

Intervals of permutations and the principal Möbius function

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Abstract

We show that the proportion of permutations of length n with principal Möbius function equal to zero, $Z(n)$, is asymptotically bounded below by 0.3995.

If a permutation π contains two intervals of length 2, where one interval is an ascent and the other a descent, then we show that the value of the principal Möbius function $\mu[1, \pi]$ is zero, and we use this result to find the lower bound for $Z(n)$.

We also show that if a permutation ϕ has certain properties, then any permutation π which contains an interval order-isomorphic to ϕ has $\mu[1, \pi] = 0$.

1 Introduction

Let σ and π be permutations of natural numbers. We say that π *contains* σ if there is a sub-sequence of points of π that is order-isomorphic to σ . As an example, 3624715 contains 3142 as the sub-sequences 6275 and 6475. If σ is contained in π , then we write $\sigma \leq \pi$.

The set of all permutations is a poset under the partial order given by containment. A closed interval $[\sigma, \pi]$ in a poset is the set defined by $\{\tau : \sigma \leq \tau \leq \pi\}$, and a half-open interval $[\sigma, \pi)$ is the set $\{\tau : \sigma \leq \tau < \pi\}$. The Möbius function is defined recursively on an interval of a poset $[\sigma, \pi]$ as:

$$\mu[\sigma, \pi] = \begin{cases} 0 & \text{if } \sigma \not\leq \pi, \\ 1 & \text{if } \sigma = \pi, \\ - \sum_{\tau \in [\sigma, \pi)} \mu[\sigma, \tau] & \text{otherwise.} \end{cases}$$

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From the definition of the Möbius function, it follows that if $\sigma < \pi$, then $\sum_{\tau \in [\sigma, \pi]} \mu[\sigma, \tau] = 0$.

In this paper we are mainly concerned with the *principal Möbius function* of a permutation π , written $\mu[\pi]$, where $\mu[\pi] = \mu[1, \pi]$. We consider cases where the value of the principal Möbius function $\mu[\pi]$ can be determined by examining small localities of π . In this section, we use “locally zero” to describe results of this form. In Section 4 we define “strongly zero” permutations which share some characteristics with locally zero permutations. We show that, asymptotically, the proportion of permutations where the principal Möbius function is zero is bounded below by 0.3995. We also show that our results for the principal Möbius function can be extended to intervals where the lower bound is not 1.

The question of the Möbius function in the permutation poset was first raised by Wilf [16]. The first result was by Sagan and Vatter [9], who determined the Möbius function on intervals of layered permutations. Steingrímsson and Tenner [15] found pairs of permutations (σ, π) where $\mu[\sigma, \pi] = 0$.

Burstein, Jelínek, Jelínková and Steingrímsson [5] found a recursion for the Möbius function for sum and skew decomposable permutations. They used this to determine the Möbius function for separable permutations. Their results for sum and skew decomposable permutations implicitly include the first locally zero result, which is that, up to symmetry, if a permutation π with length greater than two begins 12, then $\mu[\pi] = 0$.

Smith [10], found an explicit formula for the Möbius function on the interval $[1, \pi]$ for all permutations π with a single descent. Smith’s paper includes a lemma, reproduced here as Lemma 4, which is that if a permutation π contains an interval order-isomorphic to 123, then $\mu[\pi] = 0$. This is the second instance in the literature of a locally zero result. The result in [5] requires that the permutation starts with a particular sequence. Smith’s result is, in some sense, more general, as the critical interval (123) can occur in any position. We will see later that this lemma is the first instance of a strongly zero result.

Smith [11], has explicit expressions for the Möbius function $\mu[\sigma, \pi]$ when σ and π have the same number of descents. In [12], Smith found an expression that determines the Möbius function for all intervals in the poset, although the expression involves a rather complicated double sum, which includes $\sum_{\tau \in [\sigma, \pi]} \mu[\sigma, \tau]$.

Brignall and Marchant [4] showed that if the lower bound of an interval is indecomposable, then the Möbius function depends only on the indecomposable permutations contained in the upper bound, and used this result to find a fast polynomial algorithm for finding $\mu[\pi]$ where π is an increasing oscillation.

2 Definitions and notation

An *interval* of a permutation π is a non-empty set of contiguous indices $i, i + 1, \dots, j$ where the set of values $\{\pi_i, \pi_{i+1}, \dots, \pi_j\}$ is also contiguous. A permutation of length n that only has intervals of length 1 or n is a *simple* permutation. As an example, 419725386 is a simple permutation. This is shown in Figure 1.

An *adjacency* in a permutation is an interval of length two. If a permutation contains a monotonic interval of length three or more, then each sub-interval of length two is an adjacency. As examples, 367249815 has two adjacencies, 67 and 98; and 1432 also has two adjacencies, 43 and 32. If an adjacency is ascending, then it is an *up-adjacency*, otherwise it is a *down-adjacency*.

If a permutation π contains at least one up-adjacency, and at least one down-adjacency, then we say that π has *opposing adjacencies*. An example of a permutation with opposing adjacencies is 367249815, which is shown in Figure 1.



Figure 1: A simple permutation and a permutation with opposing adjacencies.

A *triple adjacency* in a permutation is an interval of length three that is monotonic; that is, the interval is order-isomorphic to 123 or 321.

A permutation that does not contain any adjacencies is *adjacency-free*. Some early papers use the term “strongly irreducible” for what we call adjacency-free permutations. See, for example, Atkinson and Stitt [3].

Given a permutation σ with length n , and permutations $\alpha_1, \dots, \alpha_n$, where at least one of the α_i -s is not the empty permutation, ϵ , the *inflation* of σ by $\alpha_1, \dots, \alpha_n$ is the permutation found by removing the point σ_i if $\alpha_i = \epsilon$, and replacing σ_i with an interval isomorphic to α_i otherwise. Note that this is slightly different to the standard definition of inflation, which does not allow inflation by the empty permutation. We write inflations as $\sigma[\alpha_1, \dots, \alpha_n]$. As examples, $3624715[1, 12, 1, 1, 21, 1, 1] = 367249815$, and $3624715[\epsilon, 1, 1, \epsilon, 1, \epsilon, 1] = 3142$. A *proper inflation* is an inflation $\sigma[\alpha_1, \dots, \alpha_n]$ where none of the α_i -s are the empty permutation.

