Wreath Products of Permutation Classes

Robert Brignall
School of Mathematics and Statistics
University of St Andrews
St Andrews, Fife, Scotland
robertb@mcs.st-and.ac.uk
http://turnbull.mcs.st-and.ac.uk/~robertb

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Abstract

A permutation class which is closed under pattern involvement may be described in terms of its basis. The wreath product construction $X \wr Y$ of two permutation classes X and Y is also closed, and we investigate classes Y with the property that, for any finitely based class X, the wreath product $X \wr Y$ is also finitely based.

1 Introduction and Statement of Theorem

Two finite sequences of the same length, $\alpha = a_1 a_2 \cdots a_n$ and $\beta = b_1 b_2 \cdots b_n$, are said to be *order isomorphic* if, for all i, j, we have $a_i < a_j$ if and only if $b_i < b_j$. Viewing permutations of length n as orderings on the numbers $1, 2, \ldots, n$, every sequence of n distinct symbols is order isomorphic to a unique permutation. A permutation σ is said to be *involved* in the permutation π (denoted $\sigma \leq \pi$) if there is a subsequence (or *pattern*) of π order isomorphic to σ[†]. For example, $1324 \leq 6351427$ because of the subsequence 3547. A book introducing the study of these permutation patterns has been written by Bóna [6].

This involvement order forms a partial order on the set of all finite permutations; sets of permutations which are closed downwards under this order are called *permutation classes*. These classes are specified primarily in one of three ways:

- **Pattern avoidance.** A permutation class X can be regarded as a set of permutations which *avoid* certain patterns. The set B of minimal permutations not in X forms an antichain, and is known as the *basis* of X. We write $X = \mathrm{Av}(B)$ to mean the class $X = \{\pi \mid \beta \not\leq \pi \text{ for all } \beta \in B\}$. Antichains (and hence bases) need not be finite – see, for example, Atkinson, Murphy and Ruškuc [3], Murphy [11] and Murphy and Vatter [12].

[†]For a sequence α (not necessarily a permutation) and set of permutations Y, with a slight abuse of notation, we will sometimes write statements like "$\alpha \in Y$", meaning "the permutation order isomorphic to α lies in Y."

- **Permuting machines.** Permutation classes arise naturally as a result of machines which permute an input stream of symbols. The first such class to appear was the set of stack-sortable permutations, presented by Knuth [10].

- **Constructions.** New permutation classes can be formed using constructions involving one or more old classes. Atkinson [2] gives the first study of these, and some further constructions can be found in Atkinson and Stitt [4] and Murphy [11].

In all but the first of these, a natural question to ask is if the class is finitely based. In the case of permuting machines – more specifically, stack sorting – Bóna's survey [5] reviews several answers to this question. In the case of constructions, there are many with only partial answers. Here, we will consider the question of basis for the wreath product, a construction which is intrinsically connected to simple permutations and the substitution decomposition – see Albert and Atkinson [1] and Brignall, Huczynska and Vatter [8]. A special case of the wreath product – the "profile classes" of [2] – was also used to give alternative proofs of the enumeration results in West [13].

Given a permutation $\pi \in S_n$ and nonempty permutations $\alpha_1, \alpha_2, \ldots, \alpha_n$, the *inflation of* π *by* $\alpha_1, \alpha_2, \ldots, \alpha_n$ is the permutation obtained by replacing each point $\pi(i)$ by an interval order isomorphic to α_i, and is denoted $\pi[\alpha_1, \alpha_2, \ldots, \alpha_n]$. For example, $132[21, 2413, 321] = 217968543$. Conversely, a *deflation* of π is any permutation σ arising from a decomposition $\pi = \sigma[\pi_1, \pi_2, \ldots, \pi_m]$.

The *wreath product* of two sets of permutations X and Y (not necessarily permutation classes) is the set $X \wr Y$ of all permutations which can be expressed as an inflation of a permutation in X by permutations in Y, i.e. the set of permutations of the form $\pi[\alpha_1, \alpha_2, \ldots, \alpha_n]$ with $\pi \in X$ and $\alpha_1, \alpha_2, \ldots, \alpha_n \in Y$. It is easy to check that the wreath product of two permutation classes is again a permutation class, but in only a few cases is the question of finite basis answerable. It is proved in [4] that for any finitely based class X, the wreath product $X \wr \mathrm{Av}(21)$ is also finitely based, and that $\mathrm{Av}(21) \wr \mathrm{Av}(321654)$ is not finitely based. Our primary aim here is to establish the following general theorem:

Theorem 1.1. *For any finitely based class Y not admitting arbitrarily long pin sequences, the wreath product $X \wr Y$ is finitely based for all finitely based classes X.*

The approach is constructive; first we introduce Y-profiles, which give us the ability to decompose permutations arising in wreath products into components belonging to the two original classes. For a permutation not arising in such a wreath product, we prove the existence of a subsequence order isomorphic to a basis element of the class X. Moreover, there is a basis element of Y lying within the "minimal block" defined by any two points of this subsequence. It is then a matter of using these considerations to show that, when the class Y admits only finite pin sequences, the minimal elements not in the wreath product have bounded size.

Our secondary aim, arising as a result of the above considerations, is to exhibit a number of classes of the form $Y = \mathrm{Av}(\alpha)$ for $|\alpha| \leq 3$, or $Y = \mathrm{Av}(\alpha, \beta)$ with $|\alpha| \leq 4$, $|\beta| \leq 4$ which do not satisfy Theorem 1.1, and to demonstrate how an infinitely based wreath product $X \wr Y$ can be found in each case.

