

UNCOUNTABLY MANY ENUMERATIONS OF WELL-QUASI-ORDERED PERMUTATION CLASSES

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We construct an uncountable family of well-quasi-ordered permutation classes, each with a distinct enumeration sequence. As there are only countably many algebraic generating functions that enumerate combinatorial objects, this shows that there are well-quasi-ordered permutation classes without algebraic generating functions, disproving a widely-held conjecture. Indeed, this shows that there are well-quasi-ordered permutation classes without even D-finite generating functions.

Our construction relies on an uncountably large collection of factor-closed binary languages, and this collection also enables us to exhibit an uncountably large collection of infinite binary sequences, each with distinct linear complexity functions.

1. INTRODUCTION

Well-quasi-order is a strong structural property of permutation classes, and until recently, it was conceivable that there might be only countably many classes with this property. However, it can be seen from the recent work of Oudrar, Pouzet, and Zaguia [33] that there are uncountably many well-quasi-ordered permutation classes. Our main result goes further. We show that one can find uncountably many well-quasi-ordered permutation classes with distinct enumerations, which implies that these enumerations cannot all be nice. We defer definitions and context for our main theorem until later in this introduction, but state it now.

Theorem 1.1. *There are uncountably many distinct enumerations of well-quasi-ordered permutation classes.*

Our proof of Theorem 1.1 builds on a construction from Pouzet's 1978 thesis concerning factor-closed sets of binary words [36]. Section 2 establishes a number of properties of this construction, while the work after that exploits these in the permutation context.

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Besides underpinning the proof of Theorem 1.1, the properties we establish about Pouzet’s construction also imply a result concerning subwords of infinite binary sequences. Specifically, we are able to find uncountably many such sequences, each having distinct linear complexity functions. We defer the formal statement and necessary definitions for this result until Section 9.

The rest of this introduction provides the permutation-specific definitions and context for Theorem 1.1. We think of permutations in one-line notation, so a *permutation of length n* is simply an ordering of the set $\{1, 2, \dots, n\}$. The permutation π *contains* the permutation σ of length k if π has a subsequence of length k that is *order isomorphic* to σ , by which we mean that the subsequence has the same pairwise comparisons as σ . For example, $\pi = 372694185$ contains $\sigma = 32514$, as witnessed by its subsequence 32918, but π avoids 54321 because it has no decreasing subsequence of length five. If π contains σ , then we sometimes say that σ is a *subpermutation* of π .

A *permutation class* is a downset of permutations in this order. In other words, if \mathcal{C} is a permutation class, $\pi \in \mathcal{C}$, and $\sigma \leq \pi$, then we must also have $\sigma \in \mathcal{C}$. Every permutation class can be described by a unique *basis*, which consists of the minimal permutations *not* in the class. Every basis of a class is necessarily an *antichain*, meaning that none of its members is contained in another. As there are infinite antichains of permutations, there are infinitely-based permutation classes. A permutation class is *well-quasi-ordered* (*wqo* for short, or *belordonné* in French) if it does not contain an infinite antichain.

With \mathcal{C}_n denoting the set of permutations of length n in the class \mathcal{C} , the generating function of the class is

$$\sum_{n \geq 0} |\mathcal{C}_n| x^n = \sum_{\pi \in \mathcal{C}} x^{|\pi|},$$

where here $|\pi|$ denotes the length of π . We are often interested in whether this generating function $f(x)$ is *rational* (if $f(x) = p(x)/q(x)$ for polynomials p and q), *algebraic* (if there is a polynomial $p(x, y) \in \mathbb{Q}[x, y]$ such that $p(x, f(x)) = 0$), or *D-finite* (if $f(x)$ and its derivatives span a finite dimensional vector space over $\mathbb{Q}(x)$).

In their 1996 paper, Noonan and Zeilberger [31] conjectured that every finitely-based permutation class has a D-finite generating function. Note that the finite basis hypothesis here is clearly necessary—because there are infinite antichains of permutations, there are uncountably many different enumerations of permutations, but only countably many D-finite generating functions with rational coefficients. The Noonan–Zeilberger conjecture had begun to seem unlikely, and in 2005, Zeilberger himself conjectured that it was false [22]. However, it was not until 2015 that the conjecture was disproved by Garrabrant and Pak [23], a short version of which appeared in the proceedings of *SODA 2016* [24]^{1,2}.

¹The proof of Garrabrant and Pak does not provide a concrete counterexample to the Noonan–Zeilberger conjecture. Many believe that the class of 1324-avoiding permutations does not have a D-finite generating function because it is suspected to have the wrong asymptotics for such a generating function [18, 19]. The same type of analysis applies to several other classes avoiding a single permutation of length five, as shown by Clisby, Conway, Guttmann, and Inoue [17]. That said, for the sake of providing a concrete counterexample to the Noonan–Zeilberger conjecture, it may be preferable to consider classes with more basis elements, and thus more structure. Several potential counterexamples along these lines were identified in the work of Albert, Homberger, Pantone, Shar, and Vatter [2]. Others have later been identified by Bóna and Pantone [11].

²In an interview published in 2021 [29], Stanley singled out the results of Garrabrant and Pak, saying “a topic within enumerative combinatorics that seems ripe for further investigation is developing a theory for showing that a given generating function (in one variable) does *not* have some desirable property, such as being D-finite or differentially algebraic. There are a number of results in this area, but nothing approaching a general theory. The most significant work in this direction (in my opinion) is the disproof of the Noonan–Zeilberger conjecture by Scott Garrabrant and Igor Pak.”

Thus we know that some permutation classes fail to have nice enumerations (by an elementary argument because there are infinite antichains of permutations), and the Garrabrant–Pak result shows that some *finitely-based* permutation classes even fail to have nice enumerations. In a similar direction, at the same conference at which he conjectured that the Noonan–Zeilberger conjecture was false, Zeilberger asked for necessary and sufficient conditions for classes to have rational/algebraic/D-finite/... generating functions [22]. However, the results of Albert, Brignall, and Vatter [1] indicate that this question is likely to be intractable.

Theorem 1.2 (Albert, Brignall, and Vatter [1]). *Every permutation class except for the class of all permutations is contained in a class with a rational generating function*³.

This result shows that we cannot expect nice structure from permutation classes with rational generating functions. On the other hand, certain notions of structure do have enumerative consequences; one such example is due to Albert and Atkinson [5], who showed that permutation classes containing only finitely many simple permutations have algebraic generating functions.

We say that a permutation class is *strongly algebraic* (or *héréditairement algébrique* in French [32]) if it and all its subclasses have algebraic generating functions. By an elementary counting argument, it follows that strongly-algebraic permutation classes must be wqo. Thus it was natural to wonder if the converse might hold as well, a conjecture that appears to have been first stated in the literature by the second author.

False Conjecture 1.3 (Vatter [39, Conjecture 12.3.4]). *A permutation class is strongly algebraic if and only if it is well-quasi-ordered.*

Theorem 1.1 shows that this conjecture is false, because there are too many wqo permutation classes with distinct enumerations for them all to have algebraic generating functions. Indeed, there are too many for them all to even have D-finite generating functions⁴.

We remark that while our results show that wqo is not enough to guarantee algebraicity, a stronger notion known as *labelled well-quasi-order* (*lwqo* for short, or *héréditairement belordonné* in French) might be. While this notion is implicit in the classic papers of Higman [25] and Kruskal [28], it was not until the work of Pouzet in the 1970s (in particular, his 1972 paper [35]) that it was studied explicitly. Recently it has been shown that lwqo nicely explains many structural results about permutation classes [15]. We leave the following question for future research.

Question 1.4. *Does every lwqo permutation class have an algebraic generating function?*

At the very least, Question 1.4 will not fall to an argument such as the one we use here to disprove False Conjecture 1.3, as there are only countably many lwqo permutation classes, which is a corollary of them all being finitely based, see [15, Proposition 2.3]. A slightly weaker version of Question 1.4 has been conjectured by Oudrar [32, page 87].

The rest of this paper is organized as follows. Pouzet’s construction, and properties of the resulting (uncountably large) collection of factor-closed binary languages, are given in Section 2. In Sections 3 and 4 we show how each such language can be used to construct a permutation class of a particular type, via “pin sequences”. After establishing decomposition properties of the permutations that these

³The proof of Theorem 1.2 employs Marcus and Tardos’s resolution of the Stanley–Wilf conjecture [30], and can be viewed as a strengthening of this result.

⁴Note that the classes we construct in the proof of Theorem 1.1 are almost certainly infinitely based, and thus are unlikely to represent additional counterexamples to Noonan and Zeilberger’s false conjecture.

classes contain in Sections 5 and 6, we show in Section 7 that each such class is well-quasi-ordered. To complete the proof of Theorem 1.1, in Section 8 we show that any two such permutation classes constructed from distinct languages cannot have the same enumeration sequence. Section 9, which relies only on Section 2, covers the result concerning binary sequences. Some closing remarks are given in Section 10.

2. WORDS UNDER THE FACTOR ORDER

While the construction described in this section, which is adapted from Pouzet’s thesis [36, page 64], can be defined more generally, we specialize it to the binary alphabet $\{0, 1\}$. We are interested here in the *factor order* on the set of all finite binary words, $\{0, 1\}^*$, in which $u \leq w$ if u occurs as a *factor* (or *consecutive subword*) of w . In other words, $u \leq w$ in this order if $w = w_1 u w_2$ for some (possibly empty) words $w_1, w_2 \in \{0, 1\}^*$. This poset is *not* wqo in general, because, for example, the words $11, 101, 1001, \dots$ form an infinite antichain.

For a binary word $\gamma = \gamma_1 \cdots \gamma_n$, the *complement* (sometimes referred to as the *ones complement*) is the word $\bar{\gamma} = \bar{\gamma}_1 \cdots \bar{\gamma}_n$ formed by inverting every letter of γ , so $\bar{\gamma}_i = 1 - \gamma_i$ for all $1 \leq i \leq n$. We now describe Pouzet’s construction. While it takes a bit of preparation, this will eventually give us uncountably many wqo sets of binary words that are all closed downward in the factor order, each of which has a different enumeration sequence.