In many cases we will be interested in permutations where most positions are inflated by the singleton permutation 1. If $\sigma = 3624715$, then we will write $\sigma[1, 12, 1, 1, 21, 1, 1] = 367249815$ as $\sigma_{2,5}[12, 21]$. Formally, $\sigma_{i_1, \dots, i_k}[\alpha_1, \dots, \alpha_k]$ is the inflation of σ , where σ_{i_j} is inflated by α_j for $j = 1, \dots, k$; and all other positions of σ are inflated by 1. We note here that inflations are not necessarily unique. This is in contrast to the standard definition, originally given in Albert and Atkinson [1], where, essentially, we have that every permutation can be written as a unique inflation of a simple permutation.

If α is a permutation, then the *closure* of α is the set of permutations contained in α , including α itself and ϵ , which we write as $\text{Cl}(\alpha)$. If we have a permutation σ of length n , and permutations $\alpha_1, \dots, \alpha_n$, then the *inflation set* $\sigma\langle\alpha_1, \dots, \alpha_n\rangle$ is the set of all possible permutations that are inflations of σ , where each σ_i is inflated by an element of $\text{Cl}(\alpha_i)$. In line with our definition of inflation, we

assume throughout this paper that at least one of the permutations chosen from $\text{Cl}(\alpha_1), \dots, \text{Cl}(\alpha_n)$ is non-empty, thus the set $\sigma\langle\alpha_1, \dots, \alpha_n\rangle$ does not include the empty permutation.

We will mainly be discussing inflation sets where most positions are inflated by $\text{Cl}(1) = \{1, \epsilon\}$. We use the same style of notation that we use for inflations, indicating the positions that are not inflated by an element of $\text{Cl}(1)$ as subscripts, so, for example, we may write $\sigma_{\ell,r}\langle 312, 132\rangle$ for the inflation set of σ , where the ℓ -th position is inflated by an element of $\text{Cl}(312)$, the r -th position is inflated by an element of $\text{Cl}(132)$, and all other positions are inflated by an element of $\text{Cl}(1)$.

Our argument includes discussing sets of permutations that are an inflation of some σ , where one position is inflated by a specific permutation, and all other positions are inflated by an element of $\text{Cl}(1)$. The permutations in these sets are used as witnesses to the presence of the specific permutation. If we are inflating with the specific permutation α , then we write the set of permutations as $\sigma_{\ell}\langle\langle\alpha\rangle\rangle$, where the permutation α inflates the ℓ -th position, and all other positions are inflated by an element of $\text{Cl}(1)$.

3 Permutations with opposing adjacencies

In this section our main result is to show that if a permutation has opposing adjacencies, then the value of the principal Möbius function is zero. We then show that if σ is adjacency-free, and π contains an interval order-isomorphic to a symmetry of 1243, then $\mu[\sigma, \pi] = 0$.

We use an inductive proof. The first, rather trivial, step is to show that the base case holds. We then consider some π that has opposing adjacencies. and divide the poset $[1, \pi]$ into four sets $L, R, L \cap R$ and $T = [1, \pi] \setminus (L \cup R)$, and show that we can obtain $\mu[\pi]$ by summing over each set and then using inclusion-exclusion. We show an example of the sets for $\pi = 346215$ in Figure 2.

3.1 The principal Möbius function of permutations with an opposing adjacency

Our main theorem in this section is:

Theorem 1. *If π has opposing adjacencies, then $\mu[\pi] = 0$.*

Proof. Note first that if π has opposing adjacencies, then it must have length at least four. It is simple to see that all permutations of length four with opposing adjacencies are symmetries of 1243, and that $\mu[1243] = 0$.

Assume now that Theorem 1 applies to all permutations with length less than some $n > 4$. Let π be a permutation of length n with opposing adjacencies. Choose an up-adjacency and a down-adjacency. Without loss of generality, we can assume, by symmetry, that the first adjacency chosen is an up-adjacency.

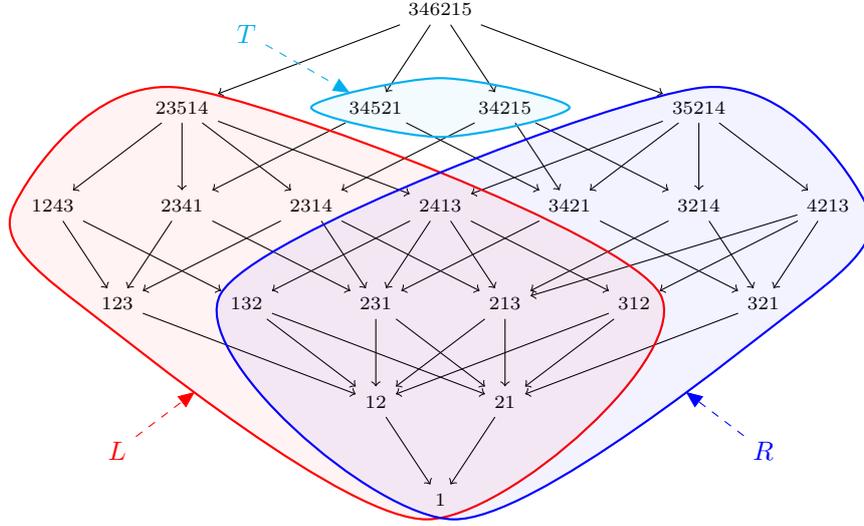


Figure 2: Four subsets of $[1, 346215]$: L in red, R in blue, $L \cap R$, and T in cyan.

