent 231 or 312. To avoid this ambiguity, we require that the permutation 1 can either be interleaved to the left or to the right, but not both. It is easy to see that this restriction restores associativity.

For the remainder of this paper, we discuss permutations in relation to direct sums and direct interleaves only. The results found can be applied to permutations with skew sums and skew interleaves by taking the reverse of the permutation. Henceforth we will drop the "direct" qualifier, and simply refer to "sums" and "interleaves". References to (in)decomposable permutations also omit the "sum" qualifier.

Let α be a permutation with length greater than 1. We will frequently want to refer to permutations that have the form $\alpha \otimes \alpha \otimes \ldots \otimes \alpha \otimes \alpha$. If there are n copies of α being interleaved, then we will write this as α^n, so, for example, we have $(21)^3 = 21 \otimes 21 \otimes 21 = 315264$. We say that α^0 is the empty permutation.

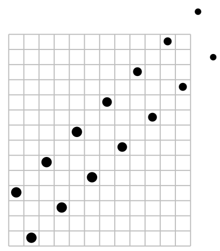


Figure 1: The start of the increasing oscillating sequence.

The *increasing oscillating sequence* is the sequence

$$4, 1, 6, 3, 8, 5, 10, 7, \ldots, 2k+2, 2k-1, \ldots.$$

The start of the sequence is shown in Figure 1.

An *interval* of a permutation is a set of continuous positions where the set of values is also contiguous. A *simple* permutation is one where the permutation contains no intervals other than those of length one, and the entire permutation. An *increasing oscillation* is a simple permutation contained in the increasing oscillating sequence. Let W_n be the increasing oscillation with n elements which starts with a descent, and let M_n be the increasing oscillation with n elements which starts with an ascent. Then

$$\begin{aligned} W_{2n} &= (21)^n, \\ W_{2n-1} &= (21)^{n-1} \oslash 1, \\ M_{2n} &= 1 \oslash (21)^{n-1} \oslash 1, \qquad \text{and} \\ M_{2n-1} &= 1 \oslash (21)^{n-1}. \end{aligned}$$

We will want to refer to the general shape of a set of permutations. In some cases, parts of the shape may be optional. We will show this by boxing the optional part and the relevant operator, so, for example, we may define the shape

$$\boxed{1\oslash}\,(21)^n\,\boxed{\oplus 1}$$

and the set being defined here is

$$\{(21)^n, 1 \oslash (21)^n, (21)^n \oplus 1, 1 \oslash (21)^n \oplus 1; n \in \mathbb{N}\}.$$

Note that the set as defined contains an infinite number of permutations, four for each natural number n.

There are also instances where, for some permutation α, we are interested in the set of permutations $\{\alpha, 1 \oplus \alpha, \alpha \oplus 1, 1 \oplus \alpha \oplus 1\}$. Given a permutation α, we refer to this set as $\mathfrak{F}(\alpha)$, and we say that this set is the *family* of α. Note that the family of α could be written as $\boxed{1\oplus}\,\alpha\,\boxed{\oplus 1}$, but we will generally use $\mathfrak{F}(\alpha)$ to emphasise that we are dealing with a family.

3 Preliminary lemmas and main theorem

In this section our aim is to show that if σ is indecomposable, then for any $\pi \geq \sigma$ there is a weighting function $W(\sigma, \alpha, \pi)$ and a set of permutations $\mathfrak{C}_{\sigma,\pi}$, such that

$$\mu[\sigma, \pi] = -\sum_{\alpha \in \mathfrak{C}_{\sigma,\pi}} \mu[\sigma, \alpha] W(\sigma, \alpha, \pi).$$

Our approach is to show that certain permutations in the interval $[\sigma, \pi]$ have a Möbius value of zero, and so can be excluded from $\mathfrak{C}_{\sigma,\pi}$. Further, we show that there are pairs of permutations λ and λ' where $\mu[\sigma, \lambda] = -\mu[\sigma, \lambda']$ and so each pair makes no net contribution, and thus we can also exclude these pairs from $\mathfrak{C}_{\sigma,\pi}$.

The Möbius function for the identity permutation $12 \ldots n$ and its reverse is trivial, and we exclude these permutations from being the upper bound of any interval under consideration.

We use Proposition 1 and 2, and Corollary 3 from Burstein, Jelínek, Jelínková and Steingrímsson [1]. We start with some required notation. If π is a non-empty permutation with decomposition $\pi_1 \oplus \ldots \oplus \pi_n$, then for any integer i with $0 \leq i \leq n$, $\pi_{\leq i}$ is the permutation $\pi_1 \oplus \ldots \oplus \pi_i$, and $\pi_{>i}$ is the permutation $\pi_{i+1} \oplus \ldots \oplus \pi_n$. An empty sum of permutations is defined as $\emptyset$, and in particular $\pi_{\leq 0} = \pi_{>n} = \emptyset$. We can see that $\mu[\emptyset, \emptyset] = 1$, $\mu[\emptyset, 1] = -1$ and $\mu[\emptyset, \tau] = 0$ for any $\tau > 1$. We now recall the results from Burstein, Jelínek, Jelínková and Steingrímsson:

Proposition 1 (Burstein, Jelínek, Jelínková and Steingrímsson [1, Proposition 1])**.** *Let σ and π be non-empty permutations with decompositions $\sigma = \sigma_1 \oplus \ldots \oplus \sigma_m$ and $\pi = \pi_1 \oplus \ldots \oplus \pi_n$, with $n \geq 2$. Assume that $\pi_1 = 1$, and let k be the largest integer such that $\pi_1, \pi_2, \ldots, \pi_k$ are all equal to 1. Let $l \geq 0$ be the largest integer such that $\sigma_1, \sigma_2, \ldots, \sigma_l$ are all equal to 1. Then*

$$\mu[\sigma, \pi] = \begin{cases} 0 & \text{if } k-1 > l, \\ -\mu[\sigma_{>k-1}, \pi_{>k}] & \text{if } k-1 = l, \\ \mu[\sigma_{>k}, \pi_{>k}] - \mu[\sigma_{>k-1}, \pi_{>k}] & \text{if } k-1 < l. \end{cases}$$