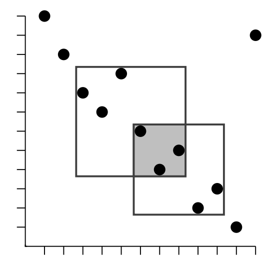

Figure 1: Two intervals and their intersection.

2 Simplicity and Substitution Decomposition

As mentioned earlier, the wreath product is closely related to simple permutations and the substitution decomposition, both of which we will need, so here we review these concepts. Often we are going to view permutations as points in a plane; the *plot* of a permutation π is the set of coordinates $\{(i, \pi(i))\}$ in the plane. This viewpoint will provide invaluable insight into many of the structural considerations discussed later on.

An *interval* or *block* of a permutation π is a segment $\pi(i)\pi(i+1)\cdots\pi(i+j)$ in which the set of values forms an interval of natural numbers. In the plot of a permutation, intervals can be seen as a set of points enclosed in an axis-parallel rectangle, with no points lying in the regions above, below, to the left or to the right. It is worth noting that the intersection of two intervals is itself an interval, an observation clearly seen in Figure 1.

The permutation π is *simple* if its only intervals are singletons, or the whole of π. Note that simple permutations have only trivial deflations, and are the only permutations with this property. As such, they can be regarded as the building blocks of permutation classes. Every permutation can be written as the inflation of a unique simple permutation, and this decomposition is known as the *substitution decomposition*. We shall refer to the unique simple permutation in this decomposition as the *skeleton*. If the skeleton has length at least 4, then the whole decomposition is unique:

Proposition 2.1. *If π has a substitution decomposition $\sigma[\pi_1, \pi_2, \ldots, \pi_m]$ with $m \geq 4$, then every π_i is determined uniquely.*

When $m = 2$, we may write $\pi = 12[\pi_1, \pi_2]$, in which case π is *sum decomposable*, or $\pi = 21[\pi_1, \pi_2]$, in which case π is *skew decomposable*, and in both cases the choice of π_1, π_2 is not necessarily unique. A permutation that is not sum (respectively, skew) decomposable is *sum* (resp. *skew*) *indecomposable*.

3 Y-Profiles

We need to be able to know when a given permutation lies in the wreath product of two permutation classes. This could be done by inspecting all possible decompositions and checking for membership of the orignal classes, but this is liable to be computationally intensive. Instead, we would prefer only to check a single decomposition, from which membership or otherwise of the wreath product is immediately obvious.

The *profile* of a permutation π is the unique permutation obtained by contracting every maximal consecutive increasing sequence in π into a single point [2]. For example, the profile of 3415672 is 3142 because of the segments 34, 1, 567 and 2.

The notion of a "Y-profile" connects this idea with the definition of the substitution decomposition $\pi = \sigma[\pi_1, \dots, \pi_m]$ of π. We want the Y-profile of π to be the shortest possible deflation of π, given we may only deflate by elements from the class Y. However, this is not clearly well-defined, so before we can proceed, we must first introduce Y-deflations.

Formally, let Y be a permutation class, and π any permutation. Then a *Y-deflation* of π is a permutation π' for which π can be expressed as $\pi'[\alpha_1, \alpha_2, \dots, \alpha_k]$ with $\alpha_1, \alpha_2, \dots, \alpha_k \in Y$. For an arbitrary permutation π, there are many different Y-deflations. However, the shortest one is unique, and it is this one that gives rise to the Y-profile.

Lemma 3.1. *For every closed class Y and permutation π, the shortest Y-deflation of π is unique.*

Proof. We proceed by induction on $n = |\pi|$. The case $n = 1$ is trivial, so now suppose $n > 1$. Fix a shortest Y-deflation of the permutation π, and label this permutation π^Y. If $\pi \in Y$ then $\pi^Y = 1$ is unique, so we will assume $\pi \notin Y$.

Let σ, of length $m \geq 2$, be the skeleton of π, and first consider the case where $m \geq 4$, whereby we have the unique substitution decomposition $\pi = \sigma[\pi_1, \pi_2, \dots, \pi_m]$. By the inductive hypothesis, the shortest Y-deflations of $\pi_1, \pi_2, \dots, \pi_m$ are unique, and we will label them $\pi_1^Y, \pi_2^Y, \dots, \pi_m^Y$. We claim that $\pi^Y = \sigma[\pi_1^Y, \pi_2^Y, \dots, \pi_m^Y]$. Consider any other Y-deflation of π, $\pi = \pi'[\alpha_1, \alpha_2, \dots, \alpha_k]$. Since $\pi \notin Y$, π' cannot be trivial, and so $\sigma \leq \pi'$, and indeed σ is the skeleton of π', giving a unique deflation $\pi' = \sigma[\pi'_1, \dots, \pi'_m]$. Moreover, π'_i is a Y-deflation of π_i for all i. Since π_i^Y is the unique shortest Y-deflation, we must have $\pi_i^Y \leq \pi'_i$, which implies $\pi^Y \leq \pi'$.