Let γ be the permutation formed by replacing the two chosen adjacencies in π by 1, and retaining all other points of π . Then we can write $\pi = \gamma_{\ell,r}[12, 21]$, where ℓ is the index of the first point of the first adjacency chosen in π , and r is one less than the index of the first point of the second adjacency chosen in π . As an example, with $\pi = 367249815$ we would have $\ell = 2$, $r = 5$, and $\pi = 3624715_{\ell,r}[12, 21]$. It is easy to see that with ℓ and r fixed, γ is unique.

Let $\lambda = \gamma_{\ell}[12]$ and let $\rho = \gamma_r[21]$.

We define four (overlapping) subsets of $[1, \pi]$ as follows:

$$\begin{aligned} L &= [1, \lambda] \\ R &= [1, \rho] \\ G &= L \cap R \\ T &= [1, \pi] \setminus (L \cup R) \end{aligned}$$

Since λ and ρ are both contained in π , it is easy to see that

$$\mu[\pi] = - \sum_{\tau \in L} \mu[\tau] - \sum_{\tau \in R} \mu[\tau] - \sum_{\tau \in T} \mu[\tau] + \sum_{\tau \in G} \mu[\tau] \quad (1)$$

To prove Theorem 1 it is sufficient to show that each of the four sums in Equation 1 is zero.

Consider first $\sum_{\tau \in L} \mu[\tau]$. This is plainly zero from the definition of the Möbius function, and the same argument applies to $\sum_{\tau \in R} \mu[\tau]$.

Now consider $\sum_{\tau \in T} \mu[\tau]$. We claim that any permutation $\tau \in T$ must have opposing adjacencies, and so, using our inductive hypothesis, $\mu[\tau] = 0$. To justify our claim, let τ be an element of $[1, \pi]$. If τ does not contain an up-adjacency, then τ must be in $\gamma_{\ell,r}\langle 1, 21 \rangle$, and so is in R . Similarly, if τ does not

contain a down-adjacency, then $\tau \in L$. Thus if $\tau \in T$, then τ has an opposing adjacency.

Finally, we consider $\sum_{\tau \in G} \mu[\tau]$. We partition G into two disjoint sets:

$$\begin{aligned} G_\gamma &= [1, \gamma] \\ G_x &= L \cap R \setminus G_\gamma \end{aligned}$$

From the construction of L and R it is easy to see that $G_\gamma \subseteq L \cap R$, so G_γ and G_x are well-defined, and we have $\sum_{\tau \in G} \mu[\tau] = \sum_{\tau \in G_\gamma} \mu[\tau] + \sum_{\tau \in G_x} \mu[\tau]$. In some cases we observe that $G_\gamma = L \cap R$, but this is not true in general as, for example, when $\pi = 53128746$, we have $G_x = \{4312\}$.

From the definition of the Möbius function, $\sum_{\tau \in G_\gamma} \mu[\tau] = 0$, so to complete the proof of Theorem 1 we simply need to show that $\sum_{\tau \in G_x} \mu[\tau] = 0$.

We claim that every permutation τ in G_x has an opposing adjacency, and so has $\mu[\tau] = 0$.

Since $L = \gamma_\ell \langle 12 \rangle$, and $G_\gamma = \gamma_\ell \langle 1 \rangle$, it follows that, as $\tau \in L \setminus G_\gamma$, then $\tau \in \gamma_\ell \langle \langle 12 \rangle \rangle$, so τ contains an up-adjacency. Similarly, if $\tau \in R \setminus G_\gamma$, then $\tau \in \gamma_r \langle \langle 21 \rangle \rangle$, so τ contains a down-adjacency. Thus if $\tau \in G_x$, then τ has opposing adjacencies, and so, by the inductive hypothesis, $\mu[\tau] = 0$, and thus $\sum_{\tau \in G_x} \mu[\tau] = 0$. \square

3.2 Extending Theorem 1

It is natural to ask if we can extend Theorem 1 to handle cases where the lower bound of the interval is not 1. This is not possible in general, as if we take any permutation $\sigma \neq 1$, and inflate any two distinct points in positions ℓ and r by 12 and 21 respectively, then $\pi = \sigma_{\ell,r}[12, 21]$ has opposing adjacencies, but $\mu[\sigma, \pi] = 1$, as can be deduced from Figure 3.

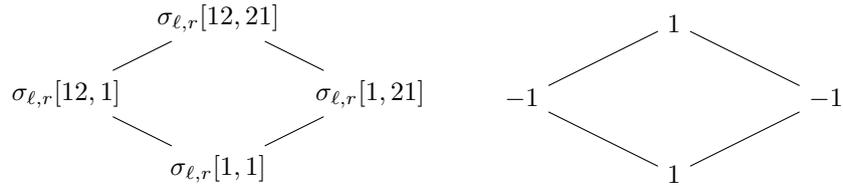


Figure 3: The Hasse diagram of the interval $[\sigma, \sigma_{\ell,r}[12, 21]]$, and the corresponding values of the Möbius function.

Although we do not have a general extension of Theorem 1, we can show that:

Theorem 2. *If σ is adjacency-free, and π contains an interval order-isomorphic to a symmetry of 1243, then $\mu[\sigma, \pi] = 0$.*

Proof. First note that if $\sigma \not\leq \pi$, then $\mu[\sigma, \pi] = 0$ from the definition of the Möbius function. Further, since σ is adjacency-free, we cannot have $\sigma = \pi$.

We can now assume that $\sigma < \pi$. Without loss of generality we can also assume, by symmetry, that the interval in π is order-isomorphic to 1243.

We start by claiming that, for any permutation σ which is adjacency-free, and any c with $1 \leq c \leq |\sigma|$, we have $\mu[\sigma, \sigma_c[1243]] = 0$. The Hasse diagram of the interval $[\sigma, \sigma_c[1243]]$ is shown in Figure 4. From the definition of the

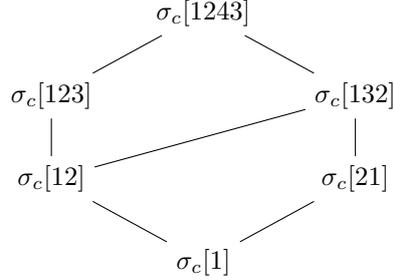


Figure 4: The Hasse diagram of the interval $[\sigma, \sigma_c[1243]]$.