Proposition 2 ([1, Proposition 2])**.** *Let σ and π be non-empty permutations with decompositions $\sigma = \sigma_1 \oplus \ldots \oplus \sigma_m$ and $\pi = \pi_1 \oplus \ldots \oplus \pi_n$, with $n \geq 2$. Assume that $\pi_1 \neq 1$, and let k be the largest integer such that $\pi_1, \pi_2, \ldots, \pi_k$ are all equal to π_1. Then*

$$\mu[\sigma, \pi] = \sum_{i=1}^{m} \sum_{j=1}^{k} \mu[\sigma_{\leq i}, \pi_1] \mu[\sigma_{>i}, \pi_{>j}].$$

Corollary 3 ([1, Corollary 3])**.** *Let σ and π be as in 2. Suppose that σ is sum indecomposable, so $m = 1$. Then*

$$\mu[\sigma, \pi] = \begin{cases} \mu[\sigma, \pi_1] & \text{if } \pi = k\pi_1, \\ -\mu[\sigma, \pi_1] & \text{if } \pi = k\pi_1 \oplus 1, \\ 0 & \text{otherwise,} \end{cases}$$

A simple consequence of Proposition 1 and 2 is the identification of some intervals of permutations where the value of the Möbius function is zero.

Lemma 4. *Let $\pi \in \{1\oplus 1\oplus\tau, \tau\oplus 1\oplus 1, \mathfrak{F}(r\alpha\oplus\tau')\}$, where τ is any permutation, r is maximal, α is sum indecomposable, and τ' is any permutation greater than 1. Let σ be a sum indecomposable permutation. Then $\mu[\sigma, \pi] = 0$.*

Proof. Consider $\pi = 1 \oplus 1 \oplus \tau$. We use Proposition 1. If $\tau = 1$ or τ begins $1 \oplus$, then $k \geq 3$, and $l \leq 1$, and the result follows immediately. Now assume that $\tau \neq 1$ and τ does not begin $1 \oplus$. Then $k = 2$. If $\sigma > 1$, then again the result follows immediately. If $\sigma = 1$, then we have $\mu[\sigma, \pi] = -\mu[\sigma_{>k-1}, \pi_{>k}] = -\mu[\emptyset, \tau] = 0$. The case for $\pi = \tau \oplus 1 \oplus 1$ follows by symmetry.

Now consider $\pi = \mathfrak{F}(r\alpha \oplus \tau')$. If $\pi = r\alpha \oplus \tau$, or $\pi = r\alpha \oplus \tau \oplus 1$, then we use Proposition 2. In that context we have $m = 1$ and $k = r$, and so $\mu[\sigma, \pi] = \sum_{j=1}^{r} \mu[\sigma, \pi_1]\mu[\emptyset, \pi_{>j}]$ For every value of j, $\pi_{>j}$ is non-empty and greater than 1, and so $\mu[\emptyset, \pi_{>j}] = 0$ for all j, and hence every term in the sum is zero. If $\pi = 1 \oplus r\alpha \oplus \tau$ or $\pi = 1 \oplus r\alpha \oplus \tau \oplus 1$, then we use proposition 1, which reduces to one of the previous cases. ☐

We now turn to identifying pairs of permutations λ, λ' such that $\mu[\sigma, \lambda] + \mu[\sigma, \lambda'] = 0$. We start by showing that, up to a sign, the permutations in $\mathfrak{F}(r\alpha)$ have the same Möbius value.

Lemma 5. *Let $\pi \in \mathfrak{F}(r\alpha)$, where $r \geq 1$ and $\alpha > 1$ and α is sum indecomposable. Let σ be a sum indecomposable permutation. Then*

$$\mu[\sigma, \pi] = \begin{cases} \mu[\sigma, \alpha] & \text{if } \pi = r\alpha \text{ or } 1 \oplus r\alpha \oplus 1, \\ -\mu[\sigma, \alpha] & \text{if } \pi = 1 \oplus r\alpha \text{ or } r\alpha \oplus 1. \end{cases}$$

As a consequence, $\mathfrak{F}(r\alpha)$ makes no net contribution to $\mu[\sigma, \pi]$ if $\mathfrak{F}(r\alpha) \subseteq [\sigma, \pi]$.

Proof. If $\pi = r\alpha$ or $\pi = r\alpha \oplus 1$, then this is immediate from Corollary 3. If $\pi = 1 \oplus r\alpha$ or $\pi = 1 \oplus r\alpha \oplus 1$, then we use Proposition 1.

For the net contribution of $\mathfrak{F}(r\alpha)$, two permutations have Möbius value $\mu[\sigma, \alpha]$, and two have Möbius value $-\mu[\sigma, \alpha]$, so the sum over all elements of $\mathfrak{F}(r\alpha)$ is zero. ☐

We now have a lemma that adds a further restriction to the permutations that have a non-zero contribution to the Möbius sum.

Lemma 6. *If π is any permutation, and $\sigma \leq \pi$ is any permutation, and $\alpha \in [\sigma, \pi]$ is sum indecomposable, and r is the smallest integer such that $1 \oplus r\alpha \oplus 1 \not\leq \pi$, then $\mathfrak{F}(k\alpha) \subseteq [\sigma, \pi]$ for all $k < r$.*

Proof. For any $k < r$, $\sigma \leq k\alpha < 1 \oplus k\alpha \oplus 1 \leq \pi$, so by Lemma 5 the net contribution of the family is zero. ☐

Observation 7. *Using the same terminology as Lemma 6, if $k > r + 1$ then we must have $k\alpha \not\leq \pi$. As a consequence, for each indecomposable $\alpha \in [\sigma, \pi]$, the only families of α that can have a non-zero net contribution to $\mu[\sigma, \pi]$ are $\mathfrak{F}(r\alpha)$ and $\mathfrak{F}((r + 1)\alpha)$.*

We now eliminate two specific permutations from the Möbius sum.