Let $(s_k)_{k \in \mathbb{N}}$ be a sequence of positive integers. We construct the infinite sequence $(\alpha_i^{(s_k)})_{i \in \mathbb{N}}$ of binary words by setting

$$\alpha_1^{(s_k)} = 01$$

and, for $i \geq 1$, defining

$$\alpha_{i+1}^{(s_k)} = \underbrace{\alpha_i^{(s_k)} \cdots \alpha_i^{(s_k)}}_{s_i \text{ copies}} \underbrace{\bar{\alpha}_i^{(s_k)} \cdots \bar{\alpha}_i^{(s_k)}}_{s_i \text{ copies}} = (\alpha_i^{(s_k)})^{s_i} (\bar{\alpha}_i^{(s_k)})^{s_i}.$$

In the case where $s = (1, 1, 1, \dots)$, the sequence $\alpha_i^{(s_k)}$ is a prefix of the well-known Prouhet–Thue–Morse sequence (see [6]):

i	$\alpha_i^{(1,1,1,\dots)}$
1	01
2	0110
3	01101001
4	0110100110010110

This example illustrates several features of the construction that hold for arbitrary sequences (s_k) . First, we can read the lengths of these words directly from the construction:

$$|\alpha_i^{(s_k)}| = 2^i s_1 s_2 \cdots s_{i-1}.$$

Second, the first letter of $\alpha_i^{(s_k)}$ is always 0, while the last letter alternates with i .

Observation 2.1. *The last letter of $\alpha_i^{(s_k)}$ is 0 if and only if i is even.*

Of particular interest to us is the language

$$\mathcal{L}^{(s_k)} = \{w : w \text{ is a factor of } \alpha_i^{(s_k)} \text{ for some } i \geq 1\}.$$

By definition, we include the empty word ε in $\mathcal{L}^{(s_k)}$.

Note that for any sequence (s_k) and indices $i \leq j$, the word $\alpha_j^{(s_k)}$ can be expressed as a sequence of concatenations of the words $\alpha_i^{(s_k)}$ and $\bar{\alpha}_i^{(s_k)}$. That is, whenever $i \leq j$, then $\alpha_j^{(s_k)}$ can be expressed as a word over the alphabet $\{\alpha_i^{(s_k)}, \bar{\alpha}_i^{(s_k)}\}$, and we use this viewpoint several times in what follows. One consequence is that if w is a factor of $\alpha_i^{(s_k)}$ for some i , then it is also a factor of $\alpha_j^{(s_k)}$ for all $j \geq i$. Thus, the language $\mathcal{L}^{(s_k)}$ can also be defined as the set of all words that appear as factors of $\alpha_i^{(s_k)}$ for all sufficiently large values of i .

In the other direction, the next result, which is also essentially from Pouzet's thesis, guarantees that all sufficiently long words in $\mathcal{L}^{(s_k)}$ actually contain $\alpha_i^{(s_k)}$.

Proposition 2.2. *For every sequence (s_k) of positive integers, there exists a function $f^{(s_k)}(i)$ such that every word in $w \in \mathcal{L}^{(s_k)}$ of length at least $f^{(s_k)}(i)$ contains $\alpha_i^{(s_k)}$ as a factor.*

Proof. Let $\alpha = \alpha_{i+1}^{(s_k)}$. We claim that we can take

$$f^{(s_k)}(i) = 2|\alpha| = 2^{i+2}s_1s_2 \cdots s_i.$$

To see this, let $w \in \mathcal{L}^{(s_k)}$ have length at least $f^{(s_k)}(i)$. As

$$|w| \geq f^{(s_k)}(i) > |\alpha_i^{(s_k)}|,$$

we see that w must be a factor of $\alpha_j^{(s_k)}$ for some $j \geq i + 1$. All such words are composed of concatenations of α with its complement $\bar{\alpha}$, and thus w must contain either α or $\bar{\alpha}$ as a factor, since w is at least twice as long as both. The result follows because each of α and $\bar{\alpha}$ contain $\alpha_i^{(s_k)}$ as a factor. \square

That these languages are wqo follows readily from Proposition 2.2.

Proposition 2.3. *For every sequence (s_k) of positive integers, the set $\mathcal{L}^{(s_k)} \subseteq \{0, 1\}^*$ of words is wqo under the factor order.*

Proof. Suppose to the contrary that there were an infinite antichain w_1, w_2, \dots of words from $\mathcal{L}^{(s_k)}$. The word w_1 is contained in $\alpha_i^{(s_k)}$ for some index i because it lies in $\mathcal{L}^{(s_k)}$. Letting $f^{(s_k)}(i)$ denote the function from Proposition 2.2, we see that every word in $\mathcal{L}^{(s_k)}$ of length at least $f^{(s_k)}(i)$ contains $\alpha_i^{(s_k)}$ as a factor, and hence also contains w_1 as a factor. It follows that there is some index j such that $|w_j| \geq f^{(s_k)}(i)$, from which we conclude that w_1 is a factor of w_j , which contradicts our assumption that these words form an infinite antichain. \square

Pouzet [36] established that when (s_k) and (t_k) are distinct sequences, then $\mathcal{L}^{(s_k)}$ and $\mathcal{L}^{(t_k)}$ are distinct languages. As there are uncountably many sequences of positive integers, there must then be uncountably many wqo factor-closed languages over a binary alphabet. For our enumerative goal, we require something a bit more precise: not only must $\mathcal{L}^{(s_k)}$ and $\mathcal{L}^{(t_k)}$ be distinct languages, but there must be some length where one language contains more words of that length than the other. In fact, the result we prove is stronger still: the set of words of some given length in one language is a *proper subset* of the words of that length in the other.

Before we state and prove this, however, we require a technical characterisation of the possible embeddings of $\alpha_i^{(s_k)}$ in $\alpha_j^{(s_k)}$ for given i and any $j > i$. We again adopt the viewpoint that $\alpha_j^{(s_k)}$ can be regarded as a word over the alphabet $\{\alpha_i^{(s_k)}, \bar{\alpha}_i^{(s_k)}\}$, but to ease exposition let us use α_i^* and $\bar{\alpha}_i^*$ to denote the *letters*, and α_j^* the word over $\{\alpha_i^*, \bar{\alpha}_i^*\}$, with the property that the binary word $\alpha_j^{(s_k)}$ is equal to α_j^* after performing the substitutions $\alpha_i^* \mapsto \alpha_i$ and $\bar{\alpha}_i^* \mapsto \bar{\alpha}_i$.

Proposition 2.4. *For every sequence (s_k) of positive integers and integers $i < j$, the only factors of $\alpha_j^{(s_k)}$ that are equal to $\alpha_i^{(s_k)}$ are given by*

- (i) *the terms corresponding to a letter α_i^* of α_j^* , and*
- (ii) *the middle terms corresponding to a pair of letters $\bar{\alpha}_i^* \bar{\alpha}_i^*$ in α_j^* .*

An analogous statement holds for factors of $\alpha_j^{(s_k)}$ that are equal to $\bar{\alpha}_i^{(s_k)}$.

Proof. Since we are dealing with a single sequence (s_k) in this proof, we drop the (s_k) superscript throughout.

It is clear that any occurrence of α_i^* in α_j^* corresponds to a factor of α_j equal to α_i . Similarly, we have

$$\bar{\alpha}_i \bar{\alpha}_i = (\bar{\alpha}_{i-1}^{s_{i-1}} \alpha_{i-1}^{s_{i-1}}) (\bar{\alpha}_{i-1}^{s_{i-1}} \alpha_{i-1}^{s_{i-1}}) = \bar{\alpha}_{i-1}^{s_{i-1}} \alpha_i \alpha_{i-1}^{s_{i-1}},$$

which demonstrates that the middle terms of $\bar{\alpha}_i \bar{\alpha}_i$ are equal to α_i .

To show that these are the only factors, we proceed by induction on i . In the case $i = 1$, we have $\alpha_i = 01$ and $\bar{\alpha}_i = 10$. For any $j > 1$, fix an occurrence of 01 in α_j . If this occurrence starts on an odd-numbered letter of α_j , then in α_j^* this occurrence corresponds either to the letter α_1^* or $\bar{\alpha}_1^*$, but it clearly cannot be the second of these. If, on the other hand, this occurrence starts on an even-numbered letter of α_j , then this occurrence straddles two letters of α_j^* . By inspection, the only possibility is $\bar{\alpha}_i^* \bar{\alpha}_i^*$. A similar argument applies for the occurrences of $\bar{\alpha}_i$ in α_j , and this completes the base case.

Suppose that the proposition is true for some $i \geq 1$. Take $j > i + 1$, and consider an occurrence of α_{i+1} in α_j . Since $\alpha_{i+1} = \alpha_i^{s_i} \bar{\alpha}_i^{s_i}$, we begin by considering the occurrences of $\alpha_i^{s_i}$ and $\bar{\alpha}_i^{s_i}$ in α_j .

By induction, the only occurrences of α_i in α_j are as given in the proposition. Consequently, the only occurrences of $\alpha_i^{s_i}$ in α_j correspond to factors $(\alpha_i^*)^{s_i}$ in α_j^* , or the middle terms of the binary word corresponding to $(\bar{\alpha}_i^*)^{s_i+1}$. A similar statement holds for the occurrences of $\bar{\alpha}_i^{s_i}$ in α_j .

We now consider the possible positions for our occurrence of $\alpha_{i+1} = \alpha_i^{s_i} \bar{\alpha}_i^{s_i}$ in α_j . If the first half (the word $\alpha_i^{s_i}$) appears in the middle of some factor of the form $\bar{\alpha}_i^{s_i+1}$, then the second half of the word, $\bar{\alpha}_i^{s_i}$ does not embed. Similarly, if the second half of the word embeds in the middle of a factor of the form $\alpha_i^{s_i+1}$ then the first half cannot embed. Therefore, the only embeddings of α_{i+1} in α_j must correspond precisely to instances of the factor $(\alpha_i^*)^{s_i} (\bar{\alpha}_i^*)^{s_i}$ in α_j^* . A similar statement applies to $\bar{\alpha}_{i+1}$.

Finally, by construction, α_j^* is composed of factors of the form $(\alpha_i^*)^{s_i} (\bar{\alpha}_i^*)^{s_i}$ and $(\bar{\alpha}_i^*)^{s_i} (\alpha_i^*)^{s_i}$. These correspond to the words α_{i+1} and $\bar{\alpha}_{i+1}$ that can be used to make up α_j , from which the inductive step follows. \square

We now state the main result of this section; we prove it after considering an example.