When $m = 2$, more care is required. In this case π is either sum or skew decomposable, and without loss of generality we may assume the former. Write $\pi = 12 \cdots t[\pi_1, \pi_2, \dots, \pi_t]$ where each π_i is sum indecomposable. If every $\pi_i \in Y$, then any shortest Y-deflation of π will be an increasing permutation of length at most t, and as there is only one increasing permutation of each length, π^Y will be unique. So now suppose that there exists at least one i such that $\pi_i \notin Y$, so that $|\pi_i^Y| \geq 2$. Since π_i is sum indecomposable, π_i^Y is also sum indecomposable. We claim the shortest Y-deflation of π will be

$$\pi^Y = (\pi_1 \oplus \cdots \oplus \pi_{i-1})^Y \oplus \pi_i^Y \oplus (\pi_{i+1} \oplus \cdots \oplus \pi_t)^Y.$$

Any other Y-deflation will also have to be written as a direct sum of three permutations in this way, and by induction each of these will involve the respective shortest Y-deflation. □

Thus, for any class Y and permutation π, the *Y-profile* of π is the unique shortest Y-deflation of π, and is denoted π^Y. Note that setting $Y = \mathrm{Av}(21)$, the set of increasing permutations, returns the original definition of the profile, but if we set $Y = S$, the set of all permutations, we do not get the substitution decomposition back, as $\pi^S = 1$ for any permutation. However, an easy consequence of the above proof is that if $\pi \notin Y$, and σ is the skeleton of π, then $\sigma \leq \pi^Y$.

As mentioned at the beginning of this section, our aim with Y-profiles is to be able to to move from the permutations of the wreath product $X \wr Y$ down to the permutations in the two classes X and Y in a single step. Thus although initially we may know very little about the structure of a permutation in the basis of $X \wr Y$, by taking its Y-profile we should be left with a permutation involving a (known) basis element of X. Conversely, we want to be able to construct basis elements of $X \wr Y$ given only the bases of X and Y. These ideas are encapsulated in the following theorem.

Theorem 3.2. *Let X and Y be two arbitrary permutation classes. Then $\pi \in X \wr Y$ if and only if $\pi^Y \in X$.*

Proof. One direction is immediate. For the converse, since $\pi \in X \wr Y$, there exists $\pi' \in X$ which is a deflation of π by permutations in Y. The proof of Lemma 3.1 then tells us that $\pi^Y \leq \pi'$, completing the proof. □

Any expression of the form $\pi = \pi^Y[\alpha_1, \ldots, \alpha_k]$ is called a *Y-profile decomposition* of π, and the blocks α_i are called the *Y-profile blocks*. These blocks are not typically uniquely defined. For example, the Av(123)-profile of 234615 is 23514, but it can be decomposed either as 23514[12, 1, 1, 1, 1] or 23514[1, 12, 1, 1, 1]. Thus it will be useful to fix a particular Y-profile decomposition, especially as later we are going to need to know about the structure of each of the Y-profile blocks.

The *left-greedy Y-profile* of π is the decomposition $\pi = \pi^Y_\lambda[\lambda_1, \lambda_2, \ldots, \lambda_\ell]$ with $\lambda_i \in Y$ for all i, in which λ_1 is first chosen maximally, then λ_2, and so on. Each λ_i is called a *left-greedy Y-profile block* of π. This yields the usual, unique, Y-profile:

Lemma 3.3. *For any class Y and permutation π, $\pi^Y = \pi^Y_\lambda$.*

Proof. Again, we use induction on $n = |\pi|$. The base case $n = 1$ is trivial, so now suppose $n > 1$. Assume further that $\pi \notin Y$, as otherwise $\pi^Y = \pi^Y_\lambda = 1$ follows immediately. Let $\pi = \pi^Y_\lambda[\lambda_1, \lambda_2, \ldots, \lambda_\ell]$ be the left-greedy Y-profile of π, let $\pi^Y[\alpha_1, \alpha_2, \ldots, \alpha_k]$ be any other Y-profile decomposition of π, and let $\sigma[\pi_1, \pi_2, \ldots, \pi_m]$ be the substitution decomposition.

Consider first the case where $m = |\sigma| \geq 4$. By the proof of Lemma 3.1, we have $\pi^Y = \sigma[\pi_1^Y, \pi_2^Y, \ldots, \pi_m^Y]$. A similar argument shows that $\pi^Y_\lambda = \sigma[(\pi_1)^Y_\lambda, (\pi_2)^Y_\lambda, \ldots, (\pi_m)^Y_\lambda]$, and by induction $\pi_i^Y = (\pi_i)^Y_\lambda$ for all i, giving the required result.

When $m = 2$, π is either sum or skew decomposable, and we may assume the former. Write $\pi = 12\cdots t[\pi_1, \pi_2, \ldots, \pi_t]$ where each π_i is sum indecomposable. In the case where every $\pi_i \in Y$, both π^Y and π^Y_λ will be increasing permutations with $k \leq \ell \leq t$. When using the left-greedy Y-profile decomposition, the block λ_1 was chosen maximally, and so $\alpha_1 \leq \lambda_1$. Then the block λ_2 was taken maximally, so the Y-profile block α_2 cannot extend further right than the end of λ_2, hence $\alpha_2 \leq \lambda_1 \oplus \lambda_2$. Continuing in this manner, we see that, for all i, $\alpha_i \leq \lambda_1 \oplus \lambda_2 \oplus \cdots \oplus \lambda_i$, and in particular $\alpha_k \leq \lambda_1 \oplus \lambda_2 \oplus \cdots \oplus \lambda_k$. But we must have $k \leq \ell$, and so $k = \ell$. The remaining case is where at least one $\pi_i \notin Y$. Pick i to be minimal with this property, and then by the proof of Lemma 3.1,the Y-profile breaks into three pieces,

$$\pi^Y = (\pi_1 \oplus \cdots \oplus \pi_{i-1})^Y \oplus \pi_i^Y \oplus (\pi_{i+1} \oplus \cdots \oplus \pi_t)^Y.$$

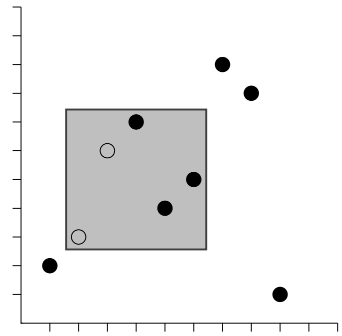

Figure 2: The minimal block $\mathrm{mb}(\pi; 2, 3)$ in $\pi = 236745981$.