Lemma 8. *If π is any permutation with $|\pi| > 3$, and σ is sum indecomposable, then the permutations 1 and $1 \oplus 1$ make no net contribution to the Möbius sum $\mu[\sigma, \pi]$.*

Proof. If $\sigma = 1$, then the interval contains both 1 and $1 \oplus 1$. Since $\mu[1, 1] = 1$ and $\mu[1, 12] = -1$, there is no net contribution to $\mu[\sigma, \pi]$. If $\sigma > 1$, then, since we exclude the reverse identity permutation, $\sigma \not\leq 1 \oplus 1$, so neither permutation is in the interval. □

Before we present the main theorem for this section, we formally define the weight function and the contributing set. Let α be a sum indecomposable permutation. The weight function, $W(\sigma, \alpha, \pi)$, is defined as

$$W(\sigma, \alpha, \pi) = \begin{cases} 1 & \text{If} \begin{cases} \sigma \leq r\alpha \leq \pi \text{ and} \\ 1 \oplus r\alpha \not\leq \pi \text{ and} \\ r\alpha \oplus 1 \not\leq \pi, \end{cases} \\ -1 & \text{If} \begin{cases} \sigma \leq r\alpha \leq \pi \text{ and} \\ 1 \oplus r\alpha \leq \pi \text{ and} \\ r\alpha \oplus 1 \leq \pi \text{ and} \\ (r+1)\alpha \not\leq \pi, \end{cases} \\ 0 & \text{Otherwise,} \end{cases} \tag{3}$$

where r is the smallest integer such that $1 \oplus r\alpha \oplus 1 \not\leq \pi$.

The contributing set $\mathfrak{C}_{\sigma,\pi}$ is defined as

$$\mathfrak{C}_{\sigma,\pi} = \left\{ \alpha : \begin{array}{l} \sigma \leq \alpha < \pi, \\ \alpha \text{ is sum indecomposable, and} \\ W(\sigma, \alpha, \pi) \neq 0 \end{array} \right\}.$$

We have one last lemma before we move on to the main theorem.

Lemma 9. *If σ and α are sum indecomposable, then for any permutation π, $\mu[\sigma, \alpha] W(\sigma, \alpha, \pi)$ gives the contribution of the set of families $\mathfrak{F}(r\alpha)$ to the Möbius sum, where r is any positive integer.*

Proof. By Lemma 6, we only need consider the contribution made by $r\alpha$ and $(r + 1)\alpha$, where r is the smallest integer such that $1 \oplus r\alpha \oplus 1 \not\leq \pi$.

If $\sigma \not\leq r\alpha$, or $r\alpha \not\leq \pi$, then $\mathfrak{F}(r\alpha)$ makes no net contribution to the Möbius sum. Now assume that $\sigma \leq r\alpha \leq \pi$. First, we can see that if $1 \oplus r\alpha \not\leq \pi$, or $r\alpha \oplus 1 \not\leq \pi$ then $(r + 1)\alpha \not\leq \pi$. We can also see that if $1 \oplus r\alpha \oplus 1 \not\leq \pi$ then

$1 \oplus r\alpha$	$r\alpha \oplus 1$	$(r+1)\alpha$	Möbius contribution
$\leq \pi$	$\leq \pi$	$\leq \pi$	0
$\leq \pi$	$\leq \pi$	$\not\leq \pi$	$-\mu[\sigma, \alpha]$
$\leq \pi$	$\not\leq \pi$	$\not\leq \pi$	0
$\not\leq \pi$	$\leq \pi$	$\not\leq \pi$	0
$\not\leq \pi$	$\not\leq \pi$	$\not\leq \pi$	$\mu[\sigma, \alpha]$

Table 1: Möbius contribution from family members.

$1 \oplus (r+1)\alpha \not\leq \pi$ and $(r+1)\alpha \not\leq \pi$. The possibilities remaining are itemised in Table 1, where the Möbius contribution is determined by applying Lemma 5. We can see that in every case $W(\sigma, \alpha, \pi)$ provides the correct weight for the Möbius function $\mu[\sigma, \alpha]$. $\square$

We are now in a position to present the main theorem for this section.

Theorem 10. *If σ is a sum indecomposable permutation, and $|\pi| > 3$, then*

$$\mu[\sigma, \pi] = -\sum_{\alpha \in \mathfrak{C}_{\sigma,\pi}} \mu[\sigma, \alpha] W(\sigma, \alpha, \pi).$$

Proof. Let $\alpha \leq \pi$ be an indecomposable permutation.

Using Lemmas 4 and 8 we can see that any permutations not in the form $\boxed{1\oplus}\ r\alpha\ \boxed{\oplus 1}$ can be excluded from $\mathfrak{C}_{\sigma,\pi}$, as these permutations make no net contribution to the Möbius sum.

For every α, by Lemma 9, $\mu[\sigma, \alpha] W(\sigma, \alpha, \pi)$ provides the contribution to the Möbius sum of all families $\mathfrak{F}(r\alpha)$, where r is a positive integer. If $W(\sigma, \alpha, \pi) = 0$, then α is excluded from $\mathfrak{C}_{\sigma,\pi}$. $\square$

Theorem 10 reduces the number of permutations that need to be considered as part of the Möbius sum. We can see that the largest permutation in $\mathfrak{C}_{\sigma,\pi}$ must have length less than $|\pi|$, and so we can apply Theorem 10 recursively to the permutations in $\mathfrak{C}_{\sigma,\pi}$ to determine their Möbius values. In this recursion, if we are attempting to determine $\mu[\sigma, \lambda]$, we can stop if $|\sigma| = |\lambda|$ or $|\sigma| = |\lambda| - 1$, as in these cases $\mu[\sigma, \lambda]$ is $+1$ and -1 respectively.

4 Increasing oscillations

We now move on to increasing oscillations. Given an indecomposable permutation σ, and an increasing oscillation π, our aim in this section is to describe $\mathfrak{C}_{\sigma,\pi}$ in precise terms. We will find a sum for the Möbius function which only requires the evaluation of simple inequalities.

We will frequently have to refer to a descending permutation of length 2 (21). We use $\delta = 21$ for simplicity.

Since the Möbius function for oscillating permutations with length less than 4 is trivial to determine, for the remainder of this section we assume that any oscillating permutation has length at least 4.

We now determine what permutations contained in an increasing oscillation have a non-zero contribution to the Möbius sum.