Proposition 2.5. *Suppose that (s_k) and (t_k) are distinct sequences of positive integers, and that (s_k) lexicographically precedes (t_k) . Then there exists an integer $M \geq 3$ such that $\mathcal{L}_n^{(s_k)} = \mathcal{L}_n^{(t_k)}$ for all $n < M$, but $\mathcal{L}_M^{(s_k)} \not\supseteq \mathcal{L}_M^{(t_k)}$.*

We note that the “inequalities” in Proposition 2.5 are in some sense reversed—we assume that (s_k) precedes (t_k) in lexicographical order, and then it follows that $\mathcal{L}_M^{(s_k)}$ is *larger* than $\mathcal{L}_M^{(t_k)}$. We illustrate this with the concrete example where the sequences are $(1, 1, 1, \dots)$ and $(2, 1, 1, \dots)$. In this case the construction gives us the table below.

i	$\alpha_i^{(1,1,1,\dots)}$	$\alpha_i^{(2,1,1,\dots)}$
1	01	01
2	0110	01011010
3	01101001	0101101010100101
4	0110100110010110	01011010101001011010010101011010

One can check that the languages $\mathcal{L}^{(1,1,1,\dots)}$ and $\mathcal{L}^{(2,1,1,\dots)}$ agree up to and including length 3:

$$\begin{aligned} \mathcal{L}_0^{(2,1,1,\dots)} &= \mathcal{L}_0^{(1,1,1,\dots)} = \{\varepsilon\}, \\ \mathcal{L}_1^{(2,1,1,\dots)} &= \mathcal{L}_1^{(1,1,1,\dots)} = \{0, 1\}, \\ \mathcal{L}_2^{(2,1,1,\dots)} &= \mathcal{L}_2^{(1,1,1,\dots)} = \{00, 01, 10, 11\}, \\ \mathcal{L}_3^{(2,1,1,\dots)} &= \mathcal{L}_3^{(1,1,1,\dots)} = \{001, 010, 011, 100, 101, 110\}; \end{aligned}$$

but they differ at length 4:

$$\begin{aligned} \mathcal{L}_4^{(2,1,1,\dots)} &= \{0010, 0100, 0101, 0110, 1001, 1010, 1011, 1101\}, \\ \mathcal{L}_4^{(1,1,1,\dots)} &= \{0010, 0100, 0101, 0110, 1001, 1010, 1011, 1101, 0011, 1100\}. \end{aligned}$$

Hence, in this case Proposition 2.5 holds with $M = 4$. We conclude this section by proving Proposition 2.5, and then we begin translating Pouzet’s construction to the permutation context in the next section.

Proof of Proposition 2.5. Let (s_k) and (t_k) be as in the statement of the result, and choose I so that $s_i = t_i$ for all indices $i < I$, but $s_I < t_I$. We claim that we may take $M = 2^I s_1 s_2 \cdots s_I + 2$. Note that $M \geq 4$. For convenience, set $\alpha = \alpha_I^{(s_k)}$, and note that $|\alpha| = 2^I s_1 \cdots s_{I-1}$, and that $M = s_I |\alpha| + 2$.

First consider a word w of length $i < M$ in $\mathcal{L}^{(s_k)}$, and let $j > I$ be such that $\alpha_j^{(s_k)}$ contains w . Now view $\alpha_j^{(s_k)}$ as a sequence of concatenations of α^{s_I} and $\bar{\alpha}^{s_I}$. Since w has length at most $s_I |\alpha| + 1$, it follows that w embeds in $\alpha_j^{(s_k)}$ in at most two terms of this concatenation. That is, w appears as a factor in one of the following words:

$$\alpha^{s_I} \alpha^{s_I}, \quad \alpha^{s_I} \bar{\alpha}^{s_I}, \quad \bar{\alpha}^{s_I} \alpha^{s_I}, \quad \bar{\alpha}^{s_I} \bar{\alpha}^{s_I}$$

In fact, containment in one of these four words is a precise characterization of the words of length at most $M - 1$ in $\mathcal{L}^{(s_k)}$, since all four words appear in $\alpha_{I+3}^{(s_k)}$:

$$\alpha_{I+3}^{(s_k)} = \left((\alpha^{s_I} \bar{\alpha}^{s_I})^{s_{I+1}} (\bar{\alpha}^{s_I} \alpha^{s_I})^{s_{I+1}} \right)^{s_{I+2}} \left((\bar{\alpha}^{s_I} \alpha^{s_I})^{s_{I+1}} (\alpha^{s_I} \bar{\alpha}^{s_I})^{s_{I+1}} \right)^{s_{I+2}}$$

Furthermore, since $t_I > s_I$, all four of these words also appear in $\mathcal{L}^{(t_k)}$, and this establishes that $\mathcal{L}^{(s_k)}$ and $\mathcal{L}^{(t_k)}$ contain the same words up to length $M - 1$.

We now turn our attention to the words of length M . First, any word w of length M that embeds in one of the four words

$$\alpha^{s_I} \alpha^{s_I}, \quad \alpha^{s_I} \bar{\alpha}^{s_I}, \quad \bar{\alpha}^{s_I} \alpha^{s_I}, \quad \bar{\alpha}^{s_I} \bar{\alpha}^{s_I}$$

will lie in both $\mathcal{L}^{(s_k)}$ and $\mathcal{L}^{(t_k)}$, by the same argument as before. Any other word of length M must be formed from a copy of α^{s_I} or $\bar{\alpha}^{s_I}$, with exactly one letter before and one after. That is, the only remaining words to consider are the following:

$$\begin{aligned} 0\alpha^{s_I}0, & \quad 1\alpha^{s_I}0, & \quad 0\alpha^{s_I}1, & \quad 1\alpha^{s_I}1, \\ 0\bar{\alpha}^{s_I}0, & \quad 1\bar{\alpha}^{s_I}0, & \quad 0\bar{\alpha}^{s_I}1, & \quad 1\bar{\alpha}^{s_I}1. \end{aligned}$$

Let us assume that I is even; the case where I is odd is analogous. By Observation 2.1, the word α both begins and ends with 0. The following table summarizes whether each of the eight words above belongs to $\mathcal{L}^{(s_k)}$ and $\mathcal{L}^{(t_k)}$, and if so illustrates how it arises (note that the words specified in the second and third columns are factors of $\alpha_{I+3}^{(s_k)}$ and $\alpha_{I+3}^{(t_k)}$, respectively).

word	in $\mathcal{L}^{(s_k)}$, factor of	in $\mathcal{L}^{(t_k)}$, factor of
$0\alpha^{s_I}0$	$\bar{\alpha}^{s_I}\bar{\alpha}^{s_I}$ (see below)	$\alpha^{t_I}\alpha^{t_I}$
$1\alpha^{s_I}0$	$\bar{\alpha}^{s_I}\alpha^{s_I}\alpha^{s_I}$	$\bar{\alpha}^{t_I}\alpha^{t_I}$
$0\alpha^{s_I}1$	$\alpha^{s_I}\alpha^{s_I}\bar{\alpha}^{s_I}$	$\alpha^{t_I}\bar{\alpha}^{t_I}$
$1\alpha^{s_I}1$	$\bar{\alpha}^{s_I}\alpha^{s_I}\bar{\alpha}^{s_I}$	Not in set (see below)
$0\bar{\alpha}^{s_I}0$	$\alpha^{s_I}\bar{\alpha}^{s_I}\alpha^{s_I}$	Not in set (see below)
$1\bar{\alpha}^{s_I}0$	$\bar{\alpha}^{s_I}\bar{\alpha}^{s_I}\alpha^{s_I}$	$\bar{\alpha}^{t_I}\alpha^{t_I}$
$0\bar{\alpha}^{s_I}1$	$\alpha^{s_I}\bar{\alpha}^{s_I}\bar{\alpha}^{s_I}$	$\alpha^{t_I}\bar{\alpha}^{t_I}$
$1\bar{\alpha}^{s_I}1$	$\alpha^{s_I}\alpha^{s_I}$ (see below)	$\bar{\alpha}^{t_I}\bar{\alpha}^{t_I}$

There remain four entries in the above table to consider. Let us begin first with the word $0\alpha^{s_I}0$. Proposition 2.4 tells us that α embeds in the middle of $\bar{\alpha}\bar{\alpha}$, and thus α^{s_I} embeds in the middle of $\bar{\alpha}^{s_I}\bar{\alpha}^{s_I}$. Furthermore, the letter immediately to the left of this embedding is the last letter of $\bar{\alpha}_{I-1}^{(s_k)}$, which is 0 by Observation 2.1 (since we are assuming that $I-1$ is odd). Similarly, the first letter after this embedding is the first letter of $\bar{\alpha}_{I-1}^{(s_k)}$, which is also 0, and hence $0\alpha^{s_I}0$ is a factor of $\bar{\alpha}^{s_I}\bar{\alpha}^{s_I}$.

A similar argument can be applied to show that $1\bar{\alpha}^{s_I}1 \in \mathcal{L}^{(s_k)}$, and this establishes that $\mathcal{L}_M^{(s_k)} \supseteq \mathcal{L}_M^{(t_k)}$.

Our final task is to show that neither $1\alpha^{s_I}1$ nor $0\bar{\alpha}^{s_I}0$ lies in $\mathcal{L}^{(t_k)}$. We consider only $1\alpha^{s_I}1$, the case for the other word being entirely analogous.

By Proposition 2.4, the only factors of $\alpha_j^{(t_k)}$ that are equal to α are either given by the *letter* α , or appear in the middle of the pair of letters $\bar{\alpha}\bar{\alpha}$, when we express $\alpha_j^{(t_k)}$ as a word over $\{\alpha, \bar{\alpha}\}$. Thus, the binary word α^{s_I} appears as a factor of $\alpha_j^{(t_k)}$ only as the sequence of letters α^{s_I} , or in the middle of $\bar{\alpha}^{s_I+1}$. In this latter embedding, the letter in $\bar{\alpha}^{s_I+1}$ that lies immediately to the left of such an embedding is the last letter of $\bar{\alpha}_{I-1}^{(s_k)}$, which we have already established is equal to 0. Thus $1\alpha^{s_I}1$ does not appear in the middle of $\bar{\alpha}^{s_I+1}$.