Möbius function, we have $\mu[\sigma, \sigma_c[1]] = 1$, $\mu[\sigma, \sigma_c[12]] = -1$, $\mu[\sigma, \sigma_c[21]] = -1$, $\mu[\sigma, \sigma_c[123]] = 0$, and $\mu[\sigma, \sigma_c[132]] = 1$, and so $\mu[\sigma, \sigma_c[1243]] = 0$, and thus our claim is true.

Our argument now follows a similar pattern to that used by Theorem 1, and we restrict ourselves to highlighting the differences.

Assume that π is a proper inflation of σ , with length greater than $|\sigma| + 4$, and π contains an interval order-isomorphic to 1243. Let γ be the permutation formed by replacing an occurrence of 1243 in π by 12, so if ℓ is the position of the first point of the 1243 selected, then $\gamma_{\ell, \ell+1}[12, 21] = \pi$. Let $\lambda = \gamma_{\ell, \ell+1}[12, 1]$, and let $\rho = \gamma_{\ell, \ell+1}[1, 21]$; Define sets $L = [\sigma, \lambda]$, $R = [\sigma, \rho]$, $G_\gamma = [\sigma, \gamma]$, $G_x = L \cap R \setminus G_\gamma$, and $T = [\sigma, \pi] \setminus (L \cup R)$.

Similarly to Theorem 1, we have

$$\mu[\sigma, \pi] = - \sum_{\tau \in L} \mu[\sigma, \tau] - \sum_{\tau \in R} \mu[\sigma, \tau] - \sum_{\tau \in T} \mu[\sigma, \tau] + \sum_{\tau \in G_\gamma} \mu[\sigma, \tau] + \sum_{\tau \in G_x} \mu[\sigma, \tau],$$

and the sums over the sets L , R and G_γ are obviously zero. Using similar arguments to Theorem 1, we can see that every permutation τ in T or G_x contains an interval order-isomorphic to 1243, and so by the inductive hypothesis, has $\mu[\sigma, \tau] = 0$, and thus we have $\mu[\sigma, \pi] = 0$. \square

Although we cannot find a general extension to Theorem 1, we can find a necessary condition for a proper inflation of certain permutations to have a Möbius function value of zero. This is

Lemma 3. *If σ is adjacency-free, and $\pi = \sigma[\alpha_1, \dots, \alpha_n]$ is a proper inflation of σ , then $\mu[\sigma, \pi] = 0$ implies that at least one $\alpha_i \notin \{1, 12, 21\}$.*

Proof. Assume that every $\alpha_i \in \{1, 12, 21\}$. Let k be the number of α_i -s that are not equal to 1, and let j_1, \dots, j_k be the indexes (i -s) where $\alpha_i \neq 1$, so $\pi = \sigma_{j_1, \dots, j_k}[\alpha_{j_1}, \dots, \alpha_{j_k}]$.

Then every permutation in the interval $[\sigma, \pi]$ has a unique representation as $\sigma_{j_1, \dots, j_k}[v_1, \dots, v_k]$, where

$$v_i \in \begin{cases} \{1, 12\} & \text{if } \alpha_{j_i} = 12, \\ \{1, 21\} & \text{if } \alpha_{j_i} = 21. \end{cases}$$

So each position j_i can be inflated by one of two permutations, and thus there is an obvious isomorphism between permutations in the interval $[\sigma, \pi]$ and binary numbers with k bits. It follows that the poset can be represented as a Boolean algebra, and so by a well-known result (see, for instance, Example 3.8.3 in Stanley [14]), $\mu[\sigma, \pi] = (-1)^{|\pi| - |\sigma|}$. Thus if $\mu[\sigma, \pi] = 0$, at least one $\alpha_i \notin \{1, 12, 21\}$. \square

4 Permutations containing a specific interval

In this section we show that if a permutation ϕ meets certain requirements, then any permutation π with an interval order-isomorphic to ϕ has $\mu[\pi] = 0$.

Recall that if π is a permutation, then a permutation σ is *covered* by π if $\sigma < \pi$, and there is no permutation τ such that $\sigma < \tau < \pi$. The set of permutations covered by π is the *cover* of π , written $\text{Cover}(\pi)$.

As with opposing adjacencies, our proof is inductive. For the induction step with some permutation π , our approach is to partition the poset into subsets and then show that the sum of the principal Möbius function values over the permutations in each subset is zero. As an example, Figure 5 shows how we would partition the interval $[1, 1324657)$. In this case we have four subsets: P , L_1 , L_2 and R . The partitioning is based around the permutations that are covered by ϕ . One of these permutations is the “core”, and the associated subset is P . Further subsets are specified by the other permutations in the cover of ϕ , making sure that no permutation is included in more than one subset, and we label these subsets L_1, \dots, L_k . Once we have iterated through the subsets defined by the permutations in the cover of ϕ , any remaining permutations are placed in a final subset R . Our proof then relies on showing that, for each set S , $\sum_{\tau \in S} \mu[\tau] = 0$.

We will need to use a lemma from Smith [10]:

Lemma 4 (Smith [10, Lemma 1]). *If a permutation π contains a triple adjacency then $\mu[\pi] = 0$.*

Given a permutation π , it is sometimes possible to determine the value of $\mu[\pi]$ by considering small localities of π . As examples, if π contains a monotonic interval of length three or more, then by Lemma 4, $\mu[\pi] = 0$. Theorem 1 in this paper is another example, as the presence of opposing adjacencies guarantees that $\mu[\pi] = 0$. The two instances mentioned can be rephrased in terms of inflations, so if a permutation π can be written as $\gamma_c[123]$, $\gamma_c[321]$, $\gamma_{\ell, r}[12, 21]$, or $\gamma_{\ell, r}[21, 12]$ with $1 \leq \ell < r \leq |\gamma|$, and $1 \leq c \leq |\gamma|$, then we know, from Lemma 4 and Theorem 1, that $\mu[\pi] = 0$.