Lemma 11. *If π is an increasing oscillation, then every permutation $\lambda \leq \pi$ that contributes to the Möbius sum has the shape*

$$\boxed{1\oplus}\ \left(r\left(\boxed{1\oslash}\ \delta^k\ \boxed{\oslash 1}\right)\right)\ \boxed{\oplus 1}.$$

Proof. We start by showing that if π is an increasing oscillation, and $\sigma = \sigma_1 \oplus \sigma_2 \oplus \ldots \oplus \sigma_m$, where each σ_i is sum indecomposable, and $\sigma \leq \pi$ then every $\sigma_i > 1$ has the shape

$$\boxed{1\oslash}\ \delta^k\ \boxed{\oslash 1}, \qquad \text{where } k > 0. \tag{4}$$

It is trivial to see that if σ is an an increasing oscillation, then σ has the shape given in (4). It is thus sufficient to show that if some $\sigma_i = \boxed{1\oslash}\ \delta^k\ \boxed{\oslash 1}$, then deleting a single point yields a permutation that can be written in the shape given in (4). If $k = 1$, then by examination we find that the four possibilities have the right shape. Assume now that $k > 1$. Deleting a point from the start of σ_i yields either $1 \oplus \delta^{k-1}\ \boxed{\oslash 1}$ or $\delta^k\ \boxed{\oslash 1}$. If $\sigma_i = \delta^k\ \boxed{\oslash 1}$, then deleting the second point gives $1 \oslash \delta^{k-1}\ \boxed{\oslash 1}$, otherwise deleting the second point gives $\delta \oplus \delta^{k-1}\ \boxed{\oslash 1}$. Deleting the last or last-but-one point from the end gives similar results. Finally, deleting a point that is not the first, second, last or last-but-one point results in $\sigma_i = \sigma_{i_1} \oplus \sigma_{i_2}$, where σ_{i_1} and σ_{i_2} both have the shape given in (4). Thus every σ_i has the shape $\boxed{1\oslash}\ \delta^k\ \boxed{\oslash 1}$.

To complete the proof, we now see that by Lemma 8, we can ignore $\lambda = 1$ and $\lambda = 1 \oplus 1$. If $\lambda = \lambda_1 \oplus \lambda_2 \oplus \ldots \oplus \lambda_m \leq \pi$, then by the argument above, every λ_i has the shape $\boxed{1\oslash}\ \delta^k\ \boxed{\oslash 1}$. Applying Lemma 4 completes the proof. □

We note here that the proof of Lemma 11 shows that every permutation contained in an increasing oscillation that contributes to the Möbius sum has the shape $\boxed{1\oplus}\ r\alpha\ \boxed{\oplus 1}$, where α is itself an increasing oscillation.

Following Lemma 6, it is clear that, with $\alpha = \boxed{1\oslash}\ \delta^k\ \boxed{\oslash 1}$, for any family $\mathfrak{F}(r\alpha)$, we only need consider the cases $r\alpha$ and $(r+1)\alpha$ where r is the smallest integer such that $1 \oplus r\alpha \oplus 1 \not\leq \pi$.

Given some $\pi = W_n$ or M_n, we will find inequalities that relate n, r and k and the "shape" of α that will allow us to find the values that contribute to the Möbius sum. We know from Lemma 11 the shape of the permutations that contribute to the Möbius sum. For each of the four types of increasing oscillation (W_{2n}, W_{2n-1}, M_{2n} and M_{2n-1}), we can examine how each shape can be embedded so that the unused points at the start of the increasing oscillation are minimised. Figure 2 shows examples of embeddings into W_{2n}. This gives

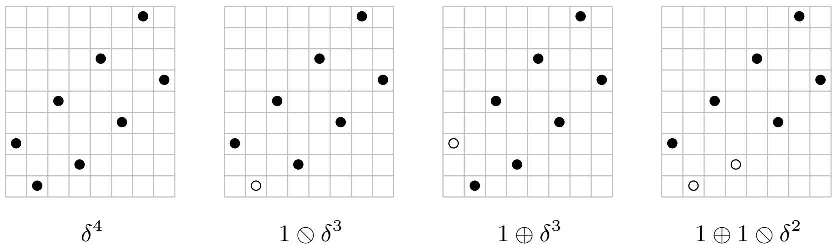


Figure 2: Embedding the start of α in W_{2n}.

us an inequality relating to the start of the embedding. Similarly, we can find inequalities for the end of the embedding. We can also find inequalities that relate to the interior (when $r > 1$), and Figures 3 and 4 show examples of this. We can use these inequalities to determine what values of k will allow the shape to be embedded. For each allowable value of k, we can then determine the maximum value of r such that $1 \oplus r\alpha \oplus 1 \not\leq \pi$. This then means that, by evaluating inequalities alone, we can identify the specific permutations that could contribute to the Möbius sum.

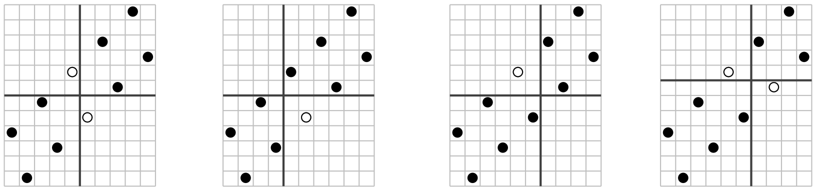
We first have two lemmas that examine inequalities at the start and end of an embedding. We adopt the convention that in δ^k, k is a positive integer, and in δ^{k^+}, k^+ is a positive integer greater than one.

Lemma 12. *If π is an increasing oscillation, and $\alpha \leq \pi$ is sum indecomposable, then in any embedding of an element λ of $\mathfrak{F}(r\alpha)$ into π, the minimum number of unused points at the start of π depends on the start of λ, and on π, and is as shown below:*

Start of λ	W_{2n}	W_{2n-1}	M_{2n}	M_{2n-1}
$\delta \ldots$	0	0	0	0
$\delta^{k^+} \ldots$	0	0	1	1
$1 \ominus \delta^k \ldots$	1	1	0	0
$1 \oplus \delta \ldots$	1	1	1	1
$1 \oplus \delta^{k^+} \ldots$	1	1	2	2
$1 \oplus 1 \ominus \delta^k \ldots$	2	2	1	1

Proof. It is clear that if we minimise the number of points at the start of an embedding, then the number of unused points depends on the shape of π, and the start of α. The values in Lemma 12 are found by considering each of the possibilities. We illustrate some of these cases in Figure 2. $\square$

Lemma 13. *If π is an increasing oscillation, and $\alpha \leq \pi$ is sum indecomposable, then in any embedding of an element λ of $\mathfrak{F}(r\alpha)$ into π, the minimum number of unused points at the end of π depends on the end of λ, and on π, and is as shown below:*



$\delta^2 \oplus \delta^2$ $\quad$ $\delta^2 \oplus 1 \oslash \delta^2$ $\quad$ $\delta^2 \oslash 1 \oplus \delta^2$ $\quad$ $\delta^2 \oslash 1 \oplus 1 \oslash \delta^1$

Figure 3: Packing α as close as possible when $\alpha \neq \delta$.