The only remaining possibility is that $1\alpha^{s_I}1$ embeds into $\alpha_j^{(t_k)}$ precisely as α^{s_I} , together with one extra letter on either side. Now $\alpha_j^{(t_k)}$ can be written as a word comprising factors of the form α^{t_I} and $\bar{\alpha}^{t_I}$. Since $t_I > s_I$, we are forced to embed either the letter 1 on the left of $1\alpha^{s_I}1$ as the rightmost letter of α , or the letter 1 on the right of $1\alpha^{s_I}1$ as the leftmost letter of α . Since α both begins and ends with 0, however, neither case is possible. Thus $1\alpha^{s_I}1$ does not embed into $\alpha_j^{(t_k)}$, from which we conclude that $1\alpha^{s_I}1 \notin \mathcal{L}^{(t_k)}$, as required. \square

3. RIGHTWARD-YEARING PIN SEQUENCES

Our tool for translating the results about Pouzet's languages $\mathcal{L}^{(s_k)}$ to the permutation context are the pin sequences first introduced by Brignall, Huczynska, and Vatter [13]. These are best described with a pictorial description of the permutation pattern order. The *plot* of the permutation π is the set $\{(i, \pi(i))\}$ of points. Clearly every plot of a permutation is *generic* in the sense that no two of its points share the same x - or y -coordinate. Conversely, every finite generic set of points in the plane is order isomorphic to the plot of a unique permutation, in the sense that two sets of points in the plane are *order isomorphic* if the axes can be stretched and shrunk to transform one of the sets into the other.

An *axis-parallel rectangle* is any rectangle in the plane with sides parallel to the x - and y -axes. The *rectangular hull* of a set of points in the plane is defined as the smallest axis-parallel rectangle containing them. Given a sequence (p_1, \dots, p_i) of points in the plane, a *proper pin* for this sequence is a point p that lies outside their rectangular hull and *separates* p_i from $\{p_1, \dots, p_{i-1}\}$, meaning that p lies either horizontally or vertically between p_i and the rectangular hull of $\{p_1, \dots, p_{i-1}\}$. A *proper pin sequence* is then constructed by starting with two points p_1 and p_2 (whose placement we discuss later), choosing p_3 to be a proper pin for (p_1, p_2) , then choosing p_4 to be a proper pin for (p_1, p_2, p_3) , and so on. We describe pins as either *left*, *right*, *up*, or *down* based on their position relative to the rectangular hull of $\{p_1, \dots, p_i\}$. Note that the direction of a pin uniquely specifies its position relative to the previous points in a pin sequence. We specify pin sequences with the alphabet $\{l, r, u, d\}$.

It follows from their definition that proper pin sequences must turn by 90° with each pin. In other words, an up pin may be immediately followed by a left or a right pin, but not by another up pin or by a down pin. As our goal is to encode binary strings as proper pin sequences, we *could* therefore translate each 0 into left or down and each 1 into right or up, with the choices determined by the previous pin. If we were to do this, we might for example have the correspondence

$$01101001 \mapsto \text{drululdr}.$$

While we suspect that our results would remain true with this translation from binary words to proper pin sequences, they would undoubtedly be more troublesome to prove. Instead, we restrict our attention to *rightward-yearning pin sequences*, which we define to be the pin sequences made up only of right, up, and down pins. This means, because proper pin sequences must turn by 90° with every pin, that every second pin in a rightward-yearning pin sequence must be a right pin. For the other pins, we translate 0 to be a down pin and 1 to be an up pin. Thus in this correspondence, we would have

$$01101001 \mapsto \text{drururdrurdrur}.$$

In fact, we encode our pin sequences slightly differently than this. For reasons that will become apparent later, we subscript the letter encoding each right pin by the type of pin immediately preceding it. This enlarges our alphabet to $\{r_d, r_u, d, u\}$, and our previous example becomes

$$01101001 \mapsto \text{dr}_d\text{ur}_u\text{ur}_u\text{dr}_d\text{ur}_u\text{dr}_d\text{dr}_d\text{ur}_u.$$

(This pin sequence is plotted in Figure 1.) We stress that the subscripts on our encodings of down pins do not affect the actual pin sequences; these subscripts instead inform us where the corresponding points lie in the plane.

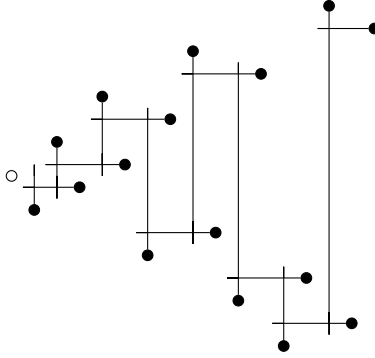


Figure 1: The rightward-yearning pin sequence associated to the word $dr_dur_uur_u dr_dur_u dr_d dr_d ur_u$.

It remains to explain how to start a pin sequence. In the present work, we start every pin sequence with a point called an *origin* and labelled by p_0 . *Importantly, we do not consider the origin p_0 to be part of the resulting rightward-yearning pin sequence.*

For the first real pin, p_1 , if it has encoding u or r_u , then p_1 appears above and to the right of p_0 . Analogously, if p_1 has encoding d or r_d , then it is placed below and to the right of p_0 . In either case, the second pin p_2 slices the rectangular hull of (p_0, p_1) in the direction indicated by its encoding.

As our pin sequences progress only to the right, this origin p_0 lies to the left of all other pins of the pin sequence, and will lie below all pins whose encoding is u or r_u , and above all pins whose encoding is d or r_d . In this way, the origin partitions each of the entries of the pin sequence into two groups: those above and below the x -axis. This means that our pin sequences could be considered a very simple type of *grid pin sequence* (as first considered by Brignall [12]), but we do not adopt this viewpoint.

Given any word $w \in \{r_d, r_u, d, u\}^n$ in which the letters alternate between $\{d, u\}$ and $\{r_d, r_u\}$, we take the *rightward-yearning pin sequence defined by w* to be the permutation τ_w° that is order isomorphic to the set $\{p_1, p_2, \dots, p_n\}$ of points defined by the word w . Thus $|\tau_w^\circ| = |w| = n$.

As is demonstrated in Section 5, in order to discuss subpermutations of rightward-yearning pin sequences, it is necessary to also introduce the permutations τ_w^\bullet that include the origin as an extra point. We take τ_w^\bullet to be the permutation that is order isomorphic to the set $\{p_0, p_1, p_2, \dots, p_n\}$ of points defined by the word w , so $|\tau_w^\bullet| = |w| + 1 = n + 1$.

4. THE CONSTRUCTION

We now have the necessary background to describe the family of permutation classes used to prove Theorem 1.1.

Given a binary word $w \in \{0, 1\}^*$, we denote by $\rho(w)$ the word (of twice the length as w) over the alphabet $\{r_d, r_u, d, u\}$ that is obtained by performing the substitutions

$$0 \leftarrow dr_d \quad \text{and} \quad 1 \leftarrow ur_u;$$

that is, replacing occurrences of 0 by dr_d and occurrences of 1 by ur_u . We are frequently interested

only in the image of ρ and factors of those words, and so we define the language

$$\mathcal{P} = \text{the factor closure of } \rho(\{0, 1\}^*).$$

Our construction itself requires us to restrict to the binary words of Section 2, for which we recall that

$$\mathcal{L}^{(s_k)} = \text{the factor-closed wqo languages of Section 2.}$$

We then consider the factors of the image under ρ of those languages, and the class of permutations to which they correspond:

$$\begin{aligned} \mathcal{P}^{(s_k)} &= \text{the factor closure of } \rho(\mathcal{L}^{(s_k)}) \text{ and} \\ \mathcal{C}^{(s_k)} &= \text{the downward closure of } \{\psi_w^\circ : w \in \mathcal{P}^{(s_k)}\}. \end{aligned}$$

It follows from the definitions that \mathcal{P} , $\mathcal{L}^{(s_k)}$, and $\mathcal{P}^{(s_k)}$ are closed under taking factors, while $\mathcal{C}^{(s_k)}$ is closed under taking subpermutations, and is thus a permutation class. What remains to do is to establish that the classes $\mathcal{C}^{(s_k)}$ are wqo, and that if the sequences (s_k) and (t_k) differ, then the resulting classes $\mathcal{C}^{(s_k)}$ and $\mathcal{C}^{(t_k)}$ have different enumerations. These two results will follow by lifting the analogous results about the languages $\mathcal{L}^{(s_k)}$, Propositions 2.3 and 2.5, to this context. In order to do this, we must first establish a decomposition result for the members of $\mathcal{C}^{(s_k)}$ in the next section. Before that, we make a simple observation now that we have the terminology to express it.

Proposition 4.1. *Suppose that v is a factor of w for words $v, w \in \mathcal{P}$. Then the permutation ψ_w° contains the permutation ψ_v° and the permutation ψ_w^\bullet contains the permutation ψ_v^\bullet .*

Proof. Fix a factor of v in w . The pins of ψ_w° corresponding to this factor are in the same relative position to each other as the pins of ψ_v° , which verifies the claim for this pair of permutations. For the version of the result with origins, we note that the origin in ψ_w^\bullet is in the same relative position to the pins that correspond to a factor of v in w as the origin in ψ_v^\bullet is relative to the rest of the pins of that permutation. \square

5. A DECOMPOSITION

We frame the discussion in this section as considering the effect of deleting points from rightward-yearning pin sequences. There are essentially three types of pins we can delete: the first pin, the last pin, or an interior pin. We handle these cases below in order of their difficulty, for a word $w \in \{r_d, r_u, d, u\}^n$.

- To delete the last pin from a pin sequence, we simply don't create it in the first place. Thus the permutation obtained from ψ_w° by deleting the point corresponding to p_n is $\psi_{w(1)\dots w(n-1)}^\circ$.
- The same argument as above holds for the first pin, so the permutation obtained from ψ_w° by deleting the point corresponding to the first pin is $\psi_{w(2)\dots w(n)}^\circ$.
- Deleting the i^{th} pin from ψ_w° for some index $2 \leq i \leq n-2$ corresponds to replacing the origin in the permutation $\psi_{w(i+1)\dots w(n)}^\bullet$ with the permutation $\psi_{w(1)\dots w(i-1)}^\circ$.

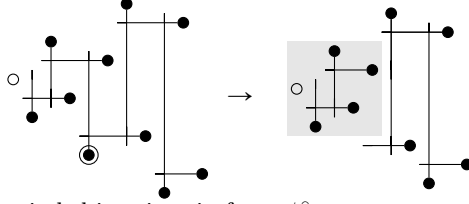


Figure 2: Deleting the circled interior pin from $\psi_{d_r_d r_u d_r_d r_u d_r_d}^\circ$ results in $\psi_{d_r_d r_u}^\circ \boxplus \psi_{r_d r_u d_r_d}^\bullet$.