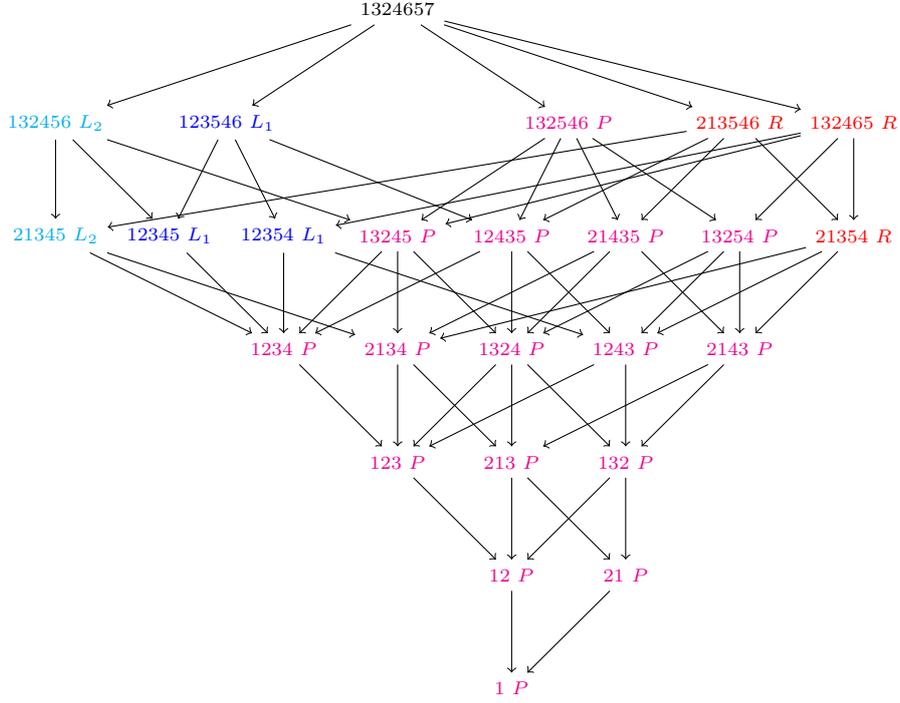


Figure 5: Four subsets of $[1, 1324657]$: P , L_1 , L_2 , and R .

Let \mathcal{SZ} be the set of permutations such that if any permutation π contains an interval order-isomorphic to some $\phi \in \mathcal{SZ}$ then $\mu[\pi] = 0$. We know that \mathcal{SZ} is non-empty, since 123 , 321 and all permutations with opposing adjacencies are elements of \mathcal{SZ} . If a permutation ϕ is an element of \mathcal{SZ} , then we say that ϕ is *strongly zero*.

There are cases where we can determine that the principal Möbius function of a permutation is zero by examining part of the permutation, but the permutation is not strongly zero. As an example, if a permutation π , with $|\pi| > 2$, begins 12 , then as a consequence of Propositions 1 and 2 in Burstein, Jelínek, Jelínková and Steingrímsson [5] (first stated explicitly as Lemma 4 in Brignall and Marchant [4]), $\mu[\pi] = 0$. Such a permutation is not strongly zero, since $12 \notin \mathcal{SZ}$.

We need one further definition before we can define “core” and proceed to the statement of our theorem.

Let $L = \{\lambda_1, \dots, \lambda_n\}$ be a set of permutations. The *ground* of L , $\text{Gr}(L)$, is the set of permutations formed by taking the union of the closure of each permutation in L , and then removing any permutation that is strongly zero, so

$$\text{Gr}(L) = \left(\bigcup_{\lambda \in L} \text{Cl}(\lambda) \right) \setminus \mathcal{SZ}.$$

As an example, if $L = \{1243, 2134\}$, then we have

$$\begin{aligned} \text{Cl}(1243) &= \{1243, 132, 123, 12, 21, 1, \epsilon\}, \\ \text{Cl}(2134) &= \{2134, 123, 213, 12, 21, 1, \epsilon\}, \\ \text{and} \quad \mathcal{SZ} &= \{123, 321, 1243, 2134, \dots\}, \\ \text{so} \quad \text{Gr}(L) &= \{132, 213, 12, 21, 1, \epsilon\}. \end{aligned}$$

Let ϕ be a permutation, and let ψ be an element of $\text{Cover}(\phi)$. Let $L = \text{Cover}(\phi) \setminus \psi$. We say that ψ is a *core* of ϕ if every permutation in the ground of L is contained in ψ , i.e., $\text{Gr}(L) \subseteq \text{Cl}(\psi)$.

If a permutation ϕ has a core ψ , then this means that every permutation in $\text{Cover}(\phi) \setminus \psi$ must be strongly zero. As a consequence, any permutation ϕ where $\text{Cover}(\phi)$ contains more than one permutation that is not strongly zero does not have a core. Further, if ϕ contains exactly one permutation ψ that is not strongly zero, then either ψ is the core, as it meets the requirement that $\text{Gr}(\text{Cover}(\phi)) \subseteq \text{Cl}(\psi)$, or ϕ does not have a core.

A further consequence of this definition, which we will use in our proof, is that if ϕ has a core ψ , then the closure of ϕ is the union of ϕ , permutations in the closure of ψ , and permutations that are structurally zero, so $\text{Cl}(\phi) \setminus (\phi \cup \text{Cl}(\psi)) \subset \mathcal{SZ}$.

It is possible for a permutation ϕ to have more than one core. For our purposes, all we require is that a core exists, and henceforth we will refer to “the core” of a permutation.

As an example, let $\phi = 21354$. Then $\text{Cover}(\phi) = \{1243, 2143, 2134\}$. The only possibility for the core is 2143, so we have

$$\begin{aligned} \text{Cl}(2143) &= \{2143, 132, 213, 12, 21, 1, \epsilon\} \\ \text{and} \quad \text{Gr}(\text{Cover}(21354) \setminus 2143) &= \{132, 213, 12, 12, 1, \epsilon\}, \\ \text{so} \quad \text{Gr}(\text{Cover}(21354)) \setminus 2143 &\subseteq \text{Cl}(2143) \end{aligned}$$

and thus 2143 is the core of 21354.