End of λ	W_{2n}	W_{2n-1}	M_{2n}	M_{2n-1}
$\ldots\delta$	0	0	0	0
$\ldots\delta^{k^+}$	0	1	1	0
$\ldots\delta^k \oslash 1$	1	0	0	1
$\ldots\delta \oplus 1$	1	1	1	1
$\ldots\delta^{k^+} \oplus 1$	1	2	2	1
$\ldots\delta^k \oslash 1 \oplus 1$	2	1	1	2

Proof. We examine all the possibilities as we did in Lemma 12. ☐

We now consider how closely copies of some sum indecomposable α can be embedded into π. This leads to two inequalities that relate α, π and the maximum number of copies of α that can be embedded in π. Where $\alpha \neq \delta$, the shape of α fixes the way the two copies can be embedded in an increasing oscillation. If $\alpha = \delta$, then we will see that there are choices for the embedding.

Lemma 14. *If π is an increasing oscillation, and $\alpha \neq \delta$, and $\alpha \leq \pi$ is sum indecomposable, then in any embedding of $r\alpha$ into π, the minimum number of points between the start and end of $r\alpha$ depends on the shape of α, and is as shown below:*

α	*Points in* $r\alpha$	*Unused points*	*Minimum points*
δ^{k^+}	$2kr$	$2r-2$	$2kr+2r-2$
$1 \oslash \delta^k$	$2kr+r$	$r-1$	$2kr+2r-1$
$\delta^k \oslash 1$	$2kr+r$	$r-1$	$2kr+2r-1$
$1 \oslash \delta^k \oslash 1$	$2kr+2r$	$2r-2$	$2kr+4r-2$

Proof. If $r = 1$, then there are no unused points, and so the minimum number of points depends solely on the points in α, and the table reflects this.

Assume now that $r > 1$. If $\alpha \neq \delta$, then we can see that the interleave fixes the layout of each copy of α, so we simply pack each copy as close as possible. This packing clearly depends on the start and end of α, and it is simple to examine the four possibilities. Examples are shown in Figure 3. ☐

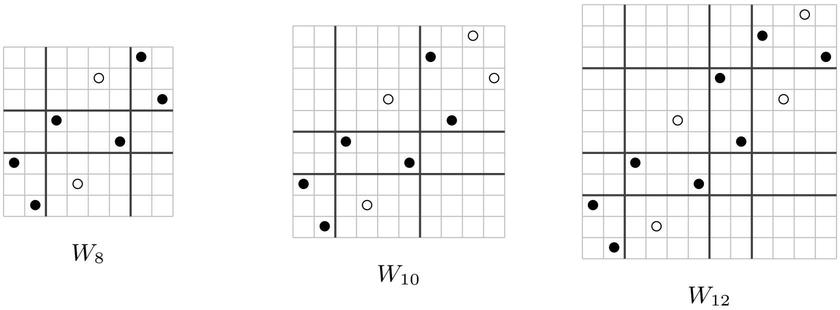


Figure 4: Examples of unused points when embedding $r\delta$.

We now turn to the case where $\alpha = \delta$. This is more complex than the previous cases. We can see that there must be at least one point between each copy of α. We can insert each copy of δ in two ways, one where the points are horizontally adjacent, and one where the points are vertically adjacent. These alternatives can be seen in Figure 4. Alternating these means that there will be exactly one point between each copy of α, so this sort of embedding minimises the number of points between the start and end of $r\alpha$. The complication in this case relates to how we start and end the embedding. We illustrate this by showing, in Figure 4, maximal embeddings where we are embedding into W_8, W_{10} and W_{12}. A detailed examination of each possible case gives us our second inequality.

Lemma 15. *If π is an increasing oscillation, and $\alpha = \delta$ then for $r\alpha$ to be contained in π we must have $3r - 1 \leq 2n$ for $\pi \in \{W_{2n}, M_{2n}\}$, and $3r \leq 2n$ for $\pi \in \{W_{2n-1}, M_{2n-1}\}$.*

Proof. In every case we start by embedding the first δ into the first two elements of the permutation. Thereafter, we embed each successive δ as close as possible to the preceding δ. The minimum number of elements to embed r copies of δ will be $2r$ elements to hold the points of the δs, and $r - 1$ intermediate empty elements. For W_{2n} andM_{2n}, this then gives $3r - 1 \leq 2n$, and for W_{2n-1} and M_{2n-1} we obtain $3r - 1 \leq 2n - 1$. $\square$

We now have a complete understanding of the number of points required to embed any permutation that contributes to the Möbius sum into an increasing oscillation. The following Lemma summarises the situation.

Lemma 16. *If π is an increasing oscillation, and $\alpha = \boxed{1 \obslash}\, \delta^k \,\boxed{\obslash 1} \leq \pi$ (so α is sum indecomposable), then for $r\alpha$ to be contained in π, the inequality in the table below must be satisfied, where $k \geq 1$, and $k^+ \geq 2$.*

π	*Shape of* α	*Inequality*
W_{2n}, M_{2n}	δ	$3r-1 \leq 2n$
W_{2n-1}, M_{2n-1}	δ	$3r \leq 2n$
W_{2n}	δ^{k^+}	$2kr+2r-2 \leq 2n$
W_{2n-1}	$1 \ominus \delta^k$	$2kr+2r+2 \leq 2n$
M_{2n-1}	$\delta^k \ominus 1$	$2kr+2r+2 \leq 2n$
M_{2n}	$1 \ominus \delta^k \ominus 1$	$2kr+4r-2 \leq 2n$
$W_{2n}, W_{2n-1}, M_{2n-1}$	$1 \ominus \delta^k \ominus 1$	$2kr+4r \leq 2n$
All other cases		$2kr+2r \leq 2n$

Proof. We apply Lemmas 12, 13, 14 and 15 to the possibilities for π and α. $\square$

As a consequence of Lemmas 12 and 13 we can define a relationship between the minimum number of points required to embed some $r\alpha$, and the minimum number of points required to embed $1 \oplus r\alpha$, $r\alpha \oplus 1$ and $1 \oplus r\alpha \oplus 1$.