We call the operation in this last case *inflating the origin*, and this case is the reason we introduced the permutations ψ_w^\bullet in Section 3 in the first place. The process of removing pins from the permutations ψ_w^\bullet is also needed, but is entirely analogous to the above, and will be handled once we have introduced some additional notation.

Inflating origins is similar to the sum of two permutations. Recall that given a permutation σ of length m and another permutation τ of length n , their *sum* is the permutation denoted by $\sigma \oplus \tau$ and defined by

$$(\sigma \oplus \tau)(i) = \begin{cases} \sigma(i) & \text{if } 1 \leq i \leq m, \\ \tau(i - m) + m & \text{if } m + 1 \leq i \leq m + n. \end{cases}$$

The origin in ψ_w^\bullet is always the leftmost point, so we define the more general operation of inflating the first entry of a permutation. Suppose that σ is a permutation of length m and that τ is a permutation of length $n + 1$. Then $\sigma \boxplus \tau$ is the permutation of length $m + n$ obtained by inflating the first entry of τ by σ . Formulaically, it is defined by

$$(\sigma \boxplus \tau)(i) = \begin{cases} \sigma(i) + \tau(1) - 1 & \text{if } 1 \leq i \leq m, \\ \tau(i - m + 1) & \text{if } m < i \leq m + n \text{ and } \tau(i - m + 1) < \tau(1), \text{ and} \\ \tau(i - m + 1) + m - 1 & \text{if } m < i \leq m + n \text{ and } \tau(i - m + 1) > \tau(1). \end{cases}$$

See Figure 2 for an example.

From our previous discussion, it follows that if we delete the i th pin from ψ_w° , where $w \in \{r_d, r_u, d, u\}^n$ and $2 \leq i \leq n - 1$, then we obtain the permutation

$$\psi_{w(1)\dots w(i-1)}^\circ \boxplus \psi_{w(i+1)\dots w(n)}^\bullet.$$

Indeed, letting ε denote the empty word or empty permutation (as dictated by the context), and with the understanding that $\psi_\varepsilon^\circ = \varepsilon$ while $\psi_\varepsilon^\bullet = 1$, we see that for *any* $1 \leq i \leq n$, the result of deleting the point corresponding to the i th pin p_i from ψ_w° is $\psi_{w(1)\dots w(i-1)}^\circ \boxplus \psi_{w(i+1)\dots w(n)}^\bullet$.

As noted earlier, we must also describe how to delete points from the permutations ψ_w^\bullet , which follows by the same analysis. If we delete the origin p_0 from ψ_w^\bullet , we obviously obtain the permutation ψ_w° . Otherwise, if $w \in \{r_d, r_u, d, u\}^n$ and $1 \leq i \leq n$, then the result of deleting the point corresponding to the i th pin p_i from ψ_w^\bullet is

$$\psi_{w(1)\dots w(i-1)}^\bullet \boxplus \psi_{w(i+1)\dots w(n)}^\bullet.$$

Our next result puts this decomposition in the form we need it later. Note that we needn't include parentheses in the statement of this result because the operation \boxplus is associative.

Proposition 5.1. *For any subpermutation π of the rightward-yearning pin sequence ψ_w° , there exist nonempty words $w_1, \dots, w_k \in \mathcal{P}$, each appearing as a factor in w , such that*

$$\pi = \psi_{w_1}^\circ \boxplus \psi_{w_2}^\bullet \boxplus \dots \boxplus \psi_{w_k}^\bullet.$$

Proof. Fix a particular embedding of π in ψ_w° , and delete the entries of ψ_w° that are not involved in this embedding one at a time. At each step of this process, we have a permutation of the form $\psi_{u_1}^\circ \boxplus \psi_{u_2}^\bullet \boxplus \cdots \boxplus \psi_{u_\ell}^\bullet$ for some choice of words $u_1, \dots, u_\ell \in \mathcal{P}$ (which we may assume, by induction, appear as factors in the word w), from which we must delete some specific entry. This entry must be a non-origin pin of either $\psi_{u_1}^\circ$ or one of the $\psi_{u_j}^\bullet$, and we've shown above that any such deletion results in another permutation of the desired form. The fact that each of the resulting words is a factor of w also follows by this inductive argument. \square

We are now able to describe the decomposition of all permutations in some class $\mathcal{C}^{(s_k)}$ in terms of permutations defined by words from the language $\mathcal{P}^{(s_k)}$.

Theorem 5.2. *Let (s_k) be a sequence of positive integers. The class $\mathcal{C}^{(s_k)}$ consists precisely of those permutations π that can be expressed as*

$$\pi = \psi_{v_1}^\circ \boxplus \psi_{v_2}^\bullet \boxplus \cdots \boxplus \psi_{v_\ell}^\bullet$$

for words $v_1, v_2, \dots, v_\ell \in \mathcal{P}^{(s_k)}$ with $|v_1| + |v_2| + \cdots + |v_\ell| = |\pi|$.

Proof. Let $\pi \in \mathcal{C}^{(s_k)}$. By the definition of $\mathcal{C}^{(s_k)}$, there is some word $w \in \mathcal{P}^{(s_k)}$ such that π is contained in the permutation ψ_w° . By Proposition 5.1, it follows that there exist words v_1, v_2, \dots, v_ℓ , each appearing as a factor of w , such that

$$\pi = \psi_{v_1}^\circ \boxplus \psi_{v_2}^\bullet \boxplus \cdots \boxplus \psi_{v_\ell}^\bullet.$$

Since each v_i is a factor of $w \in \mathcal{P}^{(s_k)}$, it follows that each $v_i \in \mathcal{P}^{(s_k)}$. It is similarly clear that $|v_1| + |v_2| + \cdots + |v_\ell| = |\pi|$.

Conversely, we consider a permutation π as in the statement of the theorem and appeal to the properties of the language $\mathcal{L}^{(s_k)}$. By definition, for each $v_i \in \mathcal{P}^{(s_k)}$, there exists an integer j_i such that v_i is a factor of $\rho(\alpha_{j_i}^{(s_k)})$. Set $m = \max\{j_1, \dots, j_\ell\}$, so that each of v_1, \dots, v_ℓ appears as a factor in $\rho(\alpha_m^{(s_k)})$.

Letting $f^{(s_k)}$ denote the function from Proposition 2.2, we see from that result that every (binary) word of length at least $f^{(s_k)}(m)$ in $\mathcal{L}^{(s_k)}$ contains $\alpha_m^{(s_k)}$. It follows that every word in $\mathcal{P}^{(s_k)}$ of length at least $2f^{(s_k)}(m) + 2$ contains $\rho(\alpha_m^{(s_k)})$, and thus also contains all of v_1, v_2, \dots, v_ℓ . (The “+2” here is due to the fact that the first and last letter of a word in \mathcal{P} need not be part of a factor of the form $\rho(u)$.) It follows that every word in $\mathcal{P}^{(s_k)}$ of length at least $\ell \cdot (2f^{(s_k)}(m) + 2) + (\ell - 1)$ must contain a word w of the form

$$w = v_1 x_1 v_2 x_2 \cdots x_{\ell-1} v_\ell$$

where $x_1, \dots, x_{\ell-1}$ are arbitrary non-empty words. Thus π is a subpermutation of ψ_w° . \square

It is tempting to conclude from Theorem 5.2 that each $\mathcal{C}^{(s_k)}$ is \boxplus -closed in the sense that $\sigma, \tau \in \mathcal{C}^{(s_k)}$ implies that $\sigma \boxplus \tau \in \mathcal{C}^{(s_k)}$, but we need to be careful: if we simply inflate the first entry of τ by σ , then what results is not guaranteed to lie in $\mathcal{C}^{(s_k)}$, since it is not necessarily the case that τ can be described in such a way that its first entry can act as a (non-phantom) origin. However, since $\tau \in \mathcal{C}^{(s_k)}$, Theorem 5.2 tells us that we can write

$$\tau = \psi_{v_1}^\circ \boxplus \psi_{v_2}^\bullet \boxplus \cdots \boxplus \psi_{v_\ell}^\bullet.$$

for some v_1, v_2, \dots, v_ℓ in $\mathcal{P}^{(s_k)}$, and then Theorem 5.2 tells us that

$$\sigma \boxplus \psi_{v_1}^\bullet \boxplus \psi_{v_2}^\bullet \boxplus \cdots \boxplus \psi_{v_\ell}^\bullet \in \mathcal{C}^{(s_k)}.$$

1	=	$\psi_u^\circ = \psi_d^\circ = \psi_r^\circ$	\boxplus -indecomposable	1234	=	$12 \boxplus 12 \boxplus 12$	
12	=	$\psi_{rd}^\circ = \psi_{dr_d}^\circ$	\boxplus -indecomposable	1243	=	$12 \boxplus 132$	
21	=	$\psi_{ru}^\circ = \psi_{ur_u}^\circ$	\boxplus -indecomposable	1324	=	$132 \boxplus 12$	
123	=	$12 \boxplus 12$		1342	=	$\psi_{rdr_du}^\circ$	\boxplus -indecomposable
132	=	$\psi_{rdr_d}^\circ = \psi_{dr_du}^\circ$	\boxplus -indecomposable	1423	=	$\psi_{dr_dur_u}^\circ$	\boxplus -indecomposable
213	=	$21 \boxplus 12 = \psi_{dr_d}^\circ$		1432			\boxplus -indecomposable
231	=	$12 \boxplus 21 = \psi_{ur_u}^\circ$		2134	=	$21 \boxplus 12 \boxplus 12$	
312	=	$\psi_{ur_d}^\circ = \psi_{rur_u}^\circ$	\boxplus -indecomposable	2143	=	$21 \boxplus 132$	
321	=	$21 \boxplus 21$		2314	=	$12 \boxplus 21 \boxplus 12$	
				2341	=	$12 \boxplus 12 \boxplus 21$	
				2413	=	$\psi_{ur_uur_u}^\circ = \psi_{rdr_d}^\circ$	\boxplus -indecomposable
				2431	=	$132 \boxplus 21$	

Table 1: Decompositions of permutations of lengths at most four. An initial letter r corresponds to either r_u or r_d .