We now have our final definition in this section. Let ϕ be any permutation. We say that ϕ is *nice* if $\mu[\phi] = 0$ and ϕ has a core. Continuing with our running example of $\phi = 21354$, we know that ϕ has a core of 2143. Since $\mu[21354] = 0$, this gives us that 21354 is nice.

We are now in a position to state our main theorem for this section.

Theorem 5. *If ϕ is a nice permutation, and π is any permutation containing an interval order-isomorphic to ϕ , then $\mu[\pi] = 0$, thus $\phi \in \mathcal{SZ}$.*

Proof. We proceed by induction. First, if $\phi = \pi$, then by definition $\mu[\pi] = 0$. Now assume that, for a given ϕ , Theorem 5 is true for all permutations with length less than some n , where $n > |\phi|$. Let π be a permutation of length n that contains at least one interval order-isomorphic to ϕ . Choose one of the intervals order-isomorphic to ϕ , and let γ be the permutation obtained by replacing the chosen interval with a single point. Let c be the index of the first point of the chosen interval in π , so $\pi = \gamma_c[\phi]$.

Let ψ be the core of ϕ ; and let $\lambda_1, \dots, \lambda_k$ be the permutations covered by ϕ excluding ψ , i.e., $\text{Cover}(\phi) \setminus \psi$.

Our approach is to divide the poset $[1, \pi)$ into disjoint sets P, L_1, \dots, L_k, R , as defined below, and then show that for each set $S \in \{P, L_1, \dots, L_k, R\}$, $\sum_{\tau \in S} \mu[\tau] = 0$, which gives us $\mu[\pi] = 0$.

Define sets of permutations as follows:

$$\begin{array}{ll}
P = \text{Cl}(\gamma_c[\psi]) & \\
L'_1 = \text{Cl}(\gamma_c[\lambda_1]) & L_1 = L'_1 \setminus (P) \\
L'_2 = \text{Cl}(\gamma_c[\lambda_2]) & L_2 = L'_2 \setminus (P \cup L'_1) \\
\vdots & \vdots \\
L'_k = \text{Cl}(\gamma_c[\lambda_k]) & L_k = L'_k \setminus (P \cup L'_1 \cup \dots \cup L'_{k-1}) \\
& R = [1, \pi) \setminus (P \cup L'_1 \cup \dots \cup L'_k)
\end{array}$$

It is easy to see that in $\{P, L_1, \dots, L_k, R\}$ the intersection of any distinct pair of sets is empty, and that $[1, \pi) = P \cup L_1 \cup \dots \cup L_k \cup R$. Thus

$$\mu[\pi] = - \sum_{\tau \in P} \mu[\tau] - \sum_{\tau \in L_1} \mu[\tau] - \dots - \sum_{\tau \in L_k} \mu[\tau] - \sum_{\tau \in R} \mu[\tau], \quad (2)$$

and so to prove Theorem 5, it is sufficient to show that each of the sums in Equation 2 is zero.

First consider $\sum_{\tau \in P} \mu[\tau]$. P is a closed interval, and so we have $\sum_{\tau \in P} \mu[\tau] = 0$ from the definition of the Möbius function.

Now consider $\sum_{\tau \in L_i} \mu[\tau]$, with $i \in [1, k]$. Recall that $P = \text{Cl}(\gamma_c[\psi]) = \gamma_c\langle\psi\rangle$, and $L'_i = \text{Cl}(\gamma_c[\lambda_i]) = \gamma_c\langle\lambda_i\rangle$; and note that $L_i \subseteq L'_i \setminus P$. From this we can see that any permutation in $L'_i \setminus P$ is in the set of permutations defined by

$$M = \{\tau : \tau \in \gamma_c\langle v \rangle, v \in (\text{Cl}(\lambda_i) \setminus \text{Cl}(\psi))\}.$$

Since ψ is the core of ϕ , it follows that every permutation in M is strongly zero. Now, since $L_i \subseteq L'_i \setminus P$, we have that every permutation in L_i is strongly zero, and so $\sum_{\tau \in L_i} \mu[\tau] = 0$.

Finally, consider $\sum_{\tau \in R} \mu[\tau]$. Recall that $\pi = \gamma_c[\phi]$. We claim that every permutation in R contains an interval order-isomorphic to ϕ . Any permutation α in $[1, \pi)$ is, by definition, contained in $\gamma_c\langle\phi\rangle$. It follows that α must be contained in $\{\tau : \tau \in \gamma_c\langle v \rangle, v \in \text{Cl}(\phi)\}$. The permutations covered by ϕ are $\psi, \lambda_1, \dots, \lambda_k$, and so we have $\text{Cl}(\phi) = \{\phi\} \cup \text{Cl}(\psi) \cup \text{Cl}(\lambda_1) \cup \dots \cup \text{Cl}(\lambda_k)$. Since $P = \text{Cl}(\gamma_c[\psi]) = \gamma_c\langle\psi\rangle$, and $L_i = \text{Cl}(\gamma_c[\lambda_i]) = \gamma_c\langle\lambda_i\rangle$, it follows that $R = \gamma_c\langle\langle\phi\rangle\rangle$, and so our claim is proved. Now, by the inductive hypothesis, we have that every permutation $\tau \in R$ has $\mu[\tau] = 0$, and so $\sum_{\tau \in R} \mu[\tau] = 0$. \square

We note here that the set of nice permutations is a subset of \mathcal{SZ} , as a permutation π with opposing adjacencies is in \mathcal{SZ} , but π may not be nice. As an example, if $\pi = 256143$, then $\pi \in \mathcal{SZ}$ since it has opposing adjacencies. The permutations covered by π are 45132, 25143, 14532 and 24513. Two of these, 25143 and 24513, are not strongly zero, and so π does not have a core, and therefore cannot be nice.