A similar argument holds for the left-greedy Y-profile, and then by induction each of the three pieces in the left-greedy Y-profile is equal to the corresponding piece in the Y-profile. □

There is, of course, nothing special about the left-greedy Y-profile; it can be seen that any algorithm to compute a Y-profile-like decomposition in which at each stage the blocks are chosen maximally will yield a Y-profile deflation. For our purposes, however, when required we will always use the left-greedy algorithm.

4 The Minimal Block

The primary aim of this section is to be able to tell if any two points in a permutation belong to the same left-greedy Y-profile block, and also a partial converse: given the Y-profile deflation, what can we say about the points "between" two specified points? To this end, we define a new concept as follows. Let π be any permutation of length n. For all $1 \leq i < j \leq n$, the *minimal block of* π *that contains* $\pi(i)$ *and* $\pi(j)$, denoted $\mathrm{mb}(\pi; i, j)$, is the set of points of π which forms the shortest interval involving both $\pi(i)$ and $\pi(j)$. In other words, there exists $k \leq i$ and $\ell \geq j - k$ such that $\mathrm{mb}(\pi; i, j) = \pi(k) \cdots \pi(k + \ell)$ forms an interval but no subsegment of this contains both $\pi(i)$ and $\pi(j)$ and forms an interval. For example, if $\pi = 236745981$, then the minimal block on $\pi(2) = 3$ and $\pi(3) = 6$ is $\mathrm{mb}(\pi; 2, 3) = 36745$ (See Figure 2).

It follows from the observation that the intersection of two intervals itself forms an interval that the minimal block is always uniquely defined. Before we can proceed to the main result, we make one further observation.

Lemma 4.1. *Let* π *be any permutation and let* $i \neq j$ *be any pair of positions in* π*. Then if* $k, \ell \in \mathrm{mb}(\pi; i, j)$ *with* $k \neq \ell$ *we have*

$$\mathrm{mb}(\pi; k, l) \subseteq \mathrm{mb}(\pi; i, j).$$

Moreover, if both i *and* j *separate* k *from* ℓ *by position, then* $\mathrm{mb}(\pi; k, \ell) = \mathrm{mb}(\pi; i, j)$.

Proof. That $\mathrm{mb}(\pi; k, \ell)$ is contained in $\mathrm{mb}(\pi; i, j)$ is obvious. Now suppose i and j separate k from ℓ by position, i.e. $k \leq i < j \leq \ell$. Then $\mathrm{mb}(\pi; k, \ell)$ is an interval of π involving both $\pi(i)$ and $\pi(j)$. As $\mathrm{mb}(\pi; i, j)$ is minimal with this property, we have $\mathrm{mb}(\pi; i, j) \subseteq \mathrm{mb}(\pi; k, \ell)$ and so $\mathrm{mb}(\pi; i, j) = \mathrm{mb}(\pi; k, \ell)$. □

We are now ready to prove our main technical result of this section.

Lemma 4.2. *Let Y be a permutation class, and let $\pi \in S_n$ be any permutation. Then for any pair i, j with $1 \leq i < j \leq n$:*

(i) If the permutation order isomorphic to $\mathrm{mb}(\pi; i, j)$ does not lie in Y, then $\pi(i)$ and $\pi(j)$ lie in different Y-profile blocks.

(ii) Conversely, if $\pi(a_i)$ and $\pi(a_j)$ are the first symbols of two distinct left greedy Y-profile blocks α_i and α_j respectively, then the permutation order isomorphic to $\mathrm{mb}(\pi; i, j)$ does not lie in Y.

Proof. (i) By minimality and uniqueness of the minimal block, every block in π containing both $\pi(i)$ and $\pi(j)$ must contain the minimal block $\mathrm{mb}(\pi; i, j)$. Hence every such block does not lie in Y, so cannot be a Y-profile block.

(ii) Write $\pi = \pi^Y[\alpha_1, \alpha_2, \ldots, \alpha_k]$, and let the sequence $\pi(a_1), \pi(a_2), \ldots, \pi(a_k)$ represent the leading points in π of the left-greedy Y-profile blocks $\alpha_1, \alpha_2, \ldots, \alpha_k$. Let α_i and α_j, $i < j$, be a pair of Y-profile blocks. We prove the statement by induction on i.

When $i = 1$, the block α_1 was picked maximally subject to $\alpha_1 \in Y$. For any $j > 1$, the minimal block $\mathrm{mb}(\pi; a_1, a_j)$ strictly contains α_1 and then the maximality of α_1 is contradicted unless $\mathrm{mb}(\pi; a_1, a_j) \notin Y$.