Corollary 17. *If π is an increasing oscillation, and $\alpha \leq \pi$ is sum indecomposable and if the minimum number of points required to embed $r\alpha$ into π is C, then the minimum number of points required to embed $1 \oplus r\alpha$ into π is $C+2$, the minimum number of points required to embed $r\alpha \oplus 1$ into π is $C+2$, and the minimum number of points required to embed $1 \oplus r\alpha \oplus 1$ into π is $C+4$.*

Proof. We can see from Lemmas 12 and 13 that adding $1 \oplus$ at the start of a permutation increases the number of points required by two – one for the new point, and one that is unused. Similarly, adding $\oplus 1$ at the end increases the points required by two. $\square$

Lemma 16 gives us inequalities that any $r\alpha$ must satisfy to ensure that $r\alpha \leq \pi$. Further, Corollary 17 gives us inequalities that, for a given $r\alpha$ allow us to determine if $1 \oplus r\alpha \leq \pi$, $r\alpha \oplus 1 \leq \pi$ and $1 \oplus r\alpha \oplus 1 \leq \pi$. We can therefore determine what values of r and k will result in $\mathfrak{F}(r\alpha)$ contributing to the Möbius function. We now consider inequalities that relate σ and α, so that we can determine if $\sigma \leq \alpha$ using an inequality.

Lemma 18. *If $\sigma > 1$ is an increasing oscillation, and $\alpha = \boxed{1 \ominus}\, \delta^k \,\boxed{\ominus 1}$, then for σ to be contained in α the inequality in the table below must be satisfied, where $k \geq 1$, and $k^+ \geq 2$.*

σ	*Shape of* α	*Inequality*
$W_{2n-1}, M_{2n}, M_{2n-1}$	δ	*False*
W_{2n-1}	$\delta^k \ominus 1$	$k \geq n-1$
M_{2n-1}	$1 \ominus \delta^k$	$k \geq n-1$
$W_{2n-1}, M_{2n}, M_{2n-1}$	$1 \ominus \delta^k \ominus 1$	$k \geq n-1$
M_{2n}	δ^{k^+}	$k \geq n+1$
All other cases		$k \geq n$

Proof. We examine all possible cases. $\square$

We are now nearly ready to present the main theorem for this section. Informally, for each possible shape of permutation α, we will first find the minimum and maximum values of k such that $\sigma \leq \alpha \leq \pi$, as any other values of k result in α being outside the interval. For each α and each k, we then determine the minimum value of r such that $1 \oplus r\alpha \oplus 1 \not\leq \pi$. We can then use this value of r (assuming it is non-zero) to determine the weight to be applied to $\mu[\sigma, \alpha]$. The set of αs with a non-zero weight is a contributing set $\mathfrak{C}_{\sigma,\pi}$. At this point we can substitute a value for any $\mu[\sigma, \alpha]$ where $|\sigma| \leq |\alpha| - 1$. We then use the same process recursively to determine the contributing set for the remaining elements of $\mathfrak{C}_{\sigma,\pi}$.

We first define some supporting functions. Let $\mathrm{RawMinK}(\sigma, \alpha)$ be the minimum value of k that satisfies the inequality in Lemma 18. For the first inequality, which is always false, we set $k = |\pi|$, as this will force the sum, defined later in Theorem 19, to be empty.

Let $\mathrm{MinK}(\sigma, \alpha)$ be defined as

$$\mathrm{MinK}(\sigma, \alpha) = \begin{cases} 1 & \text{If } \sigma = 1 \text{ and } \alpha \neq \delta^{k^+}, \\ 2 & \text{If } \sigma = 1 \text{ and } \alpha = \delta^{k^+}, \\ \mathrm{RawMinK}(\sigma, \alpha) & \text{otherwise.} \end{cases}$$

Observe that for any $k < \mathrm{MinK}(\sigma, \alpha)$, we have $\alpha < \sigma$, and so $\mathfrak{F}(k\alpha)$ makes no net contribution to the Möbius sum.

Let $\mathrm{MaxK}(\alpha, \pi)$ be defined as the maximum value of k that satisfies the inequality in Lemma 16, if the shape of α and the shape of π are different; and one less than the maximum value of k that satisfies the inequality if the shape of α and the shape of π are the same. For the first two inequalities, which do not involve k, we set $\mathrm{MaxK}(\alpha, \pi) = 1$ if the inequality is satisfied, and $\mathrm{MaxK}(\alpha, \pi) = 0$ if not. Observe here that for any $k > \mathrm{MaxK}(\alpha, \pi)$ we have $\alpha \not< \pi$, and so $\mathfrak{F}(k\alpha)$ makes no contribution to the Möbius sum.

We define the weight function for increasing oscillations, $W_{io}(\sigma, \alpha, \pi)$, as

$$W_{io}(\sigma, \alpha, \pi) = \begin{cases} 1 & \text{If } r\alpha \oplus 1 \not\leq \pi, \\ -1 & \text{If } r\alpha \oplus 1 \leq \pi \text{ and } (r+1)\alpha \not\leq \pi, \\ 0 & \text{Otherwise,} \end{cases}$$

where r is the smallest integer such that $1 \oplus r\alpha \oplus 1 \not\leq \pi$. These conditions are simpler than those given in the weight function (3) for Theorem 10 as, by Corollary 17, if $r\alpha \oplus 1 \not\leq \pi$ then $1 \oplus r\alpha \not\leq \pi$ and vice-versa. Furthermore, we will see that this weight function is only used when $\sigma \leq r\alpha \leq \pi$.

We are now in a position to state our main theorem for this section. In this theorem, we consider the contribution to the Möbius sum of each possible shape of some sum indecomposable α. There are five possible shapes, and, given that the expression for each shape is identical, we abuse notation slightly by writing our theorem as a sum over the shapes, thus the first sum in Theorem 19 is over the possible shapes of α, where four of the shapes have a parameter k. For each shape, the limits on the interior sum determine the minimum and maximum values of k, using the summation variable v. We use the notation α_v to represent the actual permutation that has the shape α, where the parameter k has been set to the value of v. As an example, if $\alpha = 1 \oslash \delta^k$, and $v = 2$, then $\alpha_v = 1 \oslash \delta^2 = 24153$.