4.1 Extending Theorem 5

It is natural to ask if we can extend Theorem 5 to handle the case where the lower bound of the interval is not 1. In order to do so, we need some further definitions.

We define a σ -closure of a permutation π , written $\text{Cl}_\sigma(\pi)$, to be the set of permutations contained in π that also contain σ .

Let \mathcal{SZ}_σ be the set of permutations such that if any permutation π contains an interval order-isomorphic to some $\tau \in \mathcal{SZ}_\sigma$ then $\mu[\sigma, \pi] = 0$.

If L is a set of permutations $\{\lambda_1, \dots, \lambda_n\}$, then the σ -ground of L , $\text{Gr}_\sigma(L)$, is the set of permutations formed by taking the σ -closure of each permutation in λ_i , and then removing any permutation that is contained in \mathcal{SZ}_σ .

We say that ψ is a σ -core of π if every permutation in the σ -ground of L is contained in ψ , i.e., $\text{Gr}_\sigma(L) \subseteq \text{Cl}_\sigma(\psi)$.

Finally, we say that π is σ -nice if $\mu[\sigma, \pi] = 0$ and π has a σ -core.

We now have:

Theorem 6. *If ϕ is a σ -nice permutation, and π is any permutation containing an interval order-isomorphic to ϕ , then $\mu[\sigma, \pi] = 0$.*

Proof. The proof follows the same pattern as Theorem 5, replacing \mathcal{SZ} by \mathcal{SZ}_σ , closure by σ -closure, ground by σ -ground, core by σ -core, nice by σ -nice, and where the lower bound of an interval is 1, replacing the lower bound by σ . We omit the details for brevity. \square

5 The proportion of permutations with $\mu[\pi] = 0$

Let $Z(n)$ be the proportion of permutations of length n where the principal Möbius function is zero. Let $Z_{sz}(n)$ be the proportion of permutations of length n that are strongly zero. Plainly, $Z(n) \geq Z_{sz}(n)$ for all n . Our aim in this section is to find an asymptotic lower bound for $Z(n)$ by determining an asymptotic lower bound for $Z_{sz}(n)$. To find this lower bound, we count inflations of simple permutations where the resulting permutation has opposing adjacencies, and so is structurally zero. We use a result from Albert and Atkinson [1]:

Proposition 7 (Albert and Atkinson [1, Proposition 2]). *Let π be any permutation. Then there is a unique simple permutation σ , and permutations $\alpha_1, \dots, \alpha_k$ such that $\pi = \sigma[\alpha_1, \dots, \alpha_k]$. If $\sigma \neq 12, 21$, then $\alpha_1, \dots, \alpha_k$ are also uniquely determined by π . If $\sigma = 12$ or 21 , then α_1, α_2 are unique so long as we require that α_1 is sum indecomposable or skew indecomposable respectively.*

We will also need a result from Albert, Atkinson and Klazar [2]:

Theorem 8 (Albert, Atkinson and Klazar [2, Theorem 5]). *The number of*

simple permutations of length n , $S(n)$, is given by

$$S(n) = \frac{n!}{e^2} \left(1 - \frac{1}{n} + \frac{2}{n(n-1)} + O(n^{-3}) \right).$$

We will prove:

Theorem 9. $Z(n)$ is, asymptotically, bounded below by 0.3995.

Proof. We find a lower bound for $Z_{sz}(n)$ by counting permutations that have opposing adjacencies.

Let $n \geq 6$ be an integer; and let k be an integer in the range $2, \dots, \lfloor n/2 \rfloor$. Let σ be a simple permutation with length $n - k$. We will count the number of ways we can inflate σ with k adjacencies to obtain a permutation with length n that has opposing adjacencies. We can choose the positions to inflate in $\binom{n-k}{k}$ ways. There are 2^k distinct inflations by adjacencies, and all but two have opposing adjacencies, thus the number of ways to inflate σ that result in a permutation with opposing adjacencies is given by

$$\binom{n-k}{k} (2^k - 2).$$

Since we are inflating simple permutations, it follows from Proposition 7 that the inflations are unique.

For an inflation to contain an opposing adjacency, we need to inflate at least two points. Further, to obtain a permutation of length n by inflating with adjacencies we can, at most, inflate $\lfloor n/2 \rfloor$ positions. Using Theorem 8, we can say that

$$Z_{sz}(n) \geq \frac{1}{n!} \sum_{k=2}^{\lfloor n/2 \rfloor} S(n-k) \binom{n-k}{k} (2^k - 2).$$

Note now that as $n \rightarrow \infty$, $S(n) \rightarrow \frac{n!}{e^2}$. Let $P(n, k)$ be the proportion of permutations of length n which are inflations of simple permutations of length $n - k$, where k positions are inflated by an adjacency, and the resulting permutation has at least one opposing adjacency. Then we have

$$\begin{aligned} \lim_{n \rightarrow \infty} P(n, 2) &= \lim_{n \rightarrow \infty} \frac{1}{n!} \frac{(n-2)!}{e^2} \binom{n-2}{2} (2^2 - 2) \\ &= \lim_{n \rightarrow \infty} \frac{1}{e^2} \frac{(n-2)(n-3)}{n(n-1)} \\ &= \frac{1}{e^2}. \end{aligned}$$

Similarly, we have

$$\begin{aligned} \lim_{n \rightarrow \infty} P(n, 3) &= \frac{1}{e^2}, & \lim_{n \rightarrow \infty} P(n, 4) &= \frac{7}{12e^2}, \\ \lim_{n \rightarrow \infty} P(n, 5) &= \frac{1}{4e^2}, & \lim_{n \rightarrow \infty} P(n, 6) &= \frac{31}{360e^2}, \\ \lim_{n \rightarrow \infty} P(n, 7) &= \frac{1}{40e^2}, & \lim_{n \rightarrow \infty} P(n, 8) &= \frac{127}{20160e^2}, \\ & \text{and} & \lim_{n \rightarrow \infty} P(n, 9) &= \frac{17}{12096e^2}. \end{aligned}$$