Suppose now that $i > 1$, and that $\mathrm{mb}(\pi; a_\ell, a_j) \notin Y$ for any $\ell < i$ and $j > \ell$. The Y-profile block α_i was picked maximally to avoid basis elements of Y, subject to starting at symbol $\pi(a_i)$. Consider, for some $j > i$, the minimal block $\mathrm{mb}(\pi; a_i, a_j)$, necessarily containing all of α_i. If the leftmost point of $\mathrm{mb}(\pi; a_i, a_j)$ is $\pi(a_i)$, then since α_i is the maximal block lying in Y which starts at $\pi(a_i)$, we must have $\mathrm{mb}(\pi; a_i, a_j) \notin Y$. So now suppose that $\mathrm{mb}(\pi; a_i, a_j)$ contains at least one symbol $\pi(h)$ from π with $h < a_i$. Let the Y-profile block containing $\pi(h)$ be α_ℓ; we claim that α_ℓ is completely contained in $\mathrm{mb}(\pi; a_i, a_j)$. If not, then part of α_ℓ lies outside $\mathrm{mb}(\pi; a_i, a_j)$ in both position and value, and so the part lying inside $\mathrm{mb}(\pi; a_i, a_j)$ itself forms an interval in either the top-left or bottom-left corner of the minimal block, but yet it contains neither $\pi(a_i)$ nor $\pi(a_j)$, contradicting the minimality of $\mathrm{mb}(\pi; a_i, a_j)$. In particular, the first symbol $\pi(a_\ell)$ of α_ℓ is in $\mathrm{mb}(\pi; a_i, a_j)$, and by Lemma 4.1, we have $\mathrm{mb}(\pi; a_\ell, a_j) = \mathrm{mb}(\pi; a_i, a_j)$. By the inductive hypothesis $\mathrm{mb}(\pi; a_\ell, a_j) \notin Y$, and so $\mathrm{mb}(\pi; a_i, a_j) \notin Y$. □

Using this result, we now know when two points of a permutation will lie in the same Y-profile block, and, more importantly for what follows, we know that a basis element of Y exists in the minimal block of the first symbols of any two Y-profile blocks. What we do not yet know is how to find it; given such a minimal block, we need a method to search through the block systematically and locate the points that form this basis element within a bounded number of steps. This is the subject of the next section.

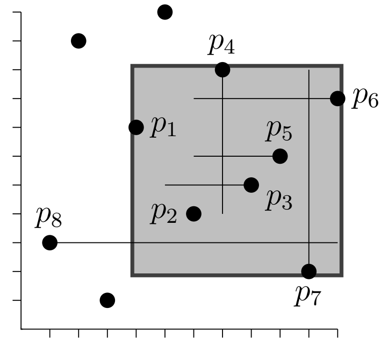

Figure 3: A pin sequence.

5 Pin Sequences and the Wreath Product

Pin sequences were introduced by Brignall, Huczynska and Vatter [7] in the study of simple permutations. The idea there is that, since simple permutations have no non-trivial intervals, if we begin with any two points we can use pin sequences to get to any chosen edge of our simple permutation. Here we are not working solely with simple permutations, and we cannot expect the same result to hold in the general case. However, we can obtain the same result for the minimal block. We begin by reviewing some terminology from [7], and to do this it is best to revert to viewing permutations as plots in the plane.

For points $p_1, p_2, \ldots, p_m$ in the plane, let $\text{rect}(p_1, p_2, \ldots, p_m)$ be the smallest axis-parallel rectangle containing them. Note that this is different to the minimal block, as we do not require that $\text{rect}(p_1, p_2, \ldots, p_m)$ be an interval.

Let π be a permutation. A *pin sequence* is a sequence of points $p_1, p_2, \ldots$ of π which for $i \geq 3$ obey, when plotted in a plane, the following two conditions.

- $p_i \notin \text{rect}(p_1, p_2, \ldots, p_{i-1})$,
- p_i *slices* $\text{rect}(p_1, p_2, \ldots, p_{i-1})$ either horizontally or vertically. That is p_i lies between two points of $\text{rect}(p_1, p_2, \ldots, p_{i-1})$ either by position or value.

For each pin p_i, $i \geq 3$, we also specify a direction, being left, right, up or down. For example, a *left pin* is one that lies between two point of $\text{rect}(p_1, p_2, \ldots, p_{i-1})$ by value, but whose position is smaller than any point of $\text{rect}(p_1, p_2, \ldots, p_{i-1})$. In Figure 3, p_3, p_5 and p_6 are right pins, p_4 is an up pin, p_7 a down pin and p_8 a left pin.

We create a *proper pin sequence* by adjoining two further conditions:

- *Maximality*: each pin must be taken maximally in its direction. For example, a proper left pin out of $\text{rect}(p_1, p_2, \ldots, p_{i-1})$ must be the left pin slicing $\text{rect}(p_1, p_2, \ldots, p_{i-1})$ with smallest position.
- *Separation*: in slicing $\text{rect}(p_1, p_2, \ldots, p_i)$, p_{i+1} must lie between p_i and $\text{rect}(p_1, p_2, \ldots, p_{i-1})$ either by position or value.

For example, in Figure 3, p_8 is a proper left pin as it slices p_7 from $\text{rect}(p_1, p_2, \ldots, p_6)$ and is maximal in its direction. Similarly, p_4 and p_7 are proper pins, but p_3, p_5 and p_6 are not, as

p_3 does not obeying maximality, p_5 does not separate p_4 from $\mathrm{rect}(p_1, p_2, p_3)$, and p_6 does not separate p_5 from $\mathrm{rect}(p_1, p_2, p_3, p_4)$.