6. INDECOMPOSABLE PERMUTATIONS

One calls a permutation *sum decomposable* if it can be expressed as the sum of two shorter permutations, and *sum indecomposable* otherwise. We analogously call a permutation \boxplus -*decomposable* if it can be expressed as $\sigma \boxplus \tau$ for two shorter permutations each of length at least two, and \boxplus -*indecomposable* otherwise. (We must require that both σ and τ have length at least 2 to avoid trivial decompositions, because $1 \boxplus \pi = \pi \boxplus 1 = \pi$.)

Table 1 shows the decomposition of permutations of lengths at most four, as well as their expressions of the form of ψ_w° , for those permutations that can be expressed that way. To cut the number of cases in half, when considering permutations of length four we utilize the fact that these concepts are invariant under *complementation* of permutations (flipping their plots upside down).

From Table 1 we see that there are eight \boxplus -indecomposable permutations of length four (1342, 1423, 1432, 2413, and their complements). In this table, an initial letter r corresponds to either r_u or r_d , so of the permutations of length four, only 1423 and its complement 4132 have unique representations of the form ψ_w , while the other six \boxplus -indecomposable permutations do not. The permutation 1432 and its complement 4123 in fact cannot be expressed in terms of \boxplus and permutations of the form ψ_w° at all. This means that these two permutations are not subpermutations of any rightward-yearning pin sequence, and thus they do not arise in the classes we construct.

We show next that all sufficiently long rightward-yearning pin sequences are \boxplus -indecomposable. The bound $|w| \geq 4$ is necessary because, as shown in Table 1, $\psi_{dr_d}^\circ = 213 = 21 \boxplus 12$ and, symmetrically, $\psi_{ur_u}^\circ = 231 = 12 \boxplus 21$.

Lemma 6.1. *Suppose that $w \in \mathcal{P}$. If $|w| \geq 3$, then ψ_w^\bullet is \boxplus -indecomposable. If $|w| \geq 4$, then ψ_w° is also \boxplus -indecomposable.*

Proof. We proceed by induction on the length of w . The base case for the claim about ψ_w° follows from examining Table 1 for ψ_w° . For the claim about ψ_w^\bullet , the base case follows by the following computations: $\psi_{ur_u}^\bullet = 1342$, $\psi_{ur_d}^\bullet = \psi_{r_dur_u}^\bullet = 2413$, $\psi_{ruur_u}^\bullet = 1423$, $\psi_{ru}^\bullet = \psi_{r_d}^\bullet = 3142$, $\psi_{rd}^\bullet = 4213$, and $\psi_{rd}^\bullet = 4132$.

w	ψ_w^\bullet	w	ψ_w^\bullet
ur_uur_u	13524	dr_dur_u	31524
ur_udr_d	35142	dr_ddr_d	53142
r_ur_uu	14253	r_dur_uu	24153
r_ur_ud	25314	r_dur_ud	35214
r_udr_du	31452	r_ddr_du	41352
r_udr_dd	42513	r_ddr_dd	52413

Table 2: The 12 words of length 4 in \mathcal{P} and their corresponding permutations ψ_w^\bullet .

Now let v denote the prefix of w comprising all but the last letter, and suppose by induction that ψ_u^\bullet (resp., ψ_u°) is \boxplus -indecomposable. The only way in which ψ_w° (resp., ψ_w^\bullet) could be \boxplus -decomposable is if the last letter corresponds to a point that is inserted northeast or southeast of all of ψ_v° (resp., ψ_v^\bullet), or if it is inserted next to the origin. Neither case is possible since pins must separate the predecessor pin from all the earlier ones. This means that the final pin cannot be placed in the top-right or bottom-right corner of ψ_w° (resp. ψ_w^\bullet), and also, since $|\psi_v^\circ| \geq 4$ (resp., $|\psi_v^\bullet| \geq 4$), there are at least two other pins whose positions come between the origin and the final pin. Thus ψ_w° and ψ_w^\bullet are \boxplus -indecomposable. \square

Next we show that, except for a small number of shorter permutations, all of the \boxplus -indecomposable permutations that arise in our classes $\mathcal{C}^{(s_k)}$ are of the form ψ_w° .

Proposition 6.2. *A subpermutation π of a rightward-yearning pin sequence is \boxplus -indecomposable if and only if $\pi \in \{1, 12, 21, 132, 312\}$ or $\pi = \psi_w^\circ$ for some word $w \in \mathcal{P}$ with $|w| \geq 4$.*

Proof. For permutations of lengths at most three, the result follows from an examination of Table 1. Now suppose that π is a subpermutation of a rightward-yearning pin sequence and $|\pi| \geq 4$. If $\pi = \psi_w^\circ$ for some word $w \in \mathcal{P}$, then $|w| \geq 4$, so Lemma 6.1 shows that π is \boxplus -indecomposable. Conversely, suppose that π is \boxplus -indecomposable. Since π is a subpermutation of a rightward-yearning pin sequence, Proposition 5.1 shows that there are nonempty words $w_1, \dots, w_k \in \mathcal{P}$ such that

$$\pi = \psi_{w_1}^\circ \boxplus \psi_{w_2}^\bullet \boxplus \dots \boxplus \psi_{w_k}^\bullet.$$

Obviously, the only way that π could be \boxplus -indecomposable in this case would be if all of the words w_i except one were empty. If w_j is the nonempty word then, by noting that $\psi_\varepsilon^\bullet = 1$ and $\psi_\varepsilon^\circ \boxplus \psi_w^\bullet = \psi_w^\circ$, we have $\pi = \psi_{w_j}^\circ$, and thus $|w_j| = |\pi| \geq 4$, completing the proof. \square

Finally, we investigate the uniqueness of the words encoding these \boxplus -indecomposable permutations. In this direction, we are primarily interested in the uniqueness of words for permutations without an origin, but it is easier to first establish the result for permutations with an origin. Note that the bound $|w| \geq 4$ is best possible because $\psi_{ur_d}^\bullet = \psi_{r_dur_u}^\bullet = 2413$.

Proposition 6.3. *If $\psi_v^\bullet = \psi_w^\bullet$ for words $v, w \in \mathcal{P}$ satisfying $|v| = |w| \geq 4$, then $v = w$.*

Proof. We proceed by induction on $|v| = |w|$. The base case of $|v| = |w| = 4$ follows from an examination of Table 2. Now suppose that $|v| = |w| \geq 5$. Note that the first entry of ψ_v^\bullet must correspond to the origin p_0 , and every entry of ψ_v^\bullet that lies above the origin must correspond to u or r_u , while every entry that lies below the origin must correspond to d or r_d .

Consider the rightmost entry of ψ_v^\bullet , which we may assume lies above the origin as the other case follows by a symmetrical argument. In this case, the last right step in both v and w is encoded by the letter r_u , and this letter is either the final or penultimate letter of both v and w .

If r_u is the final letter of both v and w , then we can remove it—the permutations are still equal, and thus by induction, so are the words v and w with the last letter removed from each. Similarly, if r_u is the penultimate letter in both v and w , then the final letter of each word must be the same (corresponding to the second to last entry of $\psi_v^\bullet = \psi_w^\bullet$), and again we can remove it and apply induction. In either case, we conclude that $v = w$.

It remains to consider the case where, without loss of generality, r_u is the final letter of v , and the penultimate letter of w . Thus we have, say, $v = xr_u$ and $w = yr_uz$ where x and y are words of lengths at least four and three, respectively, and $z \in \{u, d\}$. Removing the rightmost point of $\psi_v^\bullet = \psi_w^\bullet$ corresponds in each case to removing this final r_u . In the case of ψ_v^\bullet , this leaves us with ψ_x^\bullet , since r_u was the last pin, whereas for ψ_w^\bullet we obtain $\psi_y^\bullet \boxplus \psi_z^\bullet$ since r_u was an interior pin. However, we must have $\psi_x^\bullet = \psi_y^\bullet \boxplus \psi_z^\bullet$, and this is impossible since ψ_x^\bullet is \boxplus -indecomposable by Lemma 6.1, while $\psi_y^\bullet \boxplus \psi_z^\bullet$ is not. \square

For pin sequences without an origin, the analogous result must account for the fact that $\psi_{r_u x}^\circ = \psi_{r_d x}^\circ$ for all words $x \in \mathcal{P}$. Additionally, we note that $\psi_{ur_u ur_u}^\circ = \psi_{rd rd}^\circ = 2413$, so our result must start with words of length 5.

Proposition 6.4. *If $\psi_v^\circ = \psi_w^\circ$ for words $v, w \in \mathcal{P}$ satisfying $|v| = |w| \geq 5$, then either $v = w$, or $v = r_u x$ and $w = r_d x$ for some word $x \in \mathcal{P}$.*

Proof. Consider a word $v \in \mathcal{P}$ of length at least 5. We first claim that there exists a function $\phi : \mathcal{P} \rightarrow \mathcal{P}$ with $|\phi(v)| = |v| - 1$ such that $\psi_v^\circ = \psi_{\phi(v)}^\circ$.

To establish the claim, we recall that $\psi_v^\circ = \psi_\varepsilon^\circ \boxplus \psi_v^\bullet$, and consider the effect of deleting the origin from ψ_v^\bullet . The second-leftmost entry of ψ_v^\bullet becomes the new origin, and its position relative to the rest of the permutation depends upon the first few letters of v . The following chart defines the function ϕ for words of length at least 3, and in each case it is straightforward to verify that $\psi_v^\circ = \psi_{\phi(v)}^\circ$.

v	$\phi(v)$
$ur_u x$	$r_d x$
$dr_d x$	$r_u x$
$r_u ur_u x$	} $dr_d x$
$r_d ur_u x$	
$r_u dr_d x$	} $ur_u x$
$r_d dr_d x$	

(It would be possible to define ϕ explicitly for shorter words as well, but we do not need to do so.)

Now consider words v, w of length ≥ 5 such that $\psi_v^\circ = \psi_w^\circ$. We have

$$\psi_{\phi(v)}^\bullet = \psi_v^\bullet = \psi_w^\bullet = \psi_{\phi(w)}^\bullet,$$

and since $|\phi(v)| = |v| - 1 \geq 4$, Proposition 6.3 implies that $\phi(v) = \phi(w)$. By the definition of ϕ , it now follows that either $v = w$, or v and w differ only in their first letter, as required. \square

The combination of Theorem 5.2 with Propositions 6.3 and 6.4 provide us with a guarantee that any permutation $\pi \in \mathcal{C}^{(s_k)}$ corresponds to an almost-unique collection of words $v_1, v_2, \dots, v_\ell \in \mathcal{P}^{(s_k)}$, providing $|v_1| \geq 5$ and $|v_i| \geq 4$ for $i \geq 2$. If such criteria are met, then the only possible ambiguity arises in the first letter of v_1 .