We now write

$$\begin{aligned} \lim_{n \rightarrow \infty} Z_{sz}(n) &\geq \lim_{n \rightarrow \infty} \frac{1}{n!} \sum_{k=2}^{\lfloor n/2 \rfloor} S(n-k) \binom{n-k}{k} (2^k - 2) \\ &\geq \lim_{n \rightarrow \infty} \frac{1}{n!} \sum_{k=2}^9 S(n-k) \binom{n-k}{k} (2^k - 2) \\ &= \sum_{k=2}^9 P(n, k) \\ &= 0.3995299850 \end{aligned}$$

and thus $Z(n)$ is, asymptotically, bounded below by 0.3995. \square

We used the first nine terms of the sum $\frac{1}{n!} \sum_{k=2}^{\lfloor n/2 \rfloor} S(n-k) \binom{n-k}{k} (2^k - 2)$ as this gives us a lower bound for $Z_{sz}(n)$ to four significant figures. We found that evaluating the first 100 terms improves the lower bound to 0.3995764008, and evaluating larger number of terms makes no difference, to ten significant figures, to the value obtained.

Remark 10. Kaplansky [7] provides an asymptotic expression for the probability that a permutation of length n will have k adjacencies. Corteel, Louchard and Pemantle [6] show that the distribution is Poisson, with parameter 2. It is possible to find a lower bound for $Z(n)$ using a probabilistic argument based on these results. We found that, to four significant figures, the lower bound from this approach was still 0.3995, so using this more general construction did not improve our lower bound.

6 Concluding remarks

6.1 Permutations with non-opposing adjacencies

Given Theorem 1, it is natural to wonder if we can find a similar result that applies where a permutation has multiple adjacencies, but no opposing adjacencies.

We can find permutations that have multiple adjacencies, and do not have opposing adjacencies, where the principal Möbius function value is non-zero.

Table 1 shows, for lengths 4...12, the number of permutations with multiple non-opposing adjacencies broken down by whether the value of the principal Möbius function is zero or not.

Length	= 0	≠ 0
4	6	4
5	26	8
6	170	38
7	1154	212
8	8954	1502
9	78006	13088
10	757966	130066
11	8132206	1436296
12	95463532	17403612

Table 1: Number of permutations with non-opposing adjacencies, classified by the value of the principal Möbius function.

This suggests that it might be possible to find a result similar to Theorem 1 for some or all of these cases, although any such result will clearly need some additional criteria that will exclude permutations that have a non-zero principal Möbius function value.

6.2 The number of strongly zero permutations

In this paper we show that if a permutation is nice, then it is strongly zero. We also show that permutations with opposing adjacencies are strongly zero. There may be permutations that do not contain opposing adjacencies, and which are not nice, but are, nevertheless, strongly zero.

We place the strongly zero permutations we can identify into one of two categories: *obviously zero* permutations, which are those that contain either an opposing adjacency, or an interval that is order-isomorphic to a smaller nice permutation; and *new* permutations, which are those that are nice, but not obviously zero. As an example, 1243 is obviously zero, since it contains opposing adjacencies, whereas 12453 is new. The number of permutations for lengths 3 to 10 in each of the above groups are shown in Table 2.

The figures in Table 2 suggest that the number of permutations that are strongly zero grows as n increases, which is what we would naturally expect. There is also a suggestion, on the basis of the limited numerical evidence, that the proportion of permutations of length n that are strongly zero is falling as n increases. We know, however, from the proof of Theorem 9, that this proportion is, asymptotically, bounded below by 0.3995.

We suggest a factor that might explain this apparent contradiction. Our proof of Theorem 9 uses $\frac{n!}{e^2}$ as the number of simple permutations of length n . Table 3 compares the actual values of $S(n)$ against the computed value of $\frac{n!}{e^2}$ for $n =$

Length	Obviously zero	New	Obviously zero %	New %
3	0	2	0.00	33.33
4	10	0	41.67	0.00
5	40	10	33.33	8.33
6	258	16	35.83	2.22
7	1570	144	31.15	2.86
8	11366	816	28.19	2.02
9	91254	6144	25.15	1.69
10	817506	50664	22.53	1.40

Table 2: Number of known strongly zero permutations with length 3, . . . , 10.

4, . . . , 10, and, as can be seen, for these small values of n there is a significant difference between the two values. It is clear that the figures in Table 2 do not

n	4	5	6	7	8	9	10
$S(n)$	2	6	46	338	2926	28146	298526
$\frac{n!}{e^2}$	3.2	16.2	97.4	682	5456	49110	491104

Table 3: A comparison of $S(n)$ and $\frac{n!}{e^2}$ for $n = 4, \dots, 10$.

represent the asymptotic behaviour.

We say that a permutation π is *canonical* if there is no symmetry of π that is lexicographically smaller. As an example, 125634 is canonical, as the other symmetries of this permutation are 341256, 436521, and 652143. A file containing all of the known canonical strongly zero permutations with length less than or equal to ten is available from the second author.

6.3 The asymptotic behaviour of $Z(n)$

It is natural to wonder what the asymptotic behaviour of $Z(n)$ is. Based on numeric evidence supplied by Jason Smith [13] for $1 \leq n \leq 9$, and calculations performed by the second author, Table 4 shows the value of $Z(n)$ for $n = 1, \dots, 12$.

Based on this somewhat limited numeric evidence, we conjecture that:

Conjecture 11. The proportion of permutations that have principal Möbius function value equal to zero is bounded above by 0.6040.

Remark 12. In our exploration of the Möbius function, we have noted that the behaviour of the function can be erratic where the length of the permutation is small, and Conjecture 11 may not, therefore, reflect the asymptotic behaviour.

Length	$Z(n)$
1	0.0000
2	0.0000
3	0.3333
4	0.4167
5	0.4833
6	0.5361
7	0.5742
8	0.5942
9	0.6019
10	0.6040
11	0.6034
12	0.6021

Table 4: The value of $Z(n)$ for $n = 1, \dots, 12$.

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