In a proper pin sequence, the maximality and separation conditions force the pin p_{i+1} to have direction perpendicular to the direction of p_i, so for example a left pin can only be followed by an up pin or a down pin.

If a pin sequence $p_1, p_2, \ldots, p_m$ of π is such that $\mathrm{rect}(p_1, p_2, \ldots, p_m)$ encloses all of π, then we say that it is *saturated*. When we restrict to proper pin sequences this is likely to be impossible to acheive, even in simple permutations. However a weaker condition does hold. A pin sequence $p_1, p_2, \ldots, p_m$ of π is said to be *right-reaching* if $\mathrm{rect}(p_1, p_2, \ldots, p_m)$ encloses all of π:

Proposition 5.1 (Brignall, Huczynska, Vatter [7])**.** *From any pair of points in a simple permutation, there exists a proper right-reaching pin sequence.*

Since we are not working solely with simple permutations, we need to modify this proposition. Instead, we want the same to hold within a minimal block, defined as usual by two points, which also form the first two points of our proper pin sequence. Here, right-reaching means that the last pin is the right-most point of the minimal block, rather than of the whole permutation. Hence:

Lemma 5.2. *Let $\pi \in S_n$ be any permutation, and let $1 \le i < j \le n$. Then there exists a proper pin sequence with starting points $p_1 = (i, p_i)$ and $p_2 = (j, p_j)$ which is right-reaching in* $\mathrm{mb}(\pi; i, j)$*.*

Proof. In the minimal block $\mathrm{mb}(\pi; i, j)$, there exists a saturated (non-proper) pin sequence $p_1, p_2, \ldots$ starting from the pins $p_1 = (i, \pi(i))$ and $p_2 = (j, \pi(j))$. If there were no such sequence, then some corner of the minimal block, not including either $\pi(i)$ or $\pi(j)$, would form an interval by itself, contradicting the minimality of $\mathrm{mb}(\pi; i, j)$. Moreover, we may assume, by removing unnecessary pins and relabelling, that every pin is maximal in its direction.

The proof then follows the proof in [7] of Proposition 5.1. Since the pin sequence is saturated, it includes the rightmost point of π. Label this point p_{i_1}. Next, take the smallest $i_2 < i_1$ such that $p_1, p_2, \ldots, p_{i_2}, p_{i_1}$ is a valid pin sequence, and observe that p_{i_1} separates p_{i_2} from $\mathrm{rect}(p_1, p_2, \ldots, p_{i_2-1})$, as $p_1, p_2, \ldots, p_{i_2-1}, p_{i_1}$ is not a valid pin sequence. Continue in this manner, finding pins $p_{i_3}, p_{i_4}, \ldots$ until we reach $p_{i_{m+1}} = p_2$, and then $p_1, p_2, p_{i_m}, p_{i_{m-1}} \ldots, p_{i_1}$ is a proper right-reaching pin sequence. ☐

Proposition 5.1 is easily recovered from Lemma 5.2 by setting π to be a simple permutation, and observing that all minimal blocks in a simple permutation are the whole permutation.

We are now ready to prove our main result.

Theorem 5.3. *Let $Y = \mathrm{Av}(B)$ be a finitely based permutation class not admitting arbitrarily long pin sequences. Then $X \wr Y$ is finitely based for all finitely based classes $X = \mathrm{Av}(D)$.*

Proof. Let $b = \max_{\beta \in B}(|\beta|)$, $d = \max_{\delta \in D}(|\delta|)$, and π be any permutation in the basis of $X \wr Y$. By Theorem 3.2, we have $\pi^Y \notin X$, and so there exists some $\delta \in D$ such that

$\delta \le \pi^Y$. We will be done if we can identify a bounded subsequence of π order isomorphic to a permutation ω, say, for which $\delta \le \omega^Y$, as then $\omega^Y \notin X$ implies $\omega \notin X \wr Y$, and hence $\omega = \pi$.

First include in our subsequence of π the set of points order isomorphic to δ with positions $d_1, d_2, \ldots, d_k$ ($k = |\delta|$), chosen so that each $\pi(d_i)$ is the leftmost point of a distinct left greedy Y-profile block, and the choice of blocks is also leftmost. For every pair d_i, d_{i+1}, Lemma 4.2 tells us that the minimal block $\mathrm{mb}(\pi; d_i, d_{i+1})$ involves some $\beta \in B$, and we include one such occurrence of this β in our subsequence. Our aim now is to add a bounded number of points so that β still lies in the minimal block of the permutation ω on the points corresponding to $\pi(d_i)$ and $\pi(d_{i+1})$, as then these two points are preserved distinctly in ω^Y. We do this by taking a proper right-reaching and a proper left-reaching pin sequence of $\mathrm{mb}(\pi; d_i, d_{i+1})$ (which exist by Lemma 5.2), and including them in the subsequence. These pin sequences are only guaranteed to be bounded when Y does not admit arbitrarily long pin sequences, as then there exists a number N so that every pin sequence of length $N+2$ involves some basis element of Y.

Thus ω^Y still involves a subsequence order isomorphic to δ, and $|\omega| \le d + (d-1)(2(N-1)+b)$. $\square$

Brignall, Ruškuc and Vatter [9] proved that determining whether a finitely based class does not admit arbitrarily long pin sequences is decidable, and therefore given any pattern class we can tell whether Theorem 5.3 applies.