7. WELL-QUASI-ORDER

The fact that the classes $\mathcal{C}^{(s_k)}$ are, in a sense, “ \boxplus closed” enables us to establish that these classes are all wqo, although this takes a bit of preparation. Recall that the binary languages $\mathcal{L}^{(s_k)}$ are wqo under the factor order by Proposition 2.3. First we must lift this property to our pin sequence languages $\mathcal{P}^{(s_k)}$.

Proposition 7.1. *For every sequence (s_k) of positive integers, the set $\mathcal{P}^{(s_k)} \subseteq \{r_d, r_u, d, u\}^*$ of words is wqo under the factor order.*

Proof. Given any word w , we define $\Delta_L(w)$ to be the word obtained by removing the first letter of w (assuming that w is nonempty). If u is contained as a factor in w , it follows that $\Delta_L(u)$ is contained as a factor in $\Delta_L(w)$. Therefore, if the language \mathcal{S} is wqo under the factor order, then the language $\Delta_L(\mathcal{S})$ is wqo under the factor order. We similarly define $\Delta_R(w)$ to be the word obtained by removing the last letter of w .

By definition, for every word $v \in \mathcal{P}^{(s_k)}$, there is some word $w \in \mathcal{L}^{(s_k)}$ for which v is a factor of $\rho(w)$. Indeed, if we take w to be minimal, then v comprises all but possibly the first and last letter of $\rho(w)$. Thus we can express $\mathcal{P}^{(s_k)}$ as

$$\mathcal{P}^{(s_k)} = \rho(\mathcal{L}^{(s_k)}) \cup \Delta_L(\rho(\mathcal{L}^{(s_k)})) \cup \Delta_R(\rho(\mathcal{L}^{(s_k)})) \cup \Delta_L(\Delta_R(\rho(\mathcal{L}^{(s_k)}))).$$

This shows that $\mathcal{P}^{(s_k)}$ is the union of four wqo posets, and is therefore wqo itself. \square

To go from the languages $\mathcal{P}^{(s_k)} \subseteq \{r_d, r_u, d, u\}^*$ to the permutation classes $\mathcal{C}^{(s_k)}$, we need to first recall the setting and statement of Higman’s lemma. Given a poset (X, \leq) , we denote by X^* the set (or language) of all words with letters from X . The *generalized subword order* on X^* is defined by stipulating that the word $v = v(1) \cdots v(k)$ is contained in the word $w = w(1) \cdots w(n)$ if and only if w has a subsequence $w(i_1)w(i_2) \cdots w(i_k)$ such that $v(j) \leq w(i_j)$ for all indices j . The following is a weakened version of Higman’s original result.

Higman’s lemma [25]. *If (X, \leq) is wqo, then X^* is also wqo, under the generalized subword order.*

Higman’s lemma immediately implies (via Proposition 7.1) that the poset $(\mathcal{P}^{(s_k)})^*$ is wqo under the generalized subword order. Note that in this poset, the “letters” of a “word” are in fact words from $\mathcal{P}^{(s_k)}$ (which are themselves defined over the alphabet $\{r_d, r_u, d, u\}$).

Now define a mapping $\phi : (\mathcal{P}^{(s_k)})^* \rightarrow \mathcal{C}^{(s_k)}$ by

$$\Phi(w) = \psi_{w(1)}^\circ \boxplus \psi_{w(2)}^\bullet \boxplus \cdots \boxplus \psi_{w(k)}^\bullet.$$

Proposition 4.1 shows that the mappings $w \mapsto \psi_w^\circ$ and $w \mapsto \psi_w^\bullet$ are both order-preserving, and it then follows from the definition of \boxplus that ϕ is order-preserving. Theorem 5.2 further implies that ϕ maps surjectively onto $\mathcal{C}^{(s_k)}$.

The main result of this section then follows from the general fact that if the domain of an order-preserving mapping is wqo, then its range must be as well. (This fact is easily proved by contradiction, for one could pull back any infinite antichain in the range of such a mapping to find an infinite antichain in its domain.)

Proposition 7.2. *For every sequence (s_k) of positive integers, the permutation class $\mathcal{C}^{(s_k)}$ is wqo.*

8. DISTINCT ENUMERATIONS

Having shown that the classes $\mathcal{C}^{(s_k)}$ are all wqo, we now finish the proof of our main theorem by establishing that they have distinct enumerations.

Theorem 8.1. *Suppose that (s_k) and (t_k) are distinct sequences of positive integers, and that (s_k) lexicographically precedes (t_k) . Then there exists an integer N such that*

$$\bigcup_{n \leq N} \mathcal{C}_n^{(t_k)} \subsetneq \bigcup_{n \leq N} \mathcal{C}_n^{(s_k)}.$$

In particular, the classes $\mathcal{C}^{(s_k)}$ and $\mathcal{C}^{(t_k)}$ have distinct enumeration sequences.

Proof. Let M be the integer from Proposition 2.5, so $\mathcal{L}_n^{(s_k)} = \mathcal{L}_n^{(t_k)}$ for all $n < M$, but $\mathcal{L}_M^{(t_k)} \subsetneq \mathcal{L}_M^{(s_k)}$. We claim that $N = 2M - 1$ satisfies the requirements of the theorem. Since $M \geq 3$, we have $N \geq 5$.

First, let $\pi \in \mathcal{C}^{(t_k)}$ be a permutation of length $n \leq N$. By Theorem 5.2, we have

$$\pi = \psi_{v_1}^\circ \boxplus \psi_{v_2}^\bullet \boxplus \cdots \boxplus \psi_{v_\ell}^\bullet$$

for words $v_1, v_2, \dots, v_\ell \in \mathcal{P}^{(t_k)}$ with $|v_1| + \cdots + |v_\ell| = n$.

For each v_i (for $i = 1, 2, \dots, \ell$), take a shortest word $w_i \in \mathcal{L}^{(t_k)}$ such that v_i is a factor of $\rho(w_i)$. Now v_i can have at most two fewer letters than $\rho(w_i)$, and thus $2|w_i| = |\rho(w_i)| \leq |v_i| + 2$. Furthermore, since $|v_i| \leq N = 2M - 1$, we conclude that $|w_i| \leq M$. By Proposition 2.5, we have $w_i \in \mathcal{L}^{(s_k)}$. Thus $\rho(w_i) \in \mathcal{P}^{(s_k)}$, and since v_i is a factor of $\rho(w_i)$, it follows that $v_i \in \mathcal{P}^{(s_k)}$. Thus v_1, v_2, \dots, v_ℓ all lie in $\mathcal{P}^{(s_k)}$, and hence $\pi \in \mathcal{C}^{(s_k)}$ by Theorem 5.2. This establishes that $\mathcal{C}_n^{(t_k)} \subseteq \mathcal{C}_n^{(s_k)}$ for all $n \leq N$.

It remains to exhibit a permutation $\pi \in \mathcal{C}^{(s_k)} \setminus \mathcal{C}^{(t_k)}$ of length at most N . Take a word $w \in \mathcal{L}_M^{(s_k)} \setminus \mathcal{L}_M^{(t_k)}$, whose existence is guaranteed by Proposition 2.5. The word $\rho(w)$ clearly lies in $\mathcal{P}^{(s_k)}$. However, it cannot lie in $\mathcal{P}^{(t_k)}$, as this would imply that there exists $u \in \mathcal{L}^{(t_k)}$ such that $\rho(w)$ is a factor of $\rho(u)$, which would mean that w is a factor of u , and hence that $w \in \mathcal{L}^{(t_k)}$.

The word $\rho(w)$ ends with ur_u or dr_d . In either case, let x be the word obtained by removing its last letter. Again, x clearly lies in $\mathcal{P}^{(s_k)}$, but it cannot lie in $\mathcal{P}^{(t_k)}$, as any word that contains x as a proper factor would also have to contain $\rho(w)$, and we have already established that $\rho(w) \notin \mathcal{P}^{(t_k)}$. We now have a word x of length $2M - 1 = N$ that lies in $\mathcal{P}^{(s_k)} \setminus \mathcal{P}^{(t_k)}$ and which begins with u or d .

We now consider the permutation $\pi = \psi_x^\circ$, which by definition lies in $\mathcal{C}_N^{(s_k)}$. Since $N \geq 5$, by Proposition 6.2 we have that π is \boxplus -indecomposable.

Suppose that $\pi \in \mathcal{C}_N^{(t_k)}$. As π is \boxplus -indecomposable, Theorem 5.2 tells us that $\pi = \psi_y^\circ$ for some $y \in \mathcal{P}^{(t_k)}$. Now $\psi_y^\circ = \pi = \psi_x^\circ$ has length at least 5, and thus by Proposition 6.4 we must have either that $x = y$, or that x and y are identical after their first letter, which in each case must be r_u or r_d . However, by construction, x begins with u or d , and so in fact we must have $y = x$, and hence $x \in \mathcal{P}^{(t_k)}$, a contradiction. Thus, $\pi \notin \mathcal{C}_N^{(t_k)}$ and the proof is complete. \square

9. BINARY SEQUENCES WITH DISTINCT COMPLEXITY FUNCTIONS

In this short section, we consider a consequence of the work in Section 2 in the study of countably infinite binary words (commonly referred to as *sequences*). For a binary sequence \mathbf{w} , the *complexity* of \mathbf{w} is the function that counts the number of words of length n contained (as factors) in \mathbf{w} .

The main result of this section is as follows. After the statement of the theorem, we introduce only the bare minimum in terminology, and refer the reader to, for example, Durand, Leroy, and Richomme [21] for a fuller treatment.

Theorem 9.1. *Let $S = \{\sigma, \mu\}$ where σ and μ are constant-length morphisms defined by*

$$\mu : \begin{cases} 0 \mapsto 01 \\ 1 \mapsto 10 \end{cases} \quad \text{and} \quad \sigma : \begin{cases} 0 \mapsto 00 \\ 1 \mapsto 11. \end{cases}$$

The collection of S -adic sequences contains uncountably many sequences with distinct complexity functions.

Note that the morphism μ is the *Thue–Morse morphism*. That is, the Thue–Morse sequence is a fixed point of μ .