Theorem 19. *Let π be an increasing oscillation, and let $\sigma \leq \pi$ be sum indecomposable. Then*

$$\mu[\sigma, \pi] = \sum_{\alpha \in \mathfrak{S}} \sum_{v=\mathrm{MinK}(\sigma,\alpha)}^{\mathrm{MaxK}(\alpha,\pi)} \mu[\sigma, \alpha_v] W_{io}(\sigma, \alpha_v, \pi)$$

where the first sum is over the possible shapes of a sum indecomposable permutation contained in an increasing oscillation, so $\mathfrak{S} = \{\delta,\ \delta^{k^+},\ 1 \oslash \delta^k,\ \delta^k \oslash 1,\ 1 \oslash \delta^k \oslash 1\}$.

Proof. By Lemma 11 the only sum-decomposable permutations contained in an increasing oscillation that contribute to the Möbius sum are $\mathfrak{F}(r\alpha)$, where $\alpha \in \mathfrak{S}$.

If we set $r = 1$, then for each α in $\mathfrak{S}$ Lemma 18 provides the smallest value of k such that $\sigma \leq \alpha$. If there is no such value of k, then we use $|\pi|$, as the maximum value of k must be smaller than this, and so the sum is empty.

Again setting $r = 1$, for each α in $\mathfrak{S}$ Lemma 16 provides the maximum value of k such that $\alpha \leq \pi$. If there is no value of k that satisfies the inequality, then we set $\mathrm{MaxK}(\alpha, \pi) = 0$, thus forcing the sum to be empty.

Thus the permutations α_v in the sum

$$\sum_{\alpha \in \mathfrak{S}} \sum_{v=\mathrm{MinK}(\sigma,\alpha)}^{\mathrm{MaxK}(\alpha,\pi)}$$

are those that could contribute to the Möbius sum, and for any α_v not included in the sum, $\mathfrak{F}(r\alpha_v)$ has a zero contribution to the Möbius sum for any r.

Further, we can see from the construction method that any α_v included in the sum has $\sigma \leq r\alpha_v \leq \pi$ for at least one value of r, as if this was not the case, then we would have $\mathrm{MinK}(\sigma, \alpha) > \mathrm{MaxK}(\alpha, \pi)$, and so the sum would be empty.

We have therefore shown that the α_v-s included in the sum form a contributing set, and we could therefore set $\mathfrak{C}_{\sigma,\pi}$ to be those α_v-s, and use Theorem 10. We now show that the increasing oscillation weight function $W_{io}(\sigma, \alpha, \pi)$ is equivalent to $W(\sigma, \alpha, \pi)$ as defined in the general case.

By Corollary 17, if $r\alpha \oplus 1 \not\leq \pi$ then $1 \oplus r\alpha \not\leq \pi$ and vice-versa, and so the condition for $r\alpha \oplus 1$ also covers $1 \oplus r\alpha$. As discussed above, we know that there is at least one value of r such that $\sigma \leq r\alpha \leq \pi$, and so $W_{io}(\sigma, \alpha, \pi)$ does not need to include this condition. Thus the increasing oscillation weight function $W_{io}(\sigma, \alpha, \pi)$ is equivalent to $W(\sigma, \alpha, \pi)$ as defined in the general case. □

4.1 Example of Theorem 19

As an example of Theorem 19 in action, we show how to determine

$$\mu[3142, 315274968] = \mu[\delta^2, \delta^4 \oslash 1)].$$

We start by considering each possible shape of α, setting $r = 1$, and then using the inequalities in Lemmas 16 and 18 to determine the minimum and maximum values of k. This gives us

Shape of α	Minimum k	Maximum k
δ	1	1
δ^k	2	4
$1 \oslash \delta^k$	2	3
$\delta^k \oslash 1$	2	3
$1 \oslash \delta^k \oslash 1$	2	3

For each shape of α, and each value of k, we then use the inequalities in Lemma 16 to determine the minimum value of r such that $1 \oplus r\alpha \oplus 1 \not\leq \pi$, and we then calculate the weight using this value of r. This gives

α	r	Weight
δ	No possibilities	
δ^2	2	1
δ^3	1	-1
δ^4	1	-1
$1 \oslash \delta^2$	1	-1
$1 \oslash \delta^3$	1	-1
$\delta^2 \oslash 1$	2	1
$\delta^3 \oslash 1$	1	-1
$1 \oslash \delta^2 \oslash 1$	1	-1
$1 \oslash \delta^3 \oslash 1$	1	-1

This leads to the following initial expression:

$$\begin{aligned}\mu[\delta^2, \delta^4 \oslash 1)] =& \mu[\delta^2, \delta^2] - \mu[\delta^2, \delta^3] - \mu[\delta^2, \delta^4] \\ & - \mu[\delta^2, 1 \oslash \delta^2] - \mu[\delta^2, 1 \oslash \delta^3] \\ & + \mu[\delta^2, \delta^2 \oslash 1] - \mu[\delta^2, \delta^3 \oslash 1] \\ & - \mu[\delta^2, 1 \oslash \delta^2 \oslash 1] - \mu[\delta^2, 1 \oslash \delta^3 \oslash 1]\end{aligned}$$

We know that $\mu[\delta^2, \delta^2] = 1$, and that $\mu[\delta^2, 1 \oslash \delta^2] = \mu[\delta^2, \delta^2 \oslash 1] = -1$. Applying Theorem 19 recursively to the other intervals eventually yields $\mu[\delta^2, \delta^4 \oslash 1)] = -6$.

5 Concluding remarks

The results in [1] provide two recurrences to handle the case where π is decomposable. This work handles the case where σ is indecomposable. It overlaps with [1] when σ is indecomposable and π is decomposable. This leaves the case where σ is decomposable and π is indecomposable for further investigation.

We can see that by symmetry $\mu[\sigma, W_{2n}] = \mu[\sigma^{-1}, M_{2n}]$, and that $\mu[\sigma, W_{2n-1}] = \mu[\sigma^{-1}, M_{2n-1}]$.