6 Infinitely Based Examples

For a class Y which admits infinite pin sequences, Theorem 5.3 gives us no information on whether the basis of $X \wr Y$ (here for a specified class X) is finite. However, the proof does tell us what some of the basis elements look like, namely permutations built around a basis element of X, and in the minimal block between each pair of these points, there is a basis element of Y. Constructing arbitrarily long basis elements of this type is then achieved by embedding arbitrarily long pin sequences in the minimal blocks. For example, the class $\mathrm{Av}(321)$ admits the infinite pin sequence formed by alternating between up and right pins, and so we have:

Theorem 6.1. $\mathrm{Av}(25134) \wr \mathrm{Av}(321)$ *is not finitely based.*

Proof. We exhibit an antichain generated by repeatedly taking up and right pins lying in the basis of $\mathrm{Av}(25134) \wr \mathrm{Av}(321)$. The first few elements of the antichain are

$$\begin{aligned}
\beta_1 &= 2,5,1,3,7,6,4\\
\beta_2 &= 2,5,1,3,7,4,9,8,6\\
\beta_k &= 2,5,1,3,7,4 \mid 9,6,11,8,\ldots,2k+3,2k \mid 2k+5,2k+4,2k+2 \quad (k \ge 3).
\end{aligned}$$

Here, as in [3], the $|$ symbol is used only to clarify the structure of the permutation. See Figure 4 for an illustration of a typical member of this antichain. We observe:

(i) The set $\{\beta_k \mid k \ge 1\}$ is an antichain.

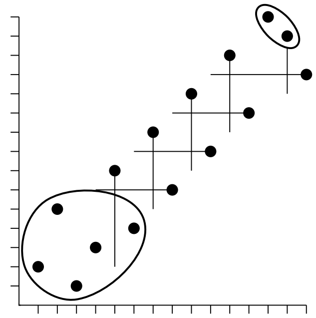

Figure 4: The element β_5 in the basis of $\mathrm{Av}(25134) \wr \mathrm{Av}(321)$.

(ii) The only occurrence of 321 in each β_k is $2k+5, 2k+4, 2k+2$.

(iii) The only occurrence of 25134 in each β_k is $2, 5, 1, 3, \cdot, 4$.

(iv) Each β_k is neither sum nor skew decomposable.

(v) The $\mathrm{Av}(321)$-profile of β_k is $2, 5, 1, 3, 7, 4, \ldots, 2k+3, 2k, 2k+4, 2k+2$ (the only nontrivial deflation occurs between $2k+5$ and $2k+4$). In particular, $25134 \prec \beta_k^{\mathrm{Av}(321)}$ for all k, hence by Theorem 3.2 $\beta_k \notin \mathrm{Av}(25134) \wr \mathrm{Av}(321)$.

It only remains to show that β_k is minimally not in $\mathrm{Av}(25134) \wr \mathrm{Av}(321)$. Consider the effect of removing any symbol j. If $j = 2k+5, 2k+4$ or $2k+2$ then by (ii) this no longer involves 321 so $\beta_k - j \in \mathrm{Av}(321) \subset \mathrm{Av}(25134) \wr \mathrm{Av}(321)$. Similarly, if $j = 2, 5, 1, 3$ or 4 then by (iii) $\beta_k - j$ no longer involves 25134 so $\beta_k - j \in \mathrm{Av}(25134) \subset \mathrm{Av}(25134) \wr \mathrm{Av}(321)$.

For any other j, $\beta_k - j$ is sum decomposable. Under the $\mathrm{Av}(321)$-profile, the first (lower) component deflates to a single point, and hence $(\beta_k - j)^{\mathrm{Av}(321)} \in \mathrm{Av}(25134)$. Thus $\beta_k - j \in \mathrm{Av}(25134) \wr \mathrm{Av}(321)$, completing the proof. $\square$

Note that in the above example, the class $X = \mathrm{Av}(25134)$ was specifically chosen so that the basis element 25134 is not contained in the repeated pin sequence used to build the antichain, but it does lie in the class Y. This ensures that 25134 acts as an "anchor" at the base of the antichain, but yet the only instance of the basis element 321 is in the upper "anchor".

As a result, for any class Y which contains both the infinite pin sequence formed by alternating between up and right pins, and the permutation 25134, the wreath product $\mathrm{Av}(25134) \wr Y$ will always contain an infinite antichain similar to the one above.

Example 6.2. (i) The classes $Y = \mathrm{Av}(321, 2341)$ and $Y = \mathrm{Av}(321, 3412)$ both avoid the permutation 321 and so the antichain in the proof of Theorem 6.1 lies in the basis of $\mathrm{Av}(25134) \wr Y$ in both cases.

(ii) All of the classes $Y = \mathrm{Av}(\alpha, \beta)$ with (α, β) being $(4321, 4312)$, $(4321, 4231)$, $(4321, 4213)$,

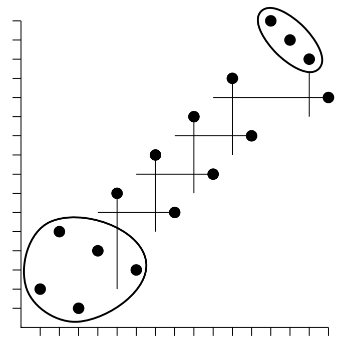

Figure 5: The element β_5 in the basis of $\mathrm{Av}(25143) \wr \mathrm{Av}(4321, 4123)$.