Given a binary sequence \mathbf{w} , an *S -adic representation* of \mathbf{w} is a sequence $(\sigma_n)_{n \in \mathbb{N}}$ of morphisms (known as a *directive word*) where $\sigma_n \in S$, together with a sequence of letters $(a_n)_{n \in \mathbb{N}}$ from $\{0, 1\}$, such that

$$\mathbf{w} = \lim_{n \rightarrow \infty} \sigma_1 \cdots \sigma_n(a_n a_n a_n \cdots).$$

In this case, we may say that \mathbf{w} is an *S -adic sequence*. Our interest in these sequences relates to the set of finite factors that they contain. Given a sequence \mathbf{w} , let

$$\mathcal{L}_n^{\mathbf{w}} = \{\text{factors of } \mathbf{w} \text{ of length } n\}.$$

Note that the union of these sets, $\mathcal{L}^{\mathbf{w}} = \bigcup \mathcal{L}_n^{\mathbf{w}}$, is a factor-closed language. We now have the following result. It is likely that the restriction to sequences comprising only powers of two in this result can be avoided if we replace our set of morphisms $S = \{\mu, \sigma\}$ by something more complicated, but we do not require this.

Proposition 9.2. *Let $(s_k)_{k \in \mathbb{N}}$ be a sequence of positive integers where each term is a power of two, that is, $s_i \in \{2^n : n \geq 0\}$. Then there exists an S -adic sequence \mathbf{w} such that $\mathcal{L}^{\mathbf{w}} = \mathcal{L}^{(s_k)}$.*

Note that Theorem 9.1 follows immediately from the above proposition, when combined with Proposition 2.5.

Proof. For $k \geq 0$, consider the morphism $\mu_k = \mu \sigma^k$. By definition, we have

$$\mu_k : \begin{cases} 0 \mapsto (01)^{2^k} \\ 1 \mapsto (10)^{2^k}. \end{cases}$$

Given a sequence $(s_k)_{k \in \mathbb{N}}$ of powers of two, let $t_k = \log_2(s_k)$. We claim that $\mathcal{L}^{\mathbf{w}} = \mathcal{L}^{(s_k)}$, where \mathbf{w} is the S -adic sequence defined by the directive word $\mu \sigma^{t_1} \mu \sigma^{t_2} \cdots$ and the all-zero sequence $0, 0, \dots$

To establish the claim, first note (for example, by induction) that

$$\mu_{t_1} \cdots \mu_{t_i}(0) = (\alpha_i^{(s_k)})^{s_{i+1}} \quad \text{and} \quad \mu_{t_1} \cdots \mu_{t_i}(1) = (\bar{\alpha}_i^{(s_k)})^{s_{i+1}}.$$

As a consequence, the sequence \mathbf{w} can be written as a word in $\{\alpha_i^{(s_k)}, \bar{\alpha}_i^{(s_k)}\}^*$, for any $i \geq 1$. It follows immediately that $\alpha_i^{(s_k)} \in \mathcal{L}^{\mathbf{w}}$ for all i , and by definition of $\mathcal{L}^{(s_k)}$, this establishes that $\mathcal{L}^{(s_k)} \subseteq \mathcal{L}^{\mathbf{w}}$.

For the reverse containment, consider a word $\gamma \in \mathcal{L}^{\mathbf{w}}$. Fix any i so that $|\gamma| \leq |\alpha_i^{(s_k)}|$, and consider \mathbf{w} as a sequence in $\{\alpha_i^{(s_k)}, \bar{\alpha}_i^{(s_k)}\}^*$. Since γ is a factor of \mathbf{w} , it follows that γ is either a factor of

$$\alpha_i^{(s_k)} \text{ or } \bar{\alpha}_i^{(s_k)},$$

or it ‘straddles’ two such factors, and thus is a factor of one of

$$\alpha_i^{(s_k)} \alpha_i^{(s_k)}, \quad \alpha_i^{(s_k)} \bar{\alpha}_i^{(s_k)}, \quad \bar{\alpha}_i^{(s_k)} \alpha_i^{(s_k)} \text{ or } \quad \bar{\alpha}_i^{(s_k)} \bar{\alpha}_i^{(s_k)}.$$

In any case, all of these words are in $\mathcal{L}^{(s_k)}$, thus $\gamma \in \mathcal{L}^{(s_k)}$. Hence $\mathcal{L}^{\mathbf{w}} \subseteq \mathcal{L}^{(s_k)}$, completing the proof. \square

We finish this section by briefly noting another property of the S -adic sequences above, namely that we can bound the complexity of all these S -adic sequences by a linear function.

Proposition 9.3. *Let $S = \{\mu, \sigma\}$, and let $\mathbf{w} \in \{0, 1\}^*$ be any S -adic sequence. Then $|\mathcal{L}_n^{\mathbf{w}}| \leq 8n$.*

Proof. This follows directly from Corollary 2.1 in [20], which states the following: for a finite alphabet A , a letter $a \in A$, and a sequence of morphisms $(\sigma_n : \{0, 1\} \rightarrow \{0, 1\}^*)_{n \in \mathbb{N}}$ of constant length ℓ , the complexity of $\mathbf{w} = \lim \sigma_1 \cdots \sigma_n(aa \cdots)$ is at most $\ell|A|^2 n$. \square

10. CONCLUDING REMARKS

Our main result, established in Sections 7 and 8, disproves False Conjecture 1.3 by showing that there are too many different enumerations of wqo permutation classes for them all to have algebraic generating functions. However, the weaker Question 1.4, which asks if every lwqo permutation class has an algebraic generating function, might still be true. Unfortunately, there are wqo-but-not-lwqo permutation classes that are strongly algebraic, so unlike False Conjecture 1.3, a positive resolution to this question would not give a structural characterization of the strongly-algebraic classes.

The permutations we have used to build our counterexamples $\mathcal{C}^{(s_k)}$ are all proper pin sequences or subpermutations thereof. As mentioned in the text, pin sequences were first defined by Brignall, Huczynska, and Vatter [13], although our language-theoretical approach is influenced by Brignall, Ruškuc, and Vatter [14]. It was conjectured [14, Conjecture 16] that the class of all proper pin sequences and their subpermutations has a rational generating function, and that conjecture was later proved by Bassino, Bouvel, and Rossin [9], who also showed that this class has an infinite basis. Thus all of the classes we have used to disprove False Conjecture 1.3 are contained in a class with a rational generating function (though this should not be too surprising, as Theorem 1.2 guarantees that such a class must exist).

For the remainder of the conclusion we discuss a much rougher notion of enumeration, that of growth rates. The *upper growth rate* of the permutation class is defined by

$$\overline{\text{gr}}(\mathcal{C}) = \limsup_{n \rightarrow \infty} \sqrt[n]{|\mathcal{C}_n|}.$$

When the limit superior above is actually a limit (as has been conjectured is always the case⁵), then we call this the *proper growth rate* of the class and denote it by $\text{gr}(\mathcal{C})$.

⁵A weaker version of this conjecture, applying only to finitely-based permutation classes, was explicitly asked by the second author at the conference *Permutation Patterns 2005* [22, Section 4], but that may not be the first time it was posed. A stronger conjecture, stated in the context of ordered graphs, has been posed by Balogh, Bollobás, and Morris [8, Conjecture 8.1].

There has been extensive research on the set of all possible growth rates of permutation classes and at what growth rates certain phenomena first appear [3, 10, 26, 27, 34, 37, 38, 40]. For example, the smallest non-wqo permutation class (equivalently, the infinite antichain whose downward closure has the smallest possible growth rate) has growth rate denoted by κ , the largest real root of $x^3 - 2x^2 - 1$, which is approximately 2.21. Furthermore, it has been established that every permutation class of growth rate smaller than κ has a rational generating function [4, Section 8]. Thus it may be interesting to determine the growth rates of the classes $\mathcal{C}^{(s_k)}$, or perhaps more ambitiously, to answer the following question.

Question 10.1. *What is the smallest real number ω such that there is a wqo permutation class of growth rate ω that fails to have an algebraic generating function?*

It is not difficult to show that the classes $\mathcal{C}^{(s_k)}$ have *proper* growth rates, as we now explain. This follows from a Fekete’s lemma argument that seems to have first been stated in the permutation context by Arratia [7]⁶. We use the multiplicative form of Fekete’s lemma, which states that if the sequence (a_n) of positive real numbers satisfies $a_{m+n} \geq a_m a_n$ for all m and n (that is, if it is supermultiplicative), then $\lim \sqrt[n]{a_n}$ exists and equals $\sup \sqrt[n]{a_n}$.

Proposition 10.2. *For every sequence (s_k) of positive integers, the permutation class $\mathcal{C}^{(s_k)}$ has a growth rate.*

Proof. Theorem 5.2 states that $\mathcal{C}^{(s_k)}$ consists precisely of the permutations of the form

$$\psi_{v_1}^\circ \boxplus \psi_{v_2}^\bullet \boxplus \cdots \boxplus \psi_{v_\ell}^\bullet$$

for words $v_1, v_2, \dots, v_\ell \in \mathcal{P}^{(s_k)}$. Take $\pi \in \mathcal{C}_m^{(s_k)}$, $\sigma \in \mathcal{C}_n^{(s_k)}$, and express σ as

$$\sigma = \tau_{w_1}^\circ \boxplus \tau_{w_2}^\bullet \boxplus \cdots \boxplus \tau_{w_\ell}^\bullet$$

for some words $w_1, \dots, w_\ell \in \mathcal{P}^{(s_k)}$. The mapping that sends the pair (π, σ) to the permutation

$$\pi \boxplus \tau_{w_1}^\bullet \boxplus \tau_{w_2}^\bullet \boxplus \cdots \boxplus \tau_{w_\ell}^\bullet \in \mathcal{C}_{m+n}^{(s_k)}$$

is an injection from $\mathcal{C}_m^{(s_k)} \times \mathcal{C}_n^{(s_k)}$ to $\mathcal{C}_{m+n}^{(s_k)}$, and thus $\lim \sqrt[n]{|\mathcal{C}_n^{(s_k)}|}$ exists by Fekete’s lemma. \square

We leave open the question of what the growth rates of the classes $\mathcal{C}^{(s_k)}$ actually are, and whether these growth rates even depend on the choice of the sequence (s_k) .

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⁶Although it seems likely given his some of his writing [16, Problem 28] that Knuth was aware of this application of Fekete’s lemma in 1972.

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