If we consider the value of the principal Möbius function, $\mu[1, \pi]$, where π is either W_n or M_n, then it is simple to show that the absolute value of the principal Möbius function is bounded above by 2^n. The weight function for increasing oscillations can be ± 1, and we can see no obvious reason why there should not be two distinct values, i and j, with the same parity, such that the signs of $\mu[1, W_i]$ and $\mu[1, W_j]$ were different. We have experimental evidence, based on the values of W_n and M_n for $n = 1 \dots 2000000$ that suggests that $\mu[1, W_{2n}] < 0$, and that $\mu[1, W_{2n-1}] > 0$.

Figure 5 is a log-log plot of the absolute values of $\mu[1, W_{2n}]$ from $n = 8000$ to $n = 10000$. As can be seen, there seems to be some evidence of "banding", and we have confirmed that this pattern continues for all values examined. Examination of the values of $\mu[1, W_{2n-1}]$ reveals the same patterns.

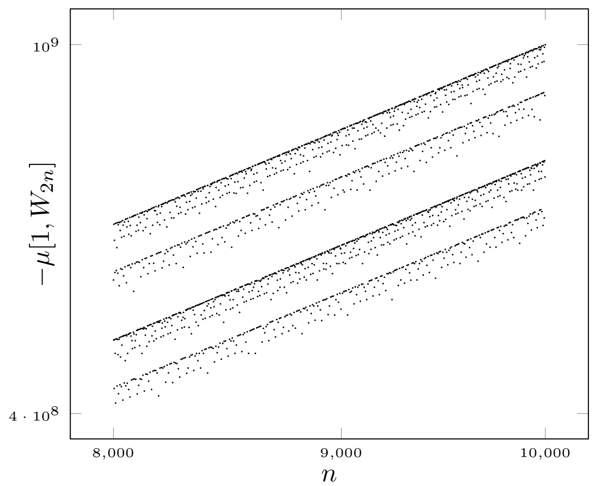


Figure 5: Log-Log plot of $|W_{2n}|$.

Following discussions at Permutation Patterns 2017, Vít Jelínek [2] provided the following conjecture (rephrased to reflect our notation).

Conjecture 20. Let $M(n)$ denote the absolute value of the Möbius function

$\mu[1, W_n] = \mu[1, M_n]$. Then for $n > 50$ we have

$$\begin{aligned}
M(2n) = n^2 &\iff n+1 \text{ is prime and } n \equiv 0 \pmod 6 \\
M(2n) = n^2 - 1 &\iff n+1 \text{ is prime and } n \equiv 4 \pmod 6 \\
M(2n+1) = n^2 - n &\iff n+1 \text{ is prime and } n \equiv 0 \pmod 6 \\
M(2n+1) = n^2 - n - 1 &\iff n+1 \text{ is prime and } n \equiv 4 \pmod 6
\end{aligned}$$

Further, Jelínek notes that there does not seem to be any other "small" constant k such that $M(n) = (n^2 - k)/4$ infinitely often.

We also have the following conjecture relating to the "banding" of the values.

Conjecture 21. Let $M(n)$ denote the absolute value of the Möbius function $\mu[1, W_n] = \mu[1, M_n]$. Let $E(n) = M(n)/(n^2)$, and let $O(n) = M(n)/(n^2 + n)$. Then there exist constants $0 < a < b < c < d < e < f < g < 1$ such that

$$\begin{aligned}
E(12n+10) &\in [a, b] & O(12n+11) &\in [a, b] \\
E(12n+2) &\in [c, d] & O(12n+3) &\in [c, d] \\
E(12n+6) &\in [c, d] & O(12n+7) &\in [c, d] \\
E(12n+4) &\in [e, f] & O(12n+5) &\in [e, f] \\
E(12n+8) &\in [g, 1] & O(12n+9) &\in [g, 1] \\
E(12n) &\in [g, 1] & O(12n+1) &\in [g, 1]
\end{aligned}$$

Examining the first 1500000 values of the Möbius function gives the following values for the constants.

a	b	c	d	e	f	g
0.615	0.680	0.692	0.760	0.821	0.896	0.923

The *complete nearly-layered* permutations are formed by interleaving descending permutations. Formally, a complete nearly-layered permutation has the form

$$\alpha_1 \oslash \alpha_2 \oslash \ldots \oslash \alpha_{k-1} \oslash \alpha_k$$

where each α_i is a descending permutation, with $\alpha_i > 1$ for $i = 2, \ldots, k-1$. If we set $\alpha_i = 21$ for $i = 2, \ldots, k-1$, and $\alpha_1, \alpha_k \in \{1, 21\}$, then we obtain the increasing oscillations.

The computational approach taken for increasing oscillations could, we think, be adapted to complete nearly-layered permutations. It is clear that the equivalent of the inequalities in Lemmas 16 and 18 would be somewhat more complex than those found here, but we believe that it should be possible to define an algorithm that could determine the Möbius function for complete nearly-layered permutations where the lower bound is sum indecomposable.

References

[1] A. Burstein, V. Jelínek, E. Jelínková, and E. Steingrímsson. The Möbius function of separable and decomposable permutations. *Journal of Combinatorial Theory, Series A*, 118(8):2346–2364, 2011.

[2] V. Jelínek. Private correspondence, 2017.

[3] P. R. McNamara and E. Steingrímsson. On the topology of the permutation pattern poset. *Journal of Combinatorial Theory, Series A*, 134:1–35, 2015.

[4] B. E. Sagan and V. Vatter. The Möbius function of a composition poset. *Journal of Algebraic Combinatorics*, 24(2):117–136, sep 2006.

[5] J. P. Smith. On the Mobius Function of Permutations With One Descent. *The Electronic Journal of Combinatorics*, 21(2):P2.11, 2014.

[6] J. P. Smith. A Formula for the Mobius Function of the Permutation Poset Based on a Topological Decomposition. *Canada DMTCS proc. BC*, pages 1099–1110, 2016.

[7] J. P. Smith. Intervals of permutations with a fixed number of descents are shellable. *Discrete Mathematics*, 339(1):118–126, 2016.

[8] E. Steingrímsson and B. E. Tenner. The Mobius function of the permutation pattern poset. *Journal of Combinatorics*, 1(1):39–52, 2010.

[9] H. S. Wilf. The patterns of permutations. *Discrete Mathematics*, 257(2):575–583, 2002.