$(4321, 3412)$ and $(4321, 3214)$ avoid 4321, and so the antichain with terms

$$\begin{aligned}
\beta_1 &= 2,5,1,3,8,7,6,4\\
\beta_2 &= 2,5,1,3,7,4,10,9,8,6\\
\beta_k &= 2,5,1,3,7,4 \mid 9,6,11,8,\ldots,2k+3,2k \mid 2k+6,2k+5,2k+4,2k+2 \quad (k \geq 3)
\end{aligned}$$

lies in the basis of $\mathrm{Av}(25134) \wr Y$ in each case.

(iii) The classes $Y = \mathrm{Av}(4312, 4231)$, $Y = \mathrm{Av}(4312, 4213)$ and $Y = \mathrm{Av}(4312, 3421)$ all avoid 4312, so reversing the final two points of each β_k in case (ii) gives the required antichain.

Example 6.3. The two classes $Y = \mathrm{Av}(4321, 4123)$ and $Y = \mathrm{Av}(4312, 4123)$ both admit the pin sequence formed by repeatedly taking up and right pins, but do not contain the permutation 25134, because of the basis element 4123. However, the class $X = \mathrm{Av}(25143)$ may be used instead. In the first case, the antichain is (see Figure 5 for an illustration):

$$\begin{aligned}
\beta_1 &= 2,5,1,4,8,7,6,3\\
\beta_2 &= 2,5,1,4,7,3,10,9,8,6\\
\beta_k &= 2,5,1,4,7,3 \mid 9,6,11,8,\ldots,2k+3,2k \mid 2k+6,2k+5,2k+4,2k+2 \quad (k \geq 3).
\end{aligned}$$

All the examples so far have admitted the same "up-right" pin sequence. Another commonly found infinite pin sequence is formed by repeating the pattern left, down, right, up*, and there are two classes of the form $Y = \mathrm{Av}(\alpha, \beta)$ with $|\alpha| = |\beta| = 4$ which admit this sequence: $Y = \mathrm{Av}(3412, 2413)$ and $Y = \mathrm{Av}(3412, 2143)$. Each one must be handled separately.

Example 6.4. (i) $Y = \mathrm{Av}(3412, 2413)$ may be paired with $X = \mathrm{Av}(31542)$ to produce the antichain with terms

$$\begin{aligned}
\beta_1 &= 8,1,6,4,9,7,5,2,3\\
\beta_k &= 4k+4,1,4k+2,4,4k,6,\ldots 2k+6,2k \mid\\
&\quad\ 2k+4,2k+2,2k+7,2k+5,2k+3 \mid\\
&\quad\ 2k+9,2k+1,\ldots,4k+5,5 \mid 2,3 \quad (k \geq 2).
\end{aligned}$$

*This repeating pattern is the foundation for the "Widdershins" antichain of [11].

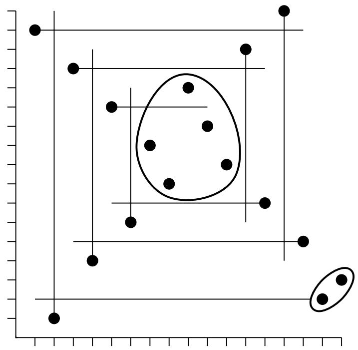

Figure 6: The basis element β_3 in $\mathrm{Av}(31542) \wr \mathrm{Av}(3412, 2413)$.

See Figure 6 for an illustration. Note that the occurrence of 3412 in any β_k is not unique, but every occurrence requires the final two symbols $2, 3$ of β_k, and so these points still behave in the same way as in previous examples.

(ii) $Y = \mathrm{Av}(3412, 2143)$ may be paired with $X = \mathrm{Av}(412563)$ to produce the antichain with terms:

$$\begin{aligned} \beta_1 &= 10, 1, 8, 4, 6, 9, 11, 7, 5, 2, 3 \\ \beta_k &= 4k+6, 1, 4k+4, 4, 4k+2, 6, \ldots, 2k+8, 2k \mid \\ &\quad\; 2k+6, 2k+2, 2k+4, 2k+7, 2k+9, 2k+5, 2k+3 \mid \\ &\quad\; 2k+11, 2k+1, \ldots, 4k+7, 5 \mid 2, 3 \quad (k \geq 2). \end{aligned}$$

7 Concluding Remarks

The above examples suggest, to some extent, a general method for finding infinite bases. However, these examples rely on just one method for constructing antichains, and there is no reason why this method should always work[‡]. Moreover, within this construction, finding a suitable class X for a given class Y is very specific in each case.

In fact, it is unlikely that we can always find such a class X. For example, the class of all subpermutations of the *increasing oscillating sequence*, $416385\cdots$, is given by $\mathrm{Av}(321, 2341, 3412, 4123)$ [9], and admits the infinite proper pin sequence alternating between an up pin and a right pin. However, there are no other permutations in this class which can be used to anchor an infinite antichain based around this pin sequence, so the method described hitherto does not work here. We therefore pose the following question.

Question 7.1. *Is there a finitely based class X for which $X \wr \mathrm{Av}(321, 2341, 3412, 4123)$ is not finitely based?*

[‡]A somewhat different construction was used by Atkinson and Stitt [4] to demonstrate an infinite antichain in the basis of $\mathrm{Av}(21) \wr \mathrm{Av}(321654)$, relying on the sum decomposability of the basis element 321654